

On Fixed Point Theorem in Fuzzy Metric Spaces

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Abstract: - The Purpose of this paper, we prove common fixed point theorem using new continuity condition in fuzzy metric spaces.
Keywords: - Compatible maps, R-weakly commuting maps, reciprocal continuity.
Mathematics subject classification: - 47H10, 54H25.

1 INTRODUCTION

The concept of fuzzy sets was introduced by Prof. Lofty Zadeh [20] in 1965 at University of California and developed a basic frame work to treat mathematically the fuzzy phenomena or systems which due to intrinsic indefiniteness, cannot themselves be characterized precisely. Fuzzy metric spaces have been introduced by Kramosil and Michalek [7] and George and Veersamani [3] modified the notion of fuzzy metric with the help of continuous t-norms. Recently many have proved fixed point theorems involving fuzzy sets [1, 2, 4-6, 8-10, 14, 16-19]. Vasuki [19] investigated the same fixed point theorems in fuzzy metric spaces for R-weakly commuting mappings and Pant [12] introduced the notion of reciprocal continuity of mappings in metric spaces. Balasubramaniam et al. and S. Muralishankar, R.P. Pant [1] proved the open problem of Rhoades [15] on the existence of a contractive definition which generals a fixed point but does not force the mapping to be continuous at the fixed point possesses an affirmative answer.

The purpose of this paper is to prove fixed point theorem in fuzzy metric spaces for using new continuity condition.

2 PRELIMINARIES

Before starting the main result we need some basic definitions and basic results, which are used to prove our main results.

Definition 2.1: A fuzzy set A in x is a function with domain X and values in [0, 1]

Definition 2.2: A binary operation *: [0, 1] → [0, 1] is called a continuous t-norm of ([0.1], *) is an abelian topological monoid with the unit 1 such that a * b ≤ c * d whenever a ≤ c and b ≤ d for all a, b, c, d ∈ [0, 1].

Example: Two typical examples of continuous t-norm are

(a) $a * b = ab,\ \ and$

(b) a * b = min {a, b}

Definition 2.3: A 3-tuple (X, M, *) is called a fuzzy metric space if X is non-empty set, * is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions for each x, y, z ∈ X and t, s > 0.

$(f1)\ M(x,y,0) > 0;$

$(f2)\ M(x,y,t) = 1, \forall\ t > 0, if\ and\ only\ if\ x = y;$

$(f3)\ M(x,y,t) = M(y,x,t);$

$(f4)\ M(x,y,t) * M(y,z,s) \le M(x,z,t+s);$

$(f5)\ M(x,y,.):(0,\infty) \to [0,1]\ is\ left\ continuous$

$(f6)\ \lim_{t\to\infty} M(x,y,t) = 1 \forall x,y \in x$

Then M is called a fuzzy metric on X. A function M(x, y, t) denote the degree of nearness between x and y with respect to t.

Example: (Induced Fuzzy metric) [3] every metric space indices a fuzzy metric space. Let (X, d) be a metric space

Define a * b = ab

$$\text{And } M(x,y,t) = \frac{kt^n}{kt^n + md(x,y)}$$

k, m, n, $t \in R^+$. Then (X, M, *) is a fuzzy metric space if we put k = m – n = 1.

We get $M(x,y,t) = \dfrac{t}{t + d(x,y)}$

The fuzzy metric induced by a metric d is referred to as a standard fuzzy metric.

Proposition 2.4 [21] in a fuzzy metric space (X, M, *), if a * a ≥a for all a ∈[0, 1]. Then a * b = min {a, b} for all a, b ∈[0, 1].

Definition 2.5 ([2])**:** Two self mappings F and S of a fuzzy metric space (X, M, *) are called compatible if $\lim_{t\to\infty}$ M (FSx$_n$, SFx$_n$, t) = 1 when ever $\{x_n\}$ is a sequence in X such that $\lim_{t\to\infty} Fx_n = \lim_{t\to\infty} Sx_n = x$ for some x in X.

Definition 2.6 ([19])**:** Two self mappings A and S of a fuzzy metric space (X, M, *) are called weakly commuting if M (FSx, SFx, t) ≥ M (Fx, Sx, t) ∀ x in X and t > 0.

Definition 2.7 ([19])**:** Two self mappings A and S of a fuzzy metric space (X, M, *) are called point wise R-weakly commuting if there exist R > 0 such that

M (FSx, SFx, t) ≥ M (Fx, Sx, t/R) for all x in X and t > 0.

Remark 1: Clearly, point R-weakly commutativity implies weak commutativity only when R ≤ 1.

Definition 2.8 ([1]): Two self maps F and S of a fuzzy metric space (X, M, *) are called reciprocally continuous on X if $\lim_{t\to\infty} FSx_n = Fx$ and $\lim_{t\to\infty} Fx_n = Sx$ when ever $\{x_n\}$ is a sequences in X such that $\lim_{t\to\infty} Fx_n = \lim_{t\to\infty} Sx_n = x$ for some x in X.

Lemma 2.9 ([16])**:** Let (X, M, *) be a fuzzy metric space. If there exists k∈(0, 1) such that M(x, y, kt) > M(x, y, t) Then x = y.

Lemma-2.10 ([2])**:** Let {y$_n$} be a sequence in a fuzzy metric space (X, M, *) with the condition (f6). If there exists, k∈ (0, 1) such that

$$M (y_n, y_{n+1}, kt) \geq M (y_{n-1}, y_n, t)$$

For all t > 0 and n∈N, Then {y$_n$} is a Cauchy sequence in X.

The following theorems are basic theorems for our result

Theorem 2.11[1]**:** Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that

1. AX⊂TX, BX⊂SX
2. M (Ax, By, ht) ≥ M(x, y, t), 0 < h < 1, x, y ∈X and t > 0.

Suppose that (A, S) and (B, T) is compatible pair of reciprocally continuous mappings X. Then A, B, S and T have a unique common fixed point.

Theorem 2.12[14]**:** Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self mappings of complete fuzzy metric space (X, M, *) such that

1. AX⊂TX, BX⊂SX
2. M (Ax, By, ht) ≥ M(x, y, t), 0 < h < 1, x, y, ∈x and t > 0.

Let (A, S) and (B, T) is compatible mappings. If any of the mappings in compatible pairs (A, S) and (B, T) is continuous then A, B, S and T have a unique common fixed point.

Remark 2: In [14], Pant and Jha proved that the theorem 2.12 is an analogue of the theorem 2.11 by obtaining connection between continuity and reciprocal continuity in fuzzy metric space.

Lemma 2.13 [21]**:** Let (X, M,*) be a complete fuzzy metric space with a*a ≥ a for all a∈ [0, 1] and the condition (f6). Let (A, S) and (B, T) be point wise R-weakly commuting pairs of self mappings of X such that

(a)AX⊂TX, BX⊂SX

There exists k ∈ (0, 1) such M (Ax, By, kt) ≥ M (x, y, t) for all x, y ∈ X, and t >0

Then the continuity of one of the mappings in compatible pair (A, S) or (B, T) on (X, M,*) implies their reciprocal continuity.

3 MAIN RESULTS

Theorem 3.1: Let (X, M,*) be a complete fuzzy metric space a*a ≥ a, for all a ∈ [0, 1].

Let (L, ST) and (M, AB) be point wise R-weakly commuting pairs of self mappings of X such that

3.1(a). L(x) ⊆ ST(x), M(x) ⊆ AB(x)

3.2(b). There exists k∈(0,1) such that

$$F^2(Lx, My, kt) * [F(ABx, Lx, kt), F(STy, My, kt)] \geq [pF(ABx, Lx, t) + qF(ABx, STy, t)], F(ABx, My, 2kt)$$

For all x, y∈X and t > 0 where p, q ∈(0, 1) such that p + q = 1.

Then A, B, S, T, L and M have a unique common fixed point in X.

Proof. Suppose $x_0 \in$ X. ∃ $x_1, x_2 \in$ X such that

$Lx_0 = STx_1$ and $Mx_1 = ABx_2$.

Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{2n} = Lx_{2n} = STx_{2n+1}$ and $y_{2n+1} = Mx_{2n+1} = ABx_{2n+2}$ for n = 0, 1, 2, ……

Step 1. Taking $x = x_{2n}$ and $y = x_{2n+1}$, we have

$$F^2(Lx_{2n}, Mx_{2n+1}, kt) * [F(ABx_{2n}, Lx_{2n}, kt), F(STx_{2n+1}, Mx_{2n+1}, kt).F(STx_{2n+1}, Mx_{2n+1}, kt)]$$
$$\geq [pF(ABx_{2n}, Lx_{2n}, t) + qF(ABx_{2n}, STx_{2n+1}, t)].F(ABx_{2n}, Mx_{2n+1}, 2kt)$$
$$F^2(y_{2n}, y_{2n+1}, kt) * [F(y_{2n-1}, y_{2n}, kt), F(y_{2n-1}, y_{2n}, kt).F(y_{2n}, y_{2n+1}, kt)]$$
$$\geq [pF(y_{2n}, y_{2n-1}, t) + qF(y_{2n-1}, y_{2n}, t)].F(y_{2n}, y_{2n+1}, 2kt)$$
$$F(y_{2n}, y_{2n+1}, kt)[F(y_{2n-1}, y_{2n}, kt) * F(y_{2n}, y_{2n+1}, kt)] \geq [(p+q)F(y_{2n}, y_{2n-1}, t).F(y_{2n-1}, y_{2n+1}, 2kt)]$$
$$F(y_{2n}, y_{2n+1}, kt)[F(y_{2n-1}, y_{2n+1}, 2kt)] \geq [F(y_{2n-1}, y_{2n}, t).F(y_{2n-1}, y_{2n+1}, 2kt)]$$

Hence, we have

$$F(y_{2n}, y_{2n+1}, kt) \geq F(y_{2n-1}, y_{2n}, t)$$

Similarly, we also have

$$F(y_{2n+1}, y_{2n+2}, kt) \geq F(y_{2n}, y_{2n+1}, t)$$

In general, for all n even or odd, we have

$$F(y_n, y_{n+1}, kt) \geq F(y_{n-1}, y_n, t)$$

for all $x, y \in X$ and $t > 0$. Thus by lemma 2.11 $\{y_n\}$ is a Cauchy sequence in X. Since (X, F, *) is complete, it converges to a point z in X. Also its subsequences converge as follows:
$\{Lx_{2n}\} \to z$, $\{ABx_{2n}\} \to z$, $\{Mx_{2n+1}\} \to z$ and $\{STx_{2n+1}\} \to z$.
Suppose AB is continuous, as AB is continuous and (L, AB) is semi-compatible, we get
$LABx_{2n+2} \to Lz$ and $LABx_{2n+2} \to ABz$.
Since the limit in Menger space is unique, we get
Lz = ABz.

Step 2. By taking $x = ABx_{2n}$ and $y = x_{2n+1}$, we have

$$F^2(LABx_{2n}, Mx_{2n+1}, kt) * [F(ABABx_{2n}, LABx_{2n}, kt).F(STx_{2n+1}, Mx_{2n+1}, kt).]$$
$$\geq [pF(ABABx_{2n}, LABx_{2n}, t) + qF(ABABx_{2n}, STx_{2n+1}, t)].F(ABABx_{2n}, Mx_{2n+1}, 2kt)$$

Taking limit $n \to \infty$

$$F^2(z, ABz, kt) * [F(ABz, ABz, kt).F(z, z, kt)] \geq [pF(ABz, ABz, t) + qF(z, ABz, t)]F(z, ABz, 2kt)$$
$$\geq [p + qF(z, ABz, t)]F(z, ABz, kt)$$
$$F(z, ABz, kt) \geq p + qF(z, ABz, kt) \geq p + qF(z, ABz, kt)$$
$$F(z, ABz, kt) \geq \frac{p}{1-q} = 1$$

For $k \in (0,1)$ and all t > 0. Thus we have
z = ABz.

Step 3. By taking x = z and $y = x_{2n+1}$, we have

$$F^2(Lz, Mx_{2n+1}, kt) * [F(ABz, Lz, kt).F(STx_{2n+1}, Mx_{2n+1}, kt)]$$
$$\geq [pF(ABz, Lz, t) + qF(ABz, STx_{2n+1}, t)].F(ABz, Mx_{2n+1}, 2kt)$$

Taking limit $n \to \infty$

$$F^2(z, Lz, kt) * [F(z, Lz, kt).F(z, z, kt)] \geq [pF(z, Lz, t) + qF(z, z, t)].F(z, z, 2kt)$$
$$F^2(z, Lz, kt) * [F(z, Lz, kt)] \geq [pF(z, Lz, t) + q]$$

Noting that $F^2(z, Lz, kt) < 1$, we have

$$F^2(z, Lz, kt) \geq pF(z, Lz, t) + q$$
$$\geq pF(z, Lz, t) + q$$
$$F(z, Lz, kt) \geq \frac{q}{1-p} = 1$$

For k $\in (0, 1)$ and all t > 0. Thus, we have z = Lz = ABz.

Step 4. By taking z = Bz and y = x_{2n+1}, we have

$$F^2(LBz, Mx_{2n+1}, kt) * [F(ABBz, LBz, kt).F(STx_{2n+1}, Mx_{2n+1}, kt)]$$
$$\geq [pF(ABBz, LBz, t) + qF(ABBz, STx_{2n+1}, t)].F(ABBz, Mx_{2n+1}, 2kt)$$

Since AB = BA and BL = LB, we have

L(Bz) = B(Lz) = Bz and AB(Bz) = B(ABz) = Bz.

Taking limit n $\to \infty$, we have

$$F^2(z, Bz, kt) * [F(Bz, Bz, kt).F(z, z, kt)] \geq [pF(Bz, Bz, t) + qF(z, Bz, t)].F(z, Bz, 2kt)$$
$$F^2(z, Bz, kt) \geq [p + qF(z, Bz, t)]F(z, Bz, 2kt)$$
$$\geq [p + qF(z, Bz, t)]F(z, Bz, kt)$$
$$F(z, Bz, kt) \geq p + qF(z, Bz, t)$$
$$\geq p + qF(z, Bz, kt)$$
$$F(z, Bz, kt) \geq \frac{p}{1-q} = 1$$

For k $\in (0,1)$ and all t > 0.

Thus, we have z = Bz.

Since z = ABz, we also have

z = Az.

Therefore, z = Az = Bz = Lz.

Step 5. Since L(X) $\subseteq$ ST(X) there exists v $\in$ X such that z = Lz = STv.

By taking x = x_{2n} and y = v, we get

$$F^2(Lx_{2n}, Mv, kt) * [F(ABx_{2n}, Lx_{2n}, kt).F(STv, Mv, kt)]$$
$$\geq [pF(ABx_{2n}, Lx_{2n}, t) + qF(ABx_{2n}, STv, t)].F(ABx_{2n}, Mv, 2kt)$$

Taking limit as n $\to \infty$, we have

$$F^2(z, Mv, kt) * [F(z, z, kt).F(z, Mv, kt)] \geq [pF(z, z, t) + qF(z, z, t)].F(z, Mv, 2kt)$$
$$F^2(z, Mv, kt) * F(z, Mv, kt) \geq (p + q)F(z, Mv, 2kt)$$

Noting that $F^2(z, Mv, kt) \leq 1$, we have

$$F(z, Mv, kt) \geq F(z, Mv, 2kt)$$
$$\geq F(z, Mv, t)$$

Thus we have

z = Mv and so z = Mv = STv.

Since (M, ST) is weakly compatible, we have

STMv = MSTv

Thus, STz = Mz.

Step 6. By taking x = x_{2n}, y = z and using step 5, we have

$$F^2(Lx_{2n}, Mz, kt) * [F(ABx_{2n}, Lx_{2n}, kt).F(STz, Mz, kt)]$$
$$\geq [pF(ABx_{2n}, Lx_{2n}, t) + qF(ABx_{2n}, STz, t)].F(ABx_{2n}, Mz, 2kt)$$

Which implies that, as n $\to \infty$

$$F^2(z, Mz, kt) * [F(z, z, kt).F(Mz, Mz, kt)] \geq [pF(z, z, t) + qF(z, Mz, t)].F(z, Mz, 2kt)$$
$$F^2(z, Mz, kt) \geq [p + qF(z, Mz, t)].F(z, Mz, 2kt)$$
$$\geq [p + qF(z, Mz, t)]F(z, Mz, kt)$$
$$F(z, Mz, kt) \geq (p + q)F(z, Mz, t)$$
$$\geq (p + q)F(z, Mz, kt)$$
$$F(z, Mz, kt) \geq \frac{p}{1-q} = 1$$

Thus, we have z = Mz and therefore z = Az = Bz = Lz = Mz = STz.

Step 7. By taking x = x_{2n}, y = Tz, we have

$$F^2(Lx_{2n}, MTz, kt) * [F(ABx_{2n}, Lx_{2n}, kt).F(STTz, MTz, kt)]$$
$$\geq [pF(ABx_{2n}, Lx_{2n}, t) + qF(ABx_{2n}, STTz, t)].F(ABx_{2n}, MTz, 2kt)$$

Since MT = TM and ST = TS, we have

MTz = TMz = Tz and ST(Tz) = TS(Tz) = Tz.

Letting n → ∞, we have

$$F^2(z, Tz, kt) * [F(z, z, kt).F(Tz, Tz, kt)] \geq [pF(z, z, t) + qF(z, Tz, t)].F(z, Tz, 2kt)$$
$$F(z, Tz, kt) \geq [p + qF(z, Tz, t)]$$
$$\geq [p + qF(z, Tz, kt)]$$
$$F(z, Tz, kt) \geq \frac{p}{1-q} = 1$$

Thus, we have z = Tz. Since Tz = STz, we also have z = Sz.
Therefore, z = Az = Bz = Lz = Mz = Sz = Tz, that is, z is the common fixed point of the six maps.

Step 8. By taking x = LLx_{2n}, y = x_{2n+1}, we have

$$F^2(LLx_{2n}, Mx_{2n+1}, kt) * [F(ABLx_{2n}, LLx_{2n}, kt).F(STx_{2n+1}, Mx_{2n+1}, kt)]$$
$$\geq [pF(ABLx_{2n}, LLx_{2n}, t) + qF(ABLx_{2n}, STx_{2n+1}, t)].F(ABLx_{2n}, Mx_{2n+1}, 2kt)$$

Letting n → ∞, we have

$$F^2(z, Lz, kt) * [F(Lz, Lz, kt).F(z, z, kt)] \geq [pF(Lz, Lz, t) + qF(z, Lz, t)].F(z, Lz, 2kt)$$
$$F^2(z, Lz, kt) \geq [p + qF(z, Lz, t)].F(z, Lz, 2kt)$$
$$\geq [p + qF(z, Lz, t)].F(z, Lz, kt)$$
$$F(z, Lz, kt) \geq p + qF(z, Lz, t)$$
$$\geq p + qF(z, Lz, kt)$$
$$F(z, Lz, kt) \geq \frac{p}{1-q} = 1$$

Thus, we have z = Lz and using steps 5-7, we have
Z = Lz = Mz = Sz = Tz.

Step 9. Since L is continuous,
LLx_{2n} →Lz and $LABx_{2n}$ →Lz
Since(L, AB) is semi-compatible,
$L(AB)x_{2n}$ → ABz.
Since limit in Menger space is unique, so Lz = ABz and using Step 4, we also have z = Bz.
Therefore, z = Az = Bz = Sz = Tz = Lz = Mz, that is, z is the common fixed point of the six maps in this case also.

Step 10. For uniqueness, let (w ≠ z)be another common fixed point of A, B, S, T, L and M.
Taking x = z, y = w, we have
F^2(Lz, Mw, kt)*[F(ABz, Lz, kt).F(STw, Mw, kt)]
≥ [pF(ABz, Lz, t)+qF(ABz, STw, t)].F(ABz, Mw, 2kt)
Which implies that
F^2(z, w, kt) ≥ [p +qF(z, w, t)]F(z, w, 2kt)
≥ [p + qF(z, w, t)]F(z, w, kt),
F(z, w, kt) ≥ p + qF(z, w, t)

$$F(z, w, kt) \geq \frac{p}{1-q} = 1$$

Thus, we have z = w.
This completes the proof of the theorem.
If we take B = T = I_X (the identity map on X) in theorem 3.1, we have the following:

References:

[1] P. Balasubramaniam, S. Muralisankar, R.P. Pant, "Common fixed' points of four mappings in a fuzzy metric space". J. Fuzzy Math. 10(2) (2002), 379-384.

[2] Y.J. Cho, H.K. Pathak, S.M. Kang. J.S. Jung, "Common fixed points of compatible maps of type (A) on fuzzy metric spates," Fuzzy Sets and Systems 93 (1998), 99-111.
[3] A. George, P. Veeramani, "On some results in fuzzy metric spaces" Fuzzy Sets and Systems, 64 (1994), 395-399.
[4] M. Grabiec, "Fixed points in fuzzy metric specs, Fuzzy Sets and Systems" 27 (1988), 385-389.
[5] O. Hadzic, "Common fixed point theorems for families of mapping in complete metric space," Math. Japon. 29 (1984). 127-134.
[6] G. Jungck, "Compatible mappings and common fixed points (2)", Internat. J. Math. Math. Sci. (1988). 285-288.
[7] O. Kramosil and J. Michalek, "Fuzzy metric and statistical metric spaces," Kybernetika 11 (1975). 326-334.
[8] S. Kutukcu, "A fixed point theorem in Menger spaces," Int. Math. F. 1(32) (2006), 1543-1554.
[9] S. Kutukcu, C. Yildiz, A.Tuna, "On common fixed points in Menger probabilistic metric spaces," Int. J. Contemp. Math. Sci. 2(8) (2007), 383-391.
[10] S. Kutukcu, "A fixed point theorem for contraction type mappings in Menger spaces," Am. J. Appl. Sci. 4(6) (2007). 371-373.
[11] S .N. Mishra, "Common fixed points of compatible mappings in PM-spaces," Math. Japon. 36 (1991), 283289.
[12] R.P. Pant, "Common fixed points of four mappings," Bull. Cal. Math. Soe. 90 (1998), 281-286.
[13] R.P. Pant, "Common fixed point theorems for contractive maps," J. Math. Anal. Appl. 226 (1998). 251-258.
[14] R.P. Pant, K. Jha, "A remark on common fixed points of four mappings in a, fuzzy metric space," J. Fuzzy Math. 12(2) (2004), 433-437.
[15] B.E. Rhoades, "Contractive definitions and continuity," Contemporary Math. 72 (1988), 233-245.
[16] S. Sharma, "Common fixed point theorems in fuzzy metric spaces", Fuzzy Sets and Systems 127 (2002). 345-352.
[17] S. Sharma, B. Deshpande, "Discontinuity and weak compatibility in fixed point consideration on non-complete fuzzy metric spaces," J. Fuzzy Math. 11(2) (2003), 671-686.
[18] S. Sharma, B. Deshpande, "Compatible multivalued mappings satisfying an implicit relation," Southeast Asian Bull. Math. 30 (2006), 585-540.
[19] R. Vasuki, "Common fixed points for R-weakly commuting maps in fuzzy metric spaces," Indian J. Pure Appl. Math. 30 (1999), 419-423.
[20] L.A. Zadeh, "Fuzzy sets" Inform and Control, 8 (1965), 338-353.
[21] S. Kutukcu, S. Sharma, S. H Tokgoz "A fixed point theorem in fuzzy metric spaces" Int . Journal of Math. Analysis vol 1, (2007), 861-872