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Common Fixed Point Theorems with Continuously Subcompatible Mappings in Fuzzy Metric Spaces

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Abstract
In this article the new concept of continuously subcompatible maps has been introduced and is used to prove common fixed point theorems in fuzzy metric spaces.
Keywords: Subcompatible maps, Continuously subcompatible maps, Subsequential continuity.

1.Introduction:

Zadeh [19] introduced the concept of fuzzy sets in 1965. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [12] and George and Veeramani [6] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. Consequently in due course of time some metric fixed points results were generalized to fuzzy metric spaces by various authors. Sessa [16] improved cmmutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space.Vasuki [18] proved fixed point theorems for R-weakly commuting mapping Pant [15,14,13] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [12] and weakly compatible maps by [11] in fuzzy metric space is generalized by A.Al Thagafi and Naseer Shahzad [3] by introducing the concept of occasionally weakly compatible mappings. Recent results on fixed point in fuzzy metric space can be viewed in [1, 2,].

In this article ,we introduce the new concepts of continuously subcompatible maps which are weaker condition than subcompatibility and subsequential continuity.

We start with some preliminaries:

2. Preliminaries

Definition 2.1 [17] - A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a continuous t-norms if $*$ satisfying conditions:

i. $*$ is commutative and associative;
ii. $*$ is continuous;
iii. $a*1=a$ for all $a\in[0,1]$;
iv. $a*b\le c*d$ whenever $a\le c$ and $b\le d$, and $a,b,c,d\in[0,1]$.

Definition 2.2[6] - A 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions , $\forall$ $x,y,z\in X, s,t>0$,

(*f*1) $M(x,y,t)>0$;
(*f*2) $M(x,y,t)=1$ if and only if $x=y$.
(*f*3) $M(x,y,t)=M(y,x,t)$;
(*f*4) $M(x,y,t)*M(y,z,s)\le M(x,z,t+s)$
(*f*5) $M(x,y,.):(0,\infty)\rightarrow(0,1]$ is continuous.

Then M is called a fuzzy metric on X. Then $M(x,y,t)$ denotes the degree of nearness between x and y with respect to t.

Example 2.3 (Induced fuzzy metric[6]) – Let (X,d) be a metric space. Denote $a*b=ab$ for all $a,b\in[0,1]$ and let M_d be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$M_d(x,y,t)=\frac{t}{t+d(x,y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.4 [9] **:**Two self mappings f and g of a fuzzy metric space $(X, M, *)$ are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t)=1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n=\lim_{n\to\infty} gx_n = x$ for some $x\in X$.

Lemma 2.5[1] **:** Let $(X, M, *)$ be fuzzy metric space. If there exists $q\in(0,1)$ such that $M(x,y,qt)\ge M(x,y,t)$ for all $x,y\in X$ and $t>0$,then $x=y$.

Definition 2.6 [10] **:** A pair of maps S and T is called weakly compatible pair if they commute at coincidence points.

The concept of occasionally weakly compatible is introduced by A. Al- Thagafi and Naseer Shahzad [3].It is stated as follows.

Definition 2.7 : Two self maps f and g of a set X are called occasionally weakly compatible (owc) iff there is a point $x\in X$ which is a coincidence point of f and g at which f and g commute.

Example 2.8 [3] **:**Let R be the usual metric space. Define $S,T: R\to R$ by $Sx=2x$ and $Tx=x^2$ for all $x\in R$.Then $Sx=Tx$ for $x=0,2$, but $ST0=TS0$, and $ST2\ne TS2$. S and T are occasionally weakly compatible self maps but not weakly compatible.

Definition 2.9 Self mappings $A\,\&\,S$ of a fuzzy metric space $(X, M, *)$ are said to be subcompatible if there exists a sequence $\{x_n\}$ in X such that

$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z, z\in X$ and satisfy $\lim_{n\to\infty} M(ASx_n, SAx_n, t)=1$

Definition 3.0 Self mappings $A\,\&\,S$ of a fuzzy metric space $(X, M, *)$ are said to be subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that.

$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z, z\in X$ and $\lim_{n\to\infty} ASx_n = Az\ \&\lim_{n\to\infty} SAx_n = Sz$

Definition 3.1 Self mappings $A\,\&\,S$ of a fuzzy metric space $(X, M, *)$ are said to be continuously subcompatible if there exists a sequence $\{x_n\}$ in X such that.

$\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z, z\in X$ and Az= Sz.

Example 3.2 Let X=R endowed with usual metric d and we define $M_d(x,y,t)=\frac{t}{t+d(x,y)}$ for all x,y in X, t>0.Let $*$ be any continuous t- norm.Then $(X, M, *)$ is FM-space. Let the Self mappings $A\,\&\,S$ of a fuzzy metric space $(X, M, *)$ such that A, S : R → R by Ax = 2x and S x = x^2 for all x ЄR if $x_n=\frac{1}{n+1}$ then $\lim_{n\to\infty} Ax_n$ =0 &$\lim_{n\to\infty} Sx_n$ =0 and A0=S0=0. A and S is continuously subcompatible.

4.Main Results

Theorem 4.1: Let $(X, M, *)$ be a fuzzy metric space and let A, B, S and T be self mappings of X.Let the pairs $\{A,S\}$ and $\{B,T\}$ be continuously subcompatible. If there exist $q\in(0,1)$ s.t.

$$M(Ax,By,qt)\geq \min\left\{\begin{array}{l} M(Sx,Ty,t),M(By,Sx,t),\\ M(Ax,Ty,t),M(Sx,Ax,t),\\ \dfrac{aM(Ax,Ty,t)+bM(Sx,Ty,t)}{aM(Sx,Ax,t)+b},\\ \dfrac{cM(Sx,Ty,t)+dM(By,Ax,t))}{c+dM(By,Ty,t)}\\ \dfrac{eM(Sx,Ax,t)+fM(By,Ty,t)}{e+f}\end{array}\right\} \quad \ldots(1)$$

for all $x,y\in X$, $t>0$ and $a,b,c,d,e,f\geq 0$ with a & b $(c\,\&\,d, e\,\&\,f)$ cannot be simultaneously 0 ,then there exist a unique point $z\in X$ such that $Az=Sz=Tz=Bz=z$.

Proof: Let the pairs $\{A,S\}$ and $\{B,T\}$ be continuously subcompatible. so there are sequences x_n & y_n in X such that $\lim_{n\to\infty} Ax_n=\lim_{n\to\infty} Sx_n=z$ and $Az=Sz$ .Again $\lim_{n\to\infty} Ty_n=\lim_{n\to\infty} By_n=w$ and $Tw=Bw$ .First we shall show that $z=w$ .For which put $x=x_n$ & $y=y_n$.

$$M(Ax_n,By_n,qt)\geq \min\left\{\begin{array}{l} M(Sx_n,Ty_n,t),M(By_n,Sx_n,t),\\ M(Ax_n,Ty_n,t),M(Sx_n,Ax_n,t),\\ \dfrac{aM(Ax_n,Ty_n,t)+bM(Sx_n,Ty_n,t)}{aM(Sx_n,Ax_n,t)+b},\\ \dfrac{cM(Sx_n,Ty_n,t)+dM(By_n,Ax_n,t))}{c+dM(By_n,Ty_n,t)}\\ \dfrac{eM(Sx_n,Ax_n,t)+fM(By_n,Ty_n,t)}{e+f}\end{array}\right\}$$

Taking limit $n\to\infty$ we get,

$$M(z,w,qt) \geq$$

$$\min\left\{\begin{array}{l} M(z,w,t), M(w,z,t), \\ M(z,w,t), M(z,z,t), \\ \dfrac{aM(z,w,t)+bM(z,w,t)}{aM(z,z,t)+b}, \\ \dfrac{cM(z,w,t)+dM(w,z,t))}{c+dM(w,w,t)} \\ \dfrac{eM(z,z,t)+fM(w,w,t)}{e+f} \end{array}\right\}$$

$$M(z,w,qt) \geq$$

$$\min\left\{\begin{array}{l} M(z,w,t), M(w,z,t), \\ M(z,w,t), 1, \\ \dfrac{aM(z,w,t)+bM(z,w,t)}{a+b}, \\ \dfrac{cM(z,w,t)+dM(w,z,t))}{c+d} \\ \dfrac{e+f}{e+f} \end{array}\right\}$$

$$M(z,w,qt) \geq$$

$$\min\left\{\begin{array}{l} M(z,w,t), M(w,z,t), \\ M(z,w,t), 1, M(w,z,t), M(w,z,t), 1 \end{array}\right\}$$

$M(z,w,qt) \geq M(z,w,t)$ so that $z = w$

Now we shall show that $Az = z$. So put x=z & y= y_n

$$M(Az, By_n, qt) \geq$$

$$\min\left\{\begin{array}{l} M(Sz,Ty_n,t), M(By_n,Sz,t), \\ M(Az,Ty_n,t), M(Sz,Az,t), \\ \dfrac{aM(Az,Ty_n,t)+bM(Sz,Ty_n,t)}{aM(Sz,Az,t)+b}, \\ \dfrac{cM(Sz,Ty_n,t)+dM(By_n,Az,t))}{c+dM(By_n,Ty_n,t)} \\ \dfrac{eM(Sz,Az,t)+fM(By_n,Ty_n,t)}{e+f} \end{array}\right\}$$

Taking limit $n \to \infty$ we get,

$$M(Az,z,qt) \geq \min\left\{\begin{array}{l} M(Az,z,t), M(z,Az,t), \\ M(Az,z,t), M(Az,Az,t), \\ \dfrac{aM(Az,z,t)+bM(Az,z,t)}{aM(Az,Az,t)+b}, \\ \dfrac{cM(Az,z,t)+dM(z,Az,t))}{c+dM(z,z,t)} \\ \dfrac{eM(Az,Az,t)+fM(z,z,t)}{e+f} \end{array}\right\}$$

$$M(Az,z,qt) \geq \min\left\{\begin{array}{l} M(Az,z,t), M(z,Az,t), \\ M(Az,z,t), 1, \\ \dfrac{aM(Az,z,t)+bM(Az,z,t)}{a+b}, \\ \dfrac{cM(Az,z,t)+dM(z,Az,t))}{c+d} \\ \dfrac{e+f}{e+f} \end{array}\right\}$$

$$M(Az,z,qt) \geq \min\left\{\begin{array}{l} M(Az,z,t), M(z,Az,t), \\ M(Az,z,t), 1, M(Az,z,t), M(Az,z,t), 1 \end{array}\right\}$$

$M(Az,z,qt) \geq M(Az,z,t)$ so that $Az = z$

.

Now we shall prove $Bw = w$.(using fact that $z = w$)

$$M(Ax_n, Bw, qt) \geq$$

$$\min\left\{\begin{array}{l} M(Sx_n, Tw, t), M(Bw, Sx_n, t), \\ M(Ax_n, Tw, t), M(Sx_n, Ax_n, t), \\ \dfrac{aM(Ax_n, Tw, t) + bM(Sx_n, Tw, t)}{aM(Sx_n, Ax_n, t) + b}, \\ \dfrac{cM(Sx_n, Tw, t) + dM(Bw, Ax_n, t))}{c + dM(Bw, Tw, t)} \\ \dfrac{eM(Sx_n, Ax_n, t) + fM(Bw, Tw, t)}{e + f} \end{array}\right\}$$

$$M(w, Bw, qt) \geq$$

$$\min\left\{\begin{array}{l} M(w, Bw, t), M(Bw, w, t), \\ M(w, Bw, t), M(w, w, t), \\ \dfrac{aM(w, Bw, t) + bM(w, Bw, t)}{aM(w, w, t) + b}, \\ \dfrac{cM(w, Bw, t) + dM(Bw, w, t))}{c + dM(Bw, Bw, t)} \\ \dfrac{eM(w, w, t) + fM(Bw, Bw, t)}{e + f} \end{array}\right\}$$

$$M(w, Bw, qt) \geq$$

$$\min\left\{\begin{array}{l} M(w, Bw, t), M(Bw, w, t), \\ M(w, Bw, t), 1, M(Bw, w, t), M(Bw, w, t), 1 \end{array}\right\}$$

$M(w, Bw, qt) \geq M(w, Bw, t)$ so that $Bw = w$.which implies $Az = Sz = Bz = Tz$ =z

Theorem 4.2: Let $(X, M, *)$ be a fuzzy metric space and let A, B, S and T be self mappings on X.Let the pairs $\{A, S\}$ and $\{B, T\}$ be continuously subcompatible and if for all $x, y \in X$, $t > 0$ and $a, b, c, d, e, f \geq 0$ with a & b $(c \& d, e \& f)$ cannot be simultaneously 0 , $\phi : [0,1] \rightarrow [0,1]$ is continuous such that $\phi(t) > t$ for all $0 < t < 1$, If there exist $q \in (0,1)$ s.t.

$$M(Ax,By,qt)\geq \phi\left(\min\left\{\begin{matrix} M(Sx,Ty,t), M(By,Sx,t), \\ M(Ax,Ty,t), M(Sx,Ax,t), \\ \dfrac{aM(Ax,Ty,t)+bM(Sx,Ty,t)}{aM(Sx,Ax,t)+b}, \\ \dfrac{cM(Sx,Ty,t)+dM(By,Ax,t)}{c+dM(By,Ty,t)} \\ \dfrac{eM(Sx,Ax,t)+fM(By,Ty,t)}{e+f} \end{matrix}\right\}\right) \quad \ldots(2)$$

then there is a unique common fixed point of A,B,S and T.

Proof: The Proof follows from theorem 4.1.

Theorem 4.3: Let $(X,M,*)$ be a fuzzy metric space and let A,B,S and T be self mappings of X.Let the pairs $\{A,S\}$ and $\{B,T\}$ be continuously subcompatible and if for all $x,y\in X$, $t>0$ and $a,b,c,d,e,f\geq 0$ with a & b $(c\,\&\,d, e\,\&\,f)$ cannot be simultaneously $0, \phi:[0,1]^5\to[0,1]$ is continuous such that $\emptyset(1,1,t,t,1)>t$ for all $0<t<1$, If there exist $q\in(0,1)$ s.t.

$$M(Ax,By,qt)\geq \phi\begin{pmatrix} M(Ax,Sx,t), M(By,Ty,t), \\ \dfrac{aM(Sx,Ty,t)+bM(Ax,Ty,t)}{aM(Sx,Ax,t)+b}, \\ \dfrac{cM(Sx,Ty,t)+dM(By,Ax,t)}{c+dM(By,Ty,t)}, \\ \dfrac{eM(Sx,Ty,t)+fM(By,Sx,t)}{e+f} \end{pmatrix} \quad (3)$$

then there is a unique common fixed point of A,B,S and T.

Proof: : Let the pairs $\{A,S\}$ and $\{B,T\}$ be continuously subcompatible. so there are sequences x_n & y_n in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ and $Az=Sz$ .Again $\lim_{n\to\infty} Ty_n = \lim_{n\to\infty} By_n = w$ and $Tw=Bw$.First we shall show that $z=w$.For which put $x=x_n$ & $y=y_n$.

$$M(Ax_n, By_n, qt) \geq$$
$$\phi\left(\begin{array}{l} M(Ax_n, Sx_n, t), M(By_n, Ty_n, t), \\ \dfrac{aM(Sx_n, Ty_n, t) + bM(Ax_n, Ty_n, t)}{aM(Sx_n, Ax_n, t) + b}, \\ \dfrac{cM(Sx_n, Ty_n, t) + dM(By_n, Ax_n, t)}{c + dM(By_n, Ty_n, t)}, \\ \dfrac{eM(Sx_n, Ty_n, t) + fM(By_n, Sx_n, t)}{e + f} \end{array}\right)$$

$$M(z, w, qt) \geq$$
$$\phi\left(\begin{array}{l} M(z, z, t), M(w, w, t), \\ \dfrac{aM(z, w, t) + bM(z, w, t)}{aM(z, z, t) + b}, \\ \dfrac{cM(z, w, t) + dM(w, z, t)}{c + dM(w, w, t)}, \\ \dfrac{eM(z, w, t) + fM(w, z, t)}{e + f} \end{array}\right)$$

$$M(z, w, qt) \geq$$
$$\phi\left(1, 1, M(z, w, t), M(z, w, t), 1\right)$$

$M(z, w, qt) \geq M(z, w, t)$ which implies $z = w$.

Now we shall shaw that $Az = z$. So put x=z & y= y_n

$$M(Az, w, qt) \geq$$
$$\phi\left(\begin{array}{l} M(Az, Az, t), M(w, w, t), \\ \dfrac{aM(Az, w, t) + bM(Az, w, t)}{aM(Az, Az, t) + b}, \\ \dfrac{cM(Az, w, t) + dM(w, Az, t)}{c + dM(w, w, t)}, \\ \dfrac{eM(Az, Az, t) + fM(w, w, t)}{e + f} \end{array}\right)$$

$$M(Az, w, qt) \geq$$
$$\phi\left(1, 1, M(Az, w, t), M(Az, w, t), 1\right)$$

$M(Az, w, qt) \geq M(Az, w, t)$ which implies that $Az = w = z$

Now we shall show that $Bw = w$. So put x= x_n & y= w

$$M(Ax_n, Bw, qt) \geq \phi\left(\begin{array}{l} M(Ax_n, Sx_n, t), M(Bw, Tw, t), \\ \dfrac{aM(Sx_n, Tw, t) + bM(Ax_n, Tw, t)}{aM(Sx_n, Ax_n, t) + b}, \\ \dfrac{cM(Sx_n, Tw, t) + dM(Bw, Ax_n, t)}{c + dM(Bw, Tw, t)}, \\ \dfrac{eM(Sx_n, Tw, t) + fM(Bw, Sx_n, t)}{e + f} \end{array}\right)$$

$$M(z, Bw, qt) \geq \phi\left(\begin{array}{l} M(z, z, t), M(Bw, Bw, t), \\ \dfrac{aM(z, Bw, t) + bM(z, Bw, t)}{aM(z, z, t) + b}, \\ \dfrac{cM(z, Bw, t) + dM(Bw, z, t)}{c + dM(Bw, Bw, t)}, \\ \dfrac{eM(z, Bw, t) + fM(Bw, z, t)}{e + f} \end{array}\right)$$

$$M(z, Bw, qt) \geq \phi(1, 1, M(z, Bw, t), M(z, Bw, t), 1)$$

$M(z, Bw, qt) \geq M(z, Bw, t)$ which implies that $Az = Sz = Bz = Tz$.

Theorem 4.4: Let $(X, M, *)$ be a fuzzy metric space and let A, B, S and T be self mappings of X.Let the pairs $\{A, S\}$ and $\{B, T\}$ be continuously subcompatible. If there exist $q \in (0,1)$ s.t.

$$M(Ax, By, qt) \geq M(Ax, Sx, t) * M(Sx, Ty, t) * \frac{aM(Sx, Ty, t) + bM(Ax, Ty, t)}{aM(Sx, Ax, t) + b} * \frac{cM(Sx, Ty, t) + dM(By, Ax, t)}{c + dM(By, Ty, t)} * \frac{eM(Sx, Ax, t) + fM(By, Ty, t)}{e + f} \quad \ldots(4)$$

for all $x, y \in X$, $t > 0$ and $a, b, c, d, e, f \geq 0$ with a & b $(c \& d, e \& f)$ cannot be simultaneously 0,then there is a unique common fixed point of A, B, S and T.

Proof: The Proof follows from theorem 4.1.

Remark: Our result are generalization of results of C.T. Aage and J.N.Salunke [1] in following manner.If we take a=d=e=0 in our theorem then our results give the results of C.T. Aage and J.N.Salunke [1].

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