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A Common Fixed Point Theorem in Menger Probibilistic Quasi-Metric Spaces

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Abstract: The aim of this paper is to present a common fixed point theorem in Menger Probabilistic Quasi-metric space for weakly compatible maps.

Introduction: 1.1:
K. Menger [1] introduced the notion of a probabilistic metric space in 1942 and since then, the theory of probabilistic metric spaces has developed in many directions, especially, in nonlinear analysis and applications [2]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar [3] studied this concept and gave some fundamental results on this space. The important development of fixed point theory in Menger spaces was due to Sehgal and Bharucha-Reid [4]. Jungck [5] introduced the concept of compatible maps. And this condition has further been weakened by introducing the notion of weakly compatible mappings by Jungck and Rhoades [6]. The concept of weakly compatible mappings is most general as each pair of compatible mappings is weakly compatible but the reverse is not true. Recently in this line, Singh and Jain [7] introduced the notion of weakly compatible maps in Menger space to establish a common fixed point theorem.

Fixed point theorems for single-valued mappings have appeared in PQM-spaces (see [5, 8, 9, 10, 11]). Cho [3] proved common fixed point theorems for set-valued mappings in quasi-metric spaces. The theory of quasi-metric spaces can be used as an efficient tool to solve so many several problems like theoretical computer science, approximation theory and topological algebra (see[2, 7, 10]).

Definitions:

2.1: A mapping T : $[0,1]\times[0,1]\rightarrow[0,1]$ is t- norm if T is satisfying the following conditions.

a. T is commutative and associative.
b. T (a, 1) = a, for all a $\in[0,1]$
c. T (a, b) $\leq$ T(c, d), whenever a $\leq$ c and b $\leq$ d, for all a,b,c,d $\in$ [0,1].

The following are some basic t- norms:

T_M (a,b) = min {a,b} ;

T_P (a,b) = ab;

T_L (a,b) = max {a+b-1, 0}

2.2: A mapping F: R $\rightarrow$ R^+ is called a ditribution function if it is non-decreasing and left continuous with inf{F(t): t $\in$ R} = 0 and sup{F(t): t $\in$ R} = 1.

2.3: A Menger PQM-space is a triplet (X,F, T), where X is non decreasing empty set, T is continuous t-norm and F is probabilistic distance satisfying the following conditions: for all x, y, z $\in$ X.

a. $F_{x,y}$ (t) = ε_0 (t) and $F_{y,x}$ (t) = ε_0 (t) then x = y.

b. $F_{x,z}$ (t+s) $\geq$ T ($F_{x,y}$ (t), $F_{y,z}$ (s))

A Menger PQM - space is called a Menger PM - space if it satisfies the symmetry condition, i.e. $F_{x,y}$ (t) = $F_{y,x}$ (t), for all x, y $\in$ X.

` The notion of a Menger space is a generalization of a notion of a metric space. So Menger PQM - spaces offers a wider framework than that of metric space and are better suited to cover even wider statistical situations.

2.4: Let (X, F, T) be a Menger PQM - space and A be a non-empty subset of X. Then A is said to be probabilistically bounded if $\sup_{t>0}\inf_{x,y\in A}F_{x,y}(t)$ = 1. If X itself is probabilistically bounded , then X is said to be a probabilistically bounded space.Throughout this paper, B (X) will denote the family of non-empty bounded subsets of a menger PQM-space X, for all A, B $\in$ B (X) and for every t > 0, we define

$_D F_{A,B}$ (t) = sup{ F $_{a,b}$ (t): a $\in$ A, b $\in$ B}

and, $_\delta F_{A,B}(t)$ = inf {F $_{a,b}$ (t): a $\in$ A, b $\in$ B}

If set A consists of a singe point a, we write $_\delta F_{A,B}(t) = {}_\delta F_{a,B}(t)$

If set B also consists of a single point b, we write $_\delta F_{A,B}(t)$ = F $_{a,b}$ (t)

It follows immediately from the definition that $_\delta F_{A,B}(t) = 1$.

Thus we conclude that A = B = {a}, for some a $\in$ X.

2.5: Let (X,F,T) be a Menger PQM-space:

a. A sequence {x $_n$ } is said to be convergent to x $\in$ X if for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that F $_{x_n,x}$ (ε) > 1- λ whenever n $\geq$ N.

b. A sequence {x $_n$ } in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda > 0$, ,thwre exists a positive integer N such that F $_{x_n x_m}$ (ε) > 1- λ whwnever n , m $\geq$ N.

c. A Menger PQM - space in which every Cauchy sequence is convergent is said to be complete.

2.6: A t- norm T is of Hadzic type (H - type in short) and $T \in \mathrm{H}$ if the family $\{T^n\}_{n\in N}$ of its iterates defined, for each x in [0,1], by $T^0(x) = 1$, $T^{n+1}(x) = T(T^n(x), x)$ for all n $\geq$ 0 is equicontinuous at x = 1, that is, $\varepsilon \in (0,1|) \exists \delta \in (0,1)$: $x > 1-\delta$.

$\Rightarrow T^n(x) > 1-\varepsilon$ for all n $\geq$ 1.

2.7: If T is a t- norm and $(x_1, x_2, \ldots\ldots\ldots, x_n) \in [0,1]^n$ (n $\in$ N), then $T^n_{i=1} x_i$ is dfined recurrently by 1, if n =0 and $T^n_{i=1} x_i = T(T^{n-1}_{i=1} x_i, x_n)$ for all n $\geq$ 1. If $(x_i)_{i\in N}$ is a sequence of numbers from [0,1], then $T^\infty_{0 i=1} x_i$ is defined as $\lim_{n\to\infty} T^n_{i=1} x_i$ (this limit always exists) and $T^\infty_{i=n} x_i$ as $T^\infty_{i=1} x_{n+i}$

2.8: The mapping f: X$\rightarrow$ X and g: X$\rightarrow$ B (X) are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, that is , gu = {fu} for some u $\in$X, then fgu = gfu.

2.9: **Proposition:**

a. If T $\geq$ T $_L$, then the following implication holds:

$$\lim_{n\to\infty} T^\infty_{i=1} x_{n+i} = 1 \Leftrightarrow \sum_{n=1}^{\infty} (1 - x_n) < \infty$$

b. If $T \in \mathrm{H}$ then for every sequence (x $_n$) $_{n\in N}$ in [0,1] such that $\lim_{n\to\infty} x_n = 1$, one has

$$\lim_{n\to\infty} T^\infty_{i=1} x_{n+i} = 1$$

Note that if T is a t-norm for which there exists (x $_n$) $\subset$ [0,1] such that $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} T^\infty_{i=1} x_{n+i} = 1$, then sup $_{t<1}$ T(t,t)= 1.

2.10: **Proposition:**

Let (x $_n$) be a sequence of numbers from [0,1] such that $\lim_{n\to\infty} x_n = 1$ and t - norms T is of H-type. Then $\lim_{n\to\infty} T^\infty_{i=n} x_i = \lim_{n\to\infty} T^\infty_{i=1} x_{n+i} = 1$.

2.11: **Lemma:**

If a Menger PQM - space (X, F, T) satisfies the follwing condition: F $_{,y}$ $(t) = C$, for all t > 0 with fixed $x, y \in X$. Then we have, C= 1 and x = y.

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2.12: **Lemma:**

Let the function ϕ(t) satisfy the following condition

ϕ(t) : [0, ∞) $\rightarrow$[0, ∞] is non-decreasing and $\sum_{n=1}^{\infty}\phi^n(t) < \infty$, for all t > 0, when $\phi^n(t)$ denotes the nth iterative function of ϕ(t). Then ϕ(t) < t for all t > 0.

3.1: Theorem

Let (X, F, T) be a complete menger PQM - space. Further, let A and B be two weakly compatible self mappings with $t*t \geq t$ such that

i. T is Hadzic type

ii. g(X) $\subset$ f(X)

iii. $F_{g(x).g(y)}(\phi(t)) \geq \min\{F_{f(x),f(y)}(t), F_{f(x),g(y)}(t), F_{g(x),f(x)}(t), F_{g(x),f(y)}(t), F_{g(y),f(y)}(t)\}$

for all x, y $\in$ X and t > 0 where the function ϕ(t) : $[0,\infty) \rightarrow [0,\infty]$ is onto and strictly increasing and satisfying the condition (Φ)

iv. f(X) is closed subset of X, then

a. g and f have a coincidence point

b. the pair (g.f) is weakly compatible. Then $\exists$ a unique common fixed point z $\in$X :

{z}= {fz} = {gz}

Proof:

Let x_0 be an arbitrary point in X. By (iii) we can find x_1 such that f(x_1)$\in$g(x_0). By induction we can find the sequences { x_n } and { y_n } such that $y_{2n} = f(x_{2n+1}) \in g(x_{2n})$ for n$\in$ N.

Putting x = x_{2n} and y = x_{2n+1} in (iv), we get

$$F_{g(x_{2n}),g(x_{2n+1})}(\phi(t)) \geq \min\{F_{f(x_{2n}),f(x_{2n+1})}(t), F_{f(x_{2n}),g(x_{2n+1})}(t), F_{g(x_{2n}),f(x_{2n})}(t),$$
$$F_{g(x_{2n}),f(x_{2n+1})}(t), F_{f(x_{2n+1}),g(x_{2n+1})}(t)\}$$

$$F_{y_{2n},y_{2n+1}}(\phi(t)) \geq \min\{F_{y_{2n-1},y_{2n}}(t), F_{y_{2n-1},y_{2n+1}}(t), F_{y_{2n},y_{2n-1}}(t), F_{y_{2n},y_{2n}}(t), F_{y_{2n},y_{2n+1}}(t)\}$$
$$\geq F_{y_{2n-1},y_{2n}}(t)$$

Similarly, we can also prove that for n $\in$ N and for all t > 0,

$$F_{y_{2n+1,2n+2}}(\phi(t)) \geq F_{y_{2n},y_{2n+1}}(t)$$

So we have, $F_{y_n,y_{n+1}}(\phi(t)) \geq F_{y_{n-1},y_n}(t)$

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1}y_n}(\phi^{-1}(t))$$
$$\geq \ldots\ldots\ldots\ldots \geq F_{y_0,y_1}(\phi^{-n}(t)).$$

We show that {y_n } is a Cauchy sequence. Let ε> 0 be given and λ $\in$(0,1) be such that $T^{m-1}(1-\lambda,\ldots\ldots\ldots,1-\lambda) > 1-\varepsilon$. Also let t > 0 be such that $F_{y_0,y_1}(t) > 1-\lambda$, ψ be a positive number and $n_1 \in$ N be such that $\sum_{n_1}^{\infty}\phi^i(t) \leq \psi$. Thus for every n $\geq n_1$ and m $\in$ N, we have

$$F_{y_n,y_{n+m}}(\psi) \geq F_{y_n,y_{n+m}}\left(\sum_{i=n}^{n+m-1}\phi^i(t)\right)$$
$$\geq T^{m-1}(F_{y_n,y_{n+1}}(\phi^n(t)) \ldots\ldots\ldots, F_{y_{n+m-1},y_{n+n}}(\phi^{n+m-1}(t))$$
$$\geq T^{m-1}(1-\lambda,\ldots\ldots\ldots, 1-\lambda)$$
$$> 1-\varepsilon$$

Hence, {y_n } is a cuchy sequence in X. Since X is complete, {y_n } converges to z in X.

Thus, $\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} fx_{2n+1} = z \in \lim_{n\to\infty} gx_{2n}$

Since f(X) is closed subset of X, there exists a point v $\in$ X such that z = fv $\in$ f(X).

Putting x = x_{2n} and y =v in (iv), we get

$F_{gx_{2n},gv}(\phi(t)) \geq \min\{F_{fx_{2n},fv}(t), F_{fx_{2n},gv}(t), F_{gx_{2n},fx_{2n}}(t), F_{gx_{2n},fv}(t), F_{gv,fv}(t)\}$

$F_{y_{2n},gv}(\phi(t)) \geq \min\{F_{y_{2n-1},z}(t), F_{y_{2n-1},gv}(t), F_{y_{2n},y_{2n-1}}(t), F_{y_{2n},z}(t), F_{gv,z}(t)\}$

$\Rightarrow F_{y_{2n},gv}(\phi(t)) \geq F_{y_{2n-1},z}(t)$

Now taking limit n $\to \infty$,we have

$F_{z,gv}(\phi(t)) \geq F_{z,z}(t) = 1$

Hence, $F_{z,gv}(\phi(t)) = 1$. We obtain g(v) = z.

It shows that v is a coincidence point of f and g. Since the pair (g,f) is weakly compatible, we have gf(v) = fg (v) , hence g (z) = {f(z)}.

Putting x = x_{2n} and y = z in (iv), we get

$F_{g(x_{2n}),gz}(\phi(t)) \geq \min\{F_{f(x_{2n}),fz}(t), F_{f(x_{2n}),gz}(t), F_{f(x_{2n}),g(x_{2n})}(t), F_{g(x_{2n}),fz}(t), F_{fz,gz}(t)\}$

$F_{y_{2n},gz}(\phi(t)) \geq \min\{F_{y_{2n-1},fz}(t), F_{y_{2n-1},gz}(t), F_{y_{2n-1},y_{2n}}(t), F_{y_{2n},fz}(t), F_{fz,gz}(t)\}$

$F_{y_{2n},gz}(\phi(t)) \geq F_{y_{2n-1},gz}(t)$

taking limit n $\to \infty$, we get

$F_{z,gz}(\phi(t)) \geq F_{z,gz}(t)$

On the other hand , since F is non decreasing, we get

$F_{z,gz}(\phi(t)) \leq F_{z,gz}(t)$

Hence, $F_{gz,z}(t)$ = C for all t > 0.

By lemma 2.1, we conclude that C= 1 , that is, g(z) = {z}.
Now combine all the results, we get g(z) = {f(z)} = {z}.
It implies that z is a common fixed point of f and g in X.

Uniqueness:

Let w ($\neq$ z) be another common fixed point of f and g. Taking x = z and y = w in (iv), we have

$F_{gz,gw}(\phi(t)) \geq \min\{F_{fz,fw}(t), F_{fz,gw}(t), F_{gz,fz}(t), F_{gz,fw}(t), F_{gw,fw}(t)\}$

$F_{gz,gw}(\phi(t)) \geq F_{fz,fw}(t)$

$F_{z,w}(\phi(t)) \geq F_{z,w}(t)$

Since F is non- decreasing we get,

$F_{z,w}(\phi(t)) \leq F_{z,w}(t)$.

Hence $F_{z,w}(t)$ = C for all t > 0. From Lemma 2.1, we conclude that C= 1, that is, z = w and so the uniqueness is proved.

4.1: References

[1] Menger K, Statistical metrics, Proc. Nat. Acad. Sci. (U.S.A.), 28 (1942), 535-537.
[2] Chang S S, Lee B S, Cho Y J, Chen Y Q, Kang S M, Jung J S, Generalized contraction mapping principles and differential equations in probabilistic metric spaces, Proc. Amer. Math. Soc., 124 (1996), 2367-2376.
[3] Schweizer B, Sklar A: Probabilistic Metric Spaces, Elsevier, North-Holland, New York, (1983), ISBN 0-444-00666-4.
[4] Sehgal V M, Bharucha-Reid A T, Fixed points of contraction mappings on probabilistic metric spaces, Math. System Theory, 6 (1972), 97-102.
[5] Jungck G, Compatible mappings and common fixed points, Int. J. Math. & Math. Sci., 9 (1986), 771-779.
[6] Jungck G, Rhoades B E, Fixed points for set valued functions without continuity, Indian J. Pure Appl. Math., 29 (1998), 227-238.

[7] Singh B, Jain S, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl., 301 (2005), 439-448.

[8] Mihet, D., A counterexample to Common fixed point theorem in probabilistic quasi- metric space [MR2475779]", J. Nonlinear Sci. Appl., 1(2) (2008), 121-122. MR2486271

[9] Mihet. D., A note on a fixed point theorem in Menger probabilistic quasi-metric spaces, Chaos, Solitons Fractals, 40(5) (2009), 2349-2352. MR2533224 (2010g:54048)

[10]Mohamad, A., Probabilistic quasi-metric spaces, Conference in Topology and Theoretical Computer Science in honour of Peter Collins and Mike Reed, August 7-10, Mathematical Institute, University of Oxford, Oxford, UK, 2006.

[11] Pant, B.D. and Chauhan, S., Fixed points theorems in Menger probabilistic quasi metric spaces using weak compatibility, Int. Math. Forum, 5(6) (2010), 283{290. MR2578969.

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