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A Fixed Point Theorem for Weakly C - Contraction Mappings of Integral Type.

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ABSTRACT
In the present paper, we shall prove a fixed point theorem by using generalized weak C- contraction of integral type. Our result is generalization of very known results.
Key words: Metric space, fixed point, weak C- contraction.
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1 Introduction and Preliminaries

Let (X, d) be a complete metric space and T: $X \to X$ a self-map of X. Suppose that $F_f = \{x \in X | \, T(x) = x\}$ is the set of fixed points of f. The classical Banach's fixed point theorem is one of the pivotal results of functional analysis. by using the following contractive definition: there exists k∈ [0, 1) such that ∀ x, y ∈ X , we have

$$d(Tx, Ty) \leq k d(x,y) . \quad (1.1)$$

If the metric space (X,d) is complete then the mapping satisfying (1.1) has a unique fixed point . Inequality (1.1) implies continuity of T. A natural question is that whether we can find contractive conditions which will imply existence of fixed point in a complete metric space but will not imply continuity .
Kannan [10,11] established the following result in which the above question has been answered in the affirmative.
If T : $X \to X$ where (X,d) is complete metric space , satisfies the inequality

$$d(Tx, Ty) \leq k[d(x,Tx) + d(y,Ty)] \quad (1.2)$$

where $0< k < \frac{1}{2}$ and x, y ∈ X, then T has a unique fixed point.
The mapping T need not be continuous .The mapping satisfying (1.2) are called Kannan type mappings. There is a large literature dealing with Kannan type mappings and their generalization some of which are noted in [8] ,[17] and [19].
A similar contractive condition has been introduced by Chatterjee [6]. We call this contraction a C-contraction.

Definition1.1 C-contraction

Let T : $X \to X$ where (X,d) is a metric space is called a C – contraction if there exists $0< k < \frac{1}{2}$ such that for all x, y ∈ X the following inequality holds:

$$d(Tx, Ty) \leq k[d(x,Ty) + d(y,Tx)] \quad (1.3)$$

Theorem 1.1 A C- contraction defined on a complete metric space has a unique fixed point.
In establishing theorem 1.1 there is no requirement of continuity of the C-contraction.
It has been established in [15] that inequalities (1.1),(1.2) and (1.3) are independent of one another. C-contraction and its generalizations have been discussed in a number of works some of which are noted in [4] ,[8], [9] and [19].
Banach's contraction mapping theorem has been generalized in a number of recent papers. As for example, asymptotic contraction has been introduced by Kirk[12] and generalized Banach contraction conjecture has been proved in [1] and [14].
Particularly a weaker contraction has been introduced in Hilbert spaces in[2].The following is the corresponding definition in metric space.

Definition1.2 Weakly contractive mapping

A mapping T : $X \to X$ where (X,d) is complete metric space is said to be weakly contractive if

$$d(Tx, Ty) \leq d(x,y) - \Psi\big(d(x,y)\big), \quad (1.4)$$

Where x, y ∈ X , Ψ :$[0,\infty) \to [0,\infty)$ is continuous and non-decreasing,
$\psi(x) = 0$ if and only if $x = 0$ and $\lim_{x\to\infty} \psi(x) = \infty$.

There are a number of works in which weakly contractive mappings have been considered. Some of these works are noted in [3],[7],[13], and [16].
In the present work in the same spirit we introduce a generalization of C- contraction.

Definition1.3 Weak C- contraction:
A mapping T : $X \to X$, where (X,d) is a metric space is said to be weakly C – contractive or a weak C-contraction if for all x, y $\in X$,

$$d(Tx,Ty) \leq \frac{1}{2}\,[d(x,Ty)+d(y,Tx)] - \Psi(d(x,Ty), d(y,Tx)) \qquad (1.5)$$

where Ψ: $[0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that Ψ(x,y) = 0 if and only if x = y= 0 .

If we take Ψ(x,y) = k(x+y) where $0< k < \frac{1}{2}$ then (1.5) reduces to (1.4), that is weak C – contractions are generalizations of C – contractions.
In a recent paper of Branciari [20] obtained a fixed point result for a single mapping satisfying an analogue of a Banach's contraction principle for integral type inequality as below: there exists c $\in$ [0,1) such that $\forall$ x, y $\in X$, we have

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq k\int_0^{d(x,y)} \varphi(t)dt$$

Where $\varphi : R^+ \to R^+$ is a Lebesgue – integrable mapping which is summable, non-negative and such that for each $\epsilon >0$, $\int_0^{\epsilon} \varphi(t)dt >0$.
Our main result is extended and modified to the weak C – contraction mapping in integral type .

MAIN RESULT
Let T : $X \to X$ where (X,d) is complete metric space be a weak C-contraction, which is satisfying the following property:

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \leq \alpha \int_0^{d(x,Ty)+d(y,Tx)} \varphi(t)dt + \beta \int_0^{\max\{d(x,Tx),d(y,Ty)\}} \varphi(t)dt$$
$$- \int_0^{\psi\{d(x,Ty),d(y,Tx),d(x,Tx),d(y,Ty)\}} \varphi(t)dt \qquad (2.1)$$

Then T has a unique fixed point.
Where $\alpha, \beta \in [0,1)$ with $2\alpha + \beta \leq 1$ and $\varphi : R^+ \to R^+$ is a Lebesgue – integrable mapping which is summable,non negative and such that for each $\epsilon >0$, $\int_0^{\epsilon} \varphi(t)dt >0$ and Ψ: $[0,\infty)^2 \to [0,\infty)$ is a continuous mapping such that Ψ(x,y) = 0 if and only if x = y= 0 .

Proof : Let x_0 ∈X and for all n≥ 1 , x_{n+1}= Tx_n .
If x_{n+1}= x_n= Tx_n. Then x_n is a fixed point of T.
So we assume,$x_{n+1} \neq x_n$.
Putting x = x_{n-1} and y = x_n in (2.1) we have for all n = 0,1,2, …….

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \int_0^{d(Tx_{n-1},Tx_n)} \varphi(t)dt$$
$$\leq \alpha \int_0^{d(x_{n-1},\ Tx_n)+d(x_n,Tx_{n-1})} \varphi(t)dt$$
$$+ \beta \int_0^{\max\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\}} \varphi(t)dt$$
$$- \int_0^{\psi\{d(x_{n-1},Tx_n),d(,x_nTx_{n-1}),d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n)\}} \varphi(t)dt$$
$$= \alpha \int_0^{d(x_{n-1},\ x_{n+1})+d(x_n,x_n)} \varphi(t)dt$$
$$+ \beta \int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \varphi(t)dt$$
$$- \int_0^{\psi\{d(x_{n-1},\ x_{n+1}),d(,x_nx_n),d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \varphi(t)dt$$

Since T is Weakly C – contraction, this gives that
$\psi\{d(x_{n-1},\ x_{n+1}), 0, d(x_{n-1},x_n), d(x_n,x_{n+1})\} = 0$ and

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \alpha \int_0^{d(x_{n-1},\ x_{n+1})} \varphi(t)dt$$
$$+ \beta \int_0^{\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}} \varphi(t)dt \qquad (2.2)$$

Now here arise two cases:
Case I: - If we choose
$\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$
Then (2.2) can be written as

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \leq \alpha \int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt + \alpha \int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt$$
$$+ \beta \int_0^{d(x_{n-1},x_n)} \varphi(t)dt$$

(1- α) $\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = (\alpha+\beta)\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$

$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \frac{\alpha+\beta}{1-\alpha}\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$

$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le K\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$ where k = $\frac{\alpha+\beta}{1-\alpha} \le 1$

***Case* 2** : : - If we choose

max { $d(x_{n-1},x_n), d(x_n,x_{n+1})$ } = $d(x_n,x_{n+1})$

Then (2.2) can be written as

$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le \alpha\int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt + \alpha\int_0^{d(x_n,\ x_{n+1})} \varphi(t)dt$

$+\beta\int_0^{d(x_n,x_{n+1})} \varphi(t)dt$

$[1-(\alpha+\beta)]\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \alpha\int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt$

$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = \frac{\alpha}{1-(\alpha+\beta)}\int_0^{d(x_{n-1},\ x_n)} \varphi(t)dt$

$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le k\int_0^{d(x_{n-1},x_n)} \varphi(t)dt$,where k = $\frac{\alpha}{1-(\alpha+\beta)} \le 1$ (2.3)

From above both cases:

$$\int_0^{d(x_n,x_{n+1})} \varphi(t)dt \le k^2\int_0^{d(x_{n-2},x_{n-1})} \varphi(t)dt$$
$$\le k^3\int_0^{d(x_{n-3},x_{n-2})} \varphi(t)dt$$
$$\le \text{------}$$
$$\le k^n\int_0^{d(x_0,x_1)} \varphi(t)dt$$

Taking limit as n → ∞ , we get

$\lim_{n\to\infty}\int_0^{d(x_n,x_{n+1})} \varphi(t)dt = 0$, as k $\in$ [0,1) (2.4)

Now we prove that $\{x_n\}$ is a Cauchysequence. Suppose it is not.Then there exists an $\varepsilon > 0$ and sub sequence $\{y_{m(p)}\}$and $\{y_{n(p)}\}$such that

M(p) < n(p) < m(p+1) with

$d(x_{n(p)},x_{m(p)}) \ge \varepsilon$, $d(x_{n(p)-1},x_{m(p)}) < \varepsilon$ (2.5)

Now

$d(x_{m(p)-1},x_{n(p)-1}) \le d(x_{m(p)-1},x_{m(p)}) + d(x_{m(p)},x_{n(p)-1})$

$< d(x_{m(p)-1},x_{m(p)}) + \varepsilon$ (2.6)

From (2.4), (2.6), we get

$\lim_{p\to\infty}\int_0^{d(x_{m(p)-1},x_{n(p)-1})} \varphi(t)dt \le \int_0^{\varepsilon} \varphi(t)dt$ (2.7)

Using (2.3), (2.5), and (2.7) we get ,

$$\int_0^{\varepsilon} \varphi(t)dt \le \int_0^{d(x_{n(p)},x_{m(p)})} \varphi(t)dt$$
$$\le k\int_0^{d(x_{n(p)-1},x_{m(p)-1})} \varphi(t)dt$$
$$\le k\int_0^{\varepsilon} \varphi(t)dt$$

Which is contradiction, since k $\in$ (0, 1). therefore $\{x_n\}$ is a Cauchy sequence Since (X,d) is complete metric space , therefore have call the limit z .

From (2.1), we get

$\int_0^{d(Tz,x_{n+1})} \varphi(t)dt = \int_0^{d(Tz,Tx_n)} \varphi(t)dt$

$\le \alpha\int_0^{d(z,Tx_n)+d(x_n,Tz)} \varphi(t)dt$

$+\beta\int_0^{\max\{d(z,Tz),d(x_n,Tx_n)\}} \varphi(t)dt$

$-\int_0^{\psi\{d(z,Tx_n),d(x_n,Tz),d(z,Tz),d(x_n,Tx_n)\}} \varphi(t)dt$

Taking limit as n → ∞ , we get

$\int_0^{d(Tz,z)} \varphi(t)dt \le \alpha\int_0^{d(z,Tz)} \varphi(t)dt + \beta\int_0^{d(z,Tz)} \varphi(t)dt$

$= (\alpha+\beta)\int_0^{d(z,Tz)} \varphi(t)dt$

Which is Contradiction

Therefore $Tz = z$

That is z is a fixed point of T in X.

Uniqueness : Let w is another fixed point of T in X such that $z \neq w$, then we have

From (2.1), we get

$$\int_0^{d(z,w)} \varphi(t)dt = \int_0^{d(Tz,Tw)} \varphi(t)dt$$

$$\leq \alpha \int_0^{d(z,Tw)+d(w,Tz)} \varphi(t)dt + \beta \int_0^{\max\{d(z,Tz),d(w,Tw)\}} \varphi(t)dt$$

$$- \int_0^{\psi\{d(z,Tw),d(w,Tz),d(z,Tz),d(w,Tw)\}} \varphi(t)dt$$

$$\int_0^{d(z,w)} \varphi(t)dt \leq 2\alpha \int_0^{d(z,w)} \varphi(t)dt$$

Which is contradiction

So z = w that is , z is unique fixed point of T in X .

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