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# Common Fixed Point Theorems under Rational Inequality

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**Abstract**
In this paper we establish common fixed point theorems for two pairs of self maps in a complete metric space by using occasionally weakly biased maps satisfying the property (E.A.) using contraction condition involving rational expressions. These results partially generalize Pachpatte [10], Jeong and Rhoades [5] and Kameswari [9].
**Keywords**: Weakly compatible, occasionally weakly compatible, property (E.A), coincidence point, point of coincidence, common fixed point.
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## 1. Introduction and Definitions

Obtaining common fixed points under weak commutative conditions using contraction conditions for more than one selfmap involving rational expressions is an interesting aspect. Sessa [10] introduced the concept of weakly commuting maps which is more general than commuting maps. Afterwards Jungck [6] initiated the concept of compatible maps which is more general than weakly commuting maps. Jungck and Rhoades [7] defined a weaker class of maps known as weakly compatible maps and obtained some common fixed point results.
Jungck and Pathak [8] introduced a new class of maps, namely biased maps and weakly biased maps whose class is larger than the class of compatible maps and weakly compatible maps respectively.
The purpose of this paper is to obtain common fixed point theorems for two pairs of selfmaps in metric space by using occasionally weakly compatible satisfying the property (E.A.) using contraction condition involving rational expressions. These results partially generalize Jeong and Rhoades [5] and Kameswari [9].

**Definitions**

**Definition 1.1:** Let S and A be selfmaps of set X. If Sx=Ax=w(say), w ϵ X, for some x in X, then x is called a coincidence point of S and A and the set of coincidence points of S and A is denoted by C(S,A) and w is called a point of coincidence of S and A.

**Definition 1.2:** The pair (A, S) is said to

(i). be compatible [6] if
$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$$
whenever $\{x_n\}$ is a sequence in X such that
$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t,$$
for some t in X.

(ii). be weakly compatible [7] if they commute at their coincidence point.

(iii). be Occasionally weakly compatible [7] if ASx = SAx for some xϵC(A,S).

*(iv).* be S-biased [10] if
$$\alpha d(SAx_n, Sx_n) \le \alpha d(ASx_n, Ax_n)$$
where $\alpha = \liminf$ and $\alpha = \limsup$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some t ϵ X.

(v). be A-biased [10] if
$$\alpha d(ASx_n, Ax_n) \le \alpha d(SAx_n, Sx_n)$$
where $\alpha = \liminf$ and $\alpha = \limsup$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some t ϵ X.

(vi). be weakly S-biased ( weakly A- biased) [10], if Ap=Sp implies
$d(ASp, Ap) \le d(SAp, Sp)$
$(d(SAp, Sp) \le d(ASp, Ap))$.

(vii). be occasionally weakly S-biased (Occasionally A-biased) [10] if and only if $d(ASp, Ap) \le d(SAp, Sp)$
$\left(d(SAp, Sp) \le d(ASp, Ap)\right)$
for some $p \epsilon C(A, S)$.

**Remark 1.3:**

(i). Every pair of compatible maps are weakly compatible but its converse need not be true [7].
(ii). Every pair of compatible maps is both S-biased and A-biased but its converse need not be true [10].
(iii). Every weakly compatible pair is occasionally weakly compatible but its converse need not be true [3].
(iv). Every pair of S-biased (A-biased) maps are weakly S-biased (A-biased) but its converse need not be true [10].
(v). Every pair of weakly S-biased (weakly A-biased) maps are Occasionally weakly S-biased (Occasionally weakly A-biased) but its converse need not be true [4].
(vi). Every pair of Occasionally weakly compatible is both occasionally weakly S-biased and occasionally A-biased, but its converse is not true [4].

**Remark 1.4:**

(i). Weakly compatible and property (E.A) are independent each other [2].
(ii). Occasionally weak compatible and property (E.A) are independent each other [1].

## 2. Main Result:

**Proposition 2.1:** Let A, B, S and T be selfmaps of a metric space (X,d) and satisfying the inequality.

($A_1$): there exists $\alpha \in [0,1)$ such that

$$d(Ax, By) \leq \alpha \max\left\{\frac{d(Ax,Sx)d(Sx,By) + d(By,Ty)d(Ty,Ax)}{d(Sx,By) + d(Ty,Ax)}, \quad d(Sx,Ty)\right\}$$

for all x, y ∈ X, if $d(Sx, By) + d(Ty, Ax) \neq 0$.
($A_2$): $d(Ax, By) = 0$ if $d(Sx, By) + d(Ty, Ax) = 0$.
Then the pairs (A, S) and (B, T) have a common point of coincidence in X,
if and only if $C(A,S) \neq \emptyset$ and $C(B,T) \neq \emptyset$.

**Proof:** If part: Let us assume that the pairs (A, S) and (B, T) satisfies common point of coincidence condition. i.e., there is a point t ∈ X such that

Au=Su=t=Bv=Tv for some u, v ∈ X.

Thus, clearly $u \in C(A,S)$ and $v \in C(B,T)$
Hence $C(A,S) \neq \emptyset$ and $C(B,T) \neq \emptyset$.

Only if part:

Assume that $C(A,S) \neq \emptyset$ and $C(B,T) \neq \emptyset$. Then there is $u \in C(A,S)$ and $v \in C(B,T)$ such that

Au=Su=p (say) ----------- (2.1.1)
Bv=Tv=q (say) ----------- (2.1.2)

Suppose $p \neq q$, then we consider x=u and y = v in our inequality (2.1.1)
since

$$d(Su, Bv) + d(Tt, Ap) = 2d(p,q) \neq 0$$

Hence from ($A_1$) we have

$$d(p,q) = d(Au, Bv)$$
$$\leq \alpha max\left\{\frac{d(Au,Su)d(Su,Bv) + d(Bv,Tv)d(Tv,Au)}{d(Su,Bv) + d(Tv,Au)}, \ d(Su,Tv)\right\}$$
$$= \alpha \max\{0, d(p,q)\}$$
$$\leq \alpha d(p,q),$$

a contradiction, thus p=q -------(2.1.3)
Therefore from (2.1.1) - (2.1.3), we have
Au=Su=Bv=Tv=p(=q).
Hence p is a common point of coincidence of A, B, S and T.

**Theorem 2.2:** In addition to the hypothesis of proposition 2.1 on A, B, S and T, if the pairs (A,S) and (B,T) are satisfies occasionally weakly S-biased and occasionally weakly T-Biased respectively then A,B,S and T have a unique common fixed point in X.

**Proof:** From the Proposition 2.1 there is a point $t \epsilon X$ such that
Au=Su=t=Bv=Tv for some u,v ϵX ---------(2.2.1)
Now we show that 't' is a common fixed point for A, B, S and T.
First we show that 't' is a fixed point of A.

Since the pair (A, S) is occasionally weakly S-biased from 2.2.1,
$d(SAu, Au) \leq d(ASu, Su)$
Implies that $d(St, t) \leq d(At, t)$. ---------(2.2.2)
Suppose At $\neq$ t
Since $d(St, Bv) + d(Tv, At) =$

$$d(St, t) + d(t, At) \leq 2d(At, t) \neq 0$$

Hence from ($A_1$) we have

$$d(At, t) = d(At, Bv)$$

$$\leq \alpha \max\left\{\frac{d(At, St)d(St, Bv) + d(Bv, Tv)d(Tv, At)}{d(St, Bv) + d(Tv, At)}, \quad d(St, Tv)\right\}$$

$$\leq \alpha \max\left\{\frac{d(At, St)d(St, t) + 0}{d(St, t) + d(t, At)}, d(St, t)\right\}$$

$$\leq \alpha \max\left\{\frac{[d(At, t) + d(St, t)]d(St, t)}{d(St, t) + d(t, At)}, d(St, t)\right\}$$

$\leq \alpha d(St, t)$ from (2.2.2)
$\leq \alpha d(At, t)$,
a contradiction. Hence we get At = t, from (2.2.2),

we have
At=St=t. ------------- (2.2.3)
Similarly, using the pair (B, T) is occasionally weakly T-biased maps and from (2.2.1), we get.
Bt=Tt=t. ------------- (2.2.4)
From (2.2.3) -- (2.2.4),
we have At=St=Bt=Tt=t.
Therefore t is a common fixed point of A, B, S and T.

**Theorem 2.3:**
Let A, B, S and T be selfmaps of a metric space (X, d) and satisfying ($A_1$) and ($A_2$) of Proposition 2.1 suppose that either
(i). B(X) ⊆ S(X), the pair (B,T) satisfying property (E.A), T(X) is closed. Or
(ii). A(X) ⊆ T(X), the pair (A,S) satisfying property (E.A) and A(X) is closed, holds.
Then the pairs (A, S) and (B, T) have a common point of coincidence in X.

**Proof:** Let us suppose (i) holds since the pair (B, T) satisfies property (E.A) there exists a sequence $\{x_n\}$ in X such that
$\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = v$, --------- (2.3.1)
for some v є X
Since B(X) ⊆ S(X) there exists a sequence $\{y_n\}$ in X
Such that $Bx_n = Sy_n$ , therefore
$\lim_{n\to\infty} Sy_n = v$ ----------- (2.3.2)
Now we show that $\lim_{n\to\infty} Ay_n = v$
Since $\lim_{n\to\infty} d(Sy_n, Bx_n) + d(Tx_n, Ay_n) \neq 0$
then from ($A_1$), we get
$d(Ay_n, Bx_n)$

$$\leq \alpha \max\left\{\frac{d(Ay_n, Sy_n)d(Sy_n, Bx_n) + d(Bx_n, Tx_n)d(Tx_n, Ay_n)}{d(Sy_n, Bx_n) + d(Tx_n, Ay_n)}, \quad d(Sy_n, Tx_n)\right\}$$

On taking limits on both sides, we get
$\lim_{n\to\infty} Ay_n = v$. ------------- (2.3.3)
From (2.3.1) and (2.3.3), we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = v \qquad \text{--------- (2.3.4)}$$

Since T(X) is complete, then there is a point z in X such that Tz=v. Now we prove that Bz=Tz,
Suppose that Bz≠Tz.
since $\lim_{n\to\infty} d(Sy_n, Bz) + d(Tz, Ay_n) \neq 0$,
then from ($A_1$) we get

$$d(Ay_n, Bz) \leq \alpha \max\left\{\frac{d(Ay_n, Sy_n)d(Sy_n, Bz) + d(Bz, Tz)d(Tz, Ay_n)}{d(Sy_n, Bz) + d(Tz, Ay_n)}, \quad d(Sy_n, Tz)\right\}$$

On taking limits as n→∞,
and using (2.3.3) and (2.3.4)

$$\lim_{n\to\infty} d(Ay_n, Bz) \leq \alpha \max\{0, 0\}$$

Therefore $d(Tz, Bz) = 0$.
implies that Tz=Bz ------------- (2.3.5)
Since B(X) ⊆ S(X) there exists a 'r' in X such that Bz=Sr. ------------ (2.3.6)
Now we show that Sr=Ar.
Suppose that Sr≠Ar, since from $A_2$

$$d(Sr, Bz) + d(Tz, Ar) \neq 0,$$

Then from $(A_1)$, we have

$$d(Ar, r) = d(Ar, Bz)$$
$$\leq \alpha \max\left\{\frac{d(Ar, Sr)d(Sr, Bz) + d(Bz, Tz)d(Tz, Ar)}{d(Sr, Bz) + d(Tz, Ar)}, \quad d(Sr, Tz)\right\}$$
$$= \alpha \max\{0,0)\}$$

Hence Ar=Sr. ----------- (2.3.7)
From (2.3.5) – (2.3.7), we get
Ar=Sr=Tz=Bz=v. ---------- (2.3.8)
Therefore 'v' is a common point of coincidence of A, B, S and T.

**Theorem 2.4:**
In addition to the hypothesis of theorem 2.3 on A, B, S and T, If the pairs (A, S) and (B, T) are satisfying Occasionally weakly S-Biased and Occasionally T-Biased respectively then A, B, S and T have a unique common fixed point in X.

The following is an example in support of Theorem 2.4.

**Example 2.5:**

Let X= [0,1) with the usual metric we defined self map A, B, S and T on X by

$$\mathrm{Ax} = \begin{cases} 0 \text{ if } x \in [0, \frac{1}{2}) \\ \frac{1}{3} \text{ if } x \in [\frac{1}{2}, 1) \end{cases}$$

$$\mathrm{Bx} = \begin{cases} 0 \text{ if } x \in [0, \frac{1}{2}) \\ \frac{1}{8} \text{ if } x \in [\frac{1}{2}, 1) \end{cases}$$

$$\mathrm{Sx} = \begin{cases} \frac{x}{2} \text{ if } x \in [0, \frac{1}{2}) \\ \frac{1}{2} \text{ if } x \in [\frac{1}{2}, 1) \end{cases}$$

$$\mathrm{Tx} = \begin{cases} 0 \text{ if } x \in [0, \frac{1}{2}) \\ \frac{3}{4} \text{ if } x \in [\frac{1}{2}, 1) \end{cases}$$

here B(X) ⊆ S(X), it is easy to verify that the pairs (A,S) and (B,T) satisfied all hypothesis of Theorem 2.4 and 0 is a common fixed point of A, B, S and T.

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