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Characterization of Function-E-Chainble Sets in Topological Space

*PRIYA CHOUDHARY,** VIJETA IYER and *** KIRAN SHRIVASTAVA
Department of Mathematics, S.N.G.G.P.G. College, Bhopal (INDIA)
*Mailing address :C- 473, Shahpura Bhopal-462016 (INDIA)
**Mailing address :C-478,Shahpura Bhopal-462016(INDIA)
*** D/79,Nehru Nagar Bhopal-462003 (INDIA)
*email id: priyavc14@gmail.com
**email id: vijeta_iyer@yahoo.com
***email id:kiranshri_1@rediffmail.com

Abstract
In this paper we introduce the concept of function-ε-chain between two sets in topological spaces through continuous function which is the extension of function-ε-chain between two points of the space. Simple characterization of function-ε-chainable sets in terms of function –ε-chains between their points has been established. In case of metric space, the equivalence of ε-chainability and function-ε-chainability of sets is also established in this paper. Further some results of [1] have been generalized.
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Throughout this paper X will stand for topological space with topology τ and $f : X \to [0,\infty)$ will be a real valued non-constant continuous function unless stated otherwise.

1. Definitions

Let A be a subset of the topological space X. For $\varepsilon > 0$,
let $V_{f\,\varepsilon}(A) = \{ x \in X : |\, f(x) - f(A)\,| < \varepsilon \}$, where $|\, f(x) - f(A)\,| = inf\,\{\, |\, f(x) - f(a)\,| : a \in A\}$

1.1. Definition

Let $A \subset X$ .Then f −diameter of A is defined to be $sup\{\, |\, f(x) - f(a)\,| : x, a \in A\,\}$ and is denoted by $\delta_f(A)$.

1.2. Definition

Let $A, B \subset X$. Then f-distance between A and B is defined to be $inf\,\{|\, f(a) - f(b)\,|: a \in A, b \in B\}$ and is denoted by $d_f(A,B)$.

1.3. Remark

$V_{f\,\varepsilon}(A) = \{x : d_f(x,A) < \varepsilon\}$

1.4. Definition

A topological space (X,τ) is said to be function-$f-\varepsilon$-chainable if for $\varepsilon > 0$ there exists a non-constant continuous function $f : X \to [0,\infty)$ such that for every pair of elements $x, y \in X$ there is a sequence $x = x_0, x_1, x_2, \dots, x_n = y$ of elements in X with $|\, f(x_i) - f(x_{i-1})\,| < \varepsilon$; $1 \le i \le n$

1.5. Definition

Let (X,τ) be a topological space and let there exist a non –constant continuous function $f : X \to [0,\infty)$ such that X is function$-f-\varepsilon$ −chainable for every $\varepsilon > 0$. Then X is said to be function$-f$ −chainable.

1.6. Definition

Let $A, B \subset X$. A function$-f-\varepsilon$ −chain of length n from A to B is a finite sequence $A_0, A_1, A_2, \dots, A_n$ of subsets of X with $A = A_0$, $A_n = B$, $A_{i-1} \subset V_{f\,\varepsilon}(A_i)$ and $A_i \subset V_{f\,\varepsilon}(A_{i-1})$. If function $-f-\varepsilon-$ chain exist between A and B we say that $< A, B >$ is function$-f-\varepsilon$ −chainable and $< A, B >$ is function$-f$ −chainable if it is function$-f-\varepsilon$ −chainable for each positive ε.

Using the notation inductively construct the set $V^n_{f\,\varepsilon}(A)$ for each $n \in Z^+$ as follows:
$V^1_{f\,\varepsilon}(A) = V_{f\,\varepsilon}(A)$ for each $n \ge 2$ set $V^n_{f\,\varepsilon}(A) = V_{f\,\varepsilon}(V^{n-1}_{f\,\varepsilon}(A))$. The following should be observed:

(1) $V^n_{f\,\varepsilon}(A) \subset V^{n+1}_{f\,\varepsilon}(A)$

(2) $V^n_{f\,\varepsilon}(A) \subset V_{f\,n\varepsilon}(A)$

We set $\Phi_{f\,\varepsilon}(< A, B >)$ to be the length of the shortest function$-f-\varepsilon$ −chain between A and B.

1.7. Example of function$-f-\varepsilon$ −chainable sets

Let X be a topological space with odd even topology which is a partition topology generated by $P = \{\{1,2\},\{3,4\},\{5,6\},\dots\}$ and $f : X \to [0,\infty)$ define by $f(2k) = k$, $f(2k-1) = k$ is continuous function. Let $A = \{1,2\}$, $B = \{3,4\}$ and $\varepsilon = 1.2$ then $V_{f\,\varepsilon}(A) = \{1,2,3,4\}$ and $V_{f\,\varepsilon}(B) = \{1,2,3,4,5,6\}$ or $A \subset V_{f\,\varepsilon}(B)$ *and* $B \subset V_{f\,\varepsilon}(A)$ *or* $A = A_0, A_1 = B$ then $< A, B >$ is $f-\varepsilon$ −chainable for $\varepsilon = 1.2$

2. Theorems

2.1. Some results whose proofs are obvious hence omitted.

2.1.1. Result

Let $A, B \subset X$, then

i. $d_f(A,B) < \varepsilon$ if $B \cap V_{f\,\varepsilon}(A) \neq \emptyset$

ii. $d_f(A,B) < \varepsilon$ if $A \cap V_{f\,\varepsilon}(B) \neq \emptyset$

iii. $A \subset V_{f\,\varepsilon}(A)$

iv. $V_{f\,\varepsilon}(A) \subset V_{f\,\varepsilon}(B)$ if $A \subset B$

v. $V_{f\,\varepsilon}(A) \cup V_{f\,\varepsilon}(B) = V_{f\,\varepsilon}(A \cup B)$

vi. $V_{f\,\varepsilon}(A \cap B) \subseteq V_{f\,\varepsilon}(A) \cap V_{f\,\varepsilon}(B)$

2.1.2. Result

If $< A,B >$ and $< C,D >$ are $f-$chainable then $< A \cup C, B \cup D >$ is also $f-$chainable where $A,B,C,D \subset X$

2.2. Theorem

Let (X,τ) be a topological space and $A \subset X$ then

$$A \subseteq \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A) = \overline{A}$$

Proof : As $A \subset V_{f\,\varepsilon}(A)$, $\varepsilon > 0$ then $A \subseteq \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A)$

Let $x \in \overline{A}$ then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$ or there exists $y \in A$ such that $|f(x) - f(y)| < \varepsilon$ or $x \in V_{f\,\varepsilon}(A)$, $\forall\ \varepsilon > 0$

$$\text{or } \overline{A} \subset \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A)$$

Suppose that $\overline{A} \nsubseteq \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A)$ or there exist $x \in \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A)$ such that $x \notin \overline{A}$

or there exist $x \in V_{f\,\varepsilon}(A)$, $\forall\, \varepsilon > 0$ such that $x \notin \overline{A}$ and hence $x \notin A$

Or $|f(x) - f(A)| \neq 0$ or $|f(x) - f(A)| = \boldsymbol{\varepsilon}'$ for some real number $\boldsymbol{\varepsilon}' > 0$ or $x \notin V_{f\,\varepsilon}(A)$ for $\varepsilon < \boldsymbol{\varepsilon}'$.
This contradicts that $x \in V_{f\,\varepsilon}(A)$ $\forall\ \varepsilon > 0$

$$\text{Hence} \quad \overline{A} = \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A)$$

2.2.1. Collolary

A is closed if and only if

$$A = \bigcap_{\varepsilon>0} V_{f\,\varepsilon}(A).$$

Characterization of function$-f-\varepsilon-$chainable sets in terms of function$-f-\varepsilon-$chains between points and sequence is given below.

2.3. Theorem

Let $A, B \subset X$ and $< A,B >$ be function$-f-\varepsilon-$chain from every point of A to some point of B and vice-versa . Also converse holds.

Proof: We prove the necessary part first. As $< A,B >$ is function$-f-\varepsilon-$chainable there exists a sequence $A_0, A_1, \dots, A_n$ of subsets of X with $A = A_0, A_n = B$, $A_i \subset V_{f\,\varepsilon}(A_{i-1})$ and $A_{i-1} \subset V_{f\,\varepsilon}(A_i)$; $1 \le i \le n$. Let $x \in A$ be arbitrary. Then $x \in A$ or $x \in V_{f\,\varepsilon}(A_1)$ *or* $|f(x) - f(x_1)| < \varepsilon$ for some $x_1 \in A_1$. Again $x \in A_1$ then $|f(x_1) - f(x_2)| < \varepsilon$ for some $x_2 \in A_2$. Repeating the above process n times we obtain a sequence of points $x = x_0, x_1, x_2, \dots, x_n = y \in B$ such that $|f(x_i) - f(x_{i-1})| < \varepsilon$; $1 \le i \le n$ and $x_i \in A_i$, showing that there exist a function$-f-\varepsilon-$chain from x to y. Likewise we can obtain a function$-f-\varepsilon-$chain from every point of B to a point of A.

We next prove the sufficient part. Let there exist a function$-f-\varepsilon-$chain from every point of A to some point of B and vice-versa. Let $A_1 = \{y \in X : |f(y) - f(x)| < \varepsilon$ for some $x \in A$ and $x \neq y\}$. Clearly $A_1 \neq \emptyset$ and $A_1 \subset V_{f\,\varepsilon}(A)$. Next we show that $A \subset V_{f\,\varepsilon}(A_1)$. If $x \in A$ then there exist a sequence $x = x_0, x_1, x_2 \dots, x_n = y \in B$ such that $|f(x) - f(x_1)| < \varepsilon$ or $x_1 \in A_1$ then $|f(x) - f(A_1)| < \varepsilon$ or $x \in V_{f\,\varepsilon}(A_1)$ or $A \subseteq V_{f\,\varepsilon}(A_1)$.

Again let $A_2 = \{y \in X : |f(y) - f(x)| < \varepsilon \textit{ for some } x \in A_1 \textit{ and } x \neq y\}$. Clearly $A_2 \neq \emptyset$, $A_2 \subset V_{f\,\varepsilon}(A_1)$ and it can be shown as above that $A_1 \subset V_{f\,\varepsilon}(A_2)$. Repeating the above process n times we obtain a sequence $A = A_0, A_1, \dots, A_n = B$ of subsets of X, such that $< A,B >$ is function$-f-\varepsilon-$chainable.

2.4. Theorem

Let $A, B \subset X$, if $\delta_f(A \cup B) \le \varepsilon$, then $< A,B >$ is function$-f-\varepsilon-$chainable.

Proof : Obvious

In next theorem the equivalence of ε −chainability and function-$f-\varepsilon$ −chainability of two sets is obtained in metric spaces.

2.5. Theorem

Let (X,d) be a metric space and (X,τ) be topological space. If $<A,B>$ is ε-chainable then $<A,B>$ is $f-\varepsilon$ −chainable for some continuous function $f : X \to [0,\infty)$.

Proof: Let $\overline{x} \in X$ and $f : X \to [0,\infty)$ be defined as $f(x) = d(x,\overline{x})\ \forall\ x \in X$. Let $<A,B>$ be ε− chainable where $A,B \subset X$. Then there exist ε−chain of finite sequence $A = A_0, A_1, A_2, \dots, A_n = B$ such that $A_{i-1} \subset V_\varepsilon(A_i)$ and $A_i \subset V_\varepsilon(A_{i-1})$. Let x $\in V_\varepsilon(A_i)$ or $d(x,A_i) < \varepsilon$

or $\inf_{a\in A_i} d(x,a) < \varepsilon$ or d(x, a) $< \varepsilon$ for some $a \in A_i$.

Now $d(x,\overline{x}) \le d(a,\overline{x}) + d(a,x)$

or $f(x) - f(a) \le d(a,x) < \varepsilon$ or $|f(x) - f(a)| < \varepsilon$

or $|f(x) - f(A_i)| < \varepsilon$ or $x \in V_{f\,\varepsilon}(A_i)$ or $V_\varepsilon(A_i) \subset V_{f\,\varepsilon}(A_i)$.

Hence $A_{i-1} \subset V_{f\,\varepsilon}(A_i)$ and $A_i \subset V_{f\,\varepsilon}(A_{i-1})$ or $<A,B>$ is $f-\varepsilon-$ chainable.

2.6. Theorem

Let $<\varepsilon_n>$ be monotonically increasing sequence of positive real number converging to ε (arbitrary). Then $<A,B>$ is function$-f-\varepsilon$-chainable if and only if there exists a subsequence $<\varepsilon_{n_k}>$ of $<\varepsilon_n>$ such that $<A,B>$ is function$-f-\varepsilon_{n_k}$-chainable for each $k \in N$.

Proof: Similar to proof of theorem 2[1]

2.7. Theorem

Let $A,B \subset X$. If $(A\cup B)$ is connected and $\varepsilon > max\,\{\delta_f(A),\delta_f(B)\}$ then $<A,B>$ is function $-f-\varepsilon$ −chainable.

Proof: Similar to proof of theorem 4[1]

2.8. Theorem

Let $A,B \subset X$ and $\varepsilon > max\,\{\delta_f(A),\delta_f(B),d_f(A,B)\}$ then $<A,B>$ is function $-f-\varepsilon-$ chainable and $\Phi_{f\varepsilon}(<A,B>) = 2$.

Proof : Similar to proof of Preposition [1]

2.9. Theorem

X is function$-f-\varepsilon$ −chainable if and only if $<A,B>$ is function$-f-\varepsilon$ −chainable for every pair of subsets A,B of X.

Proof : Similar to proof of theorem 5[1]

2.10. Theorem

Let $A,B \subset X$. Then $\overline{A} = \overline{B}$ if and only if $<A,B>$ is function$-f$ −chainable and $\Phi_{f\,\varepsilon}(<A,B>) = 1$.

Proof : Similar to theorem 7[1]

2.11. Theorem

If $V^n_{f\,\varepsilon}(A) \subset B \subseteq V^{n+1}_{f\,\varepsilon}(A)$, then $<A,B>$ is function$-f-\varepsilon$ −chainable and $\Phi_{f\,\varepsilon}(<A,B>) = n+1$.

Proof : Similar to proof of theorem 8[1]

2.12. Theorem

Let X be function$-f-\varepsilon$ −chainable. Define a relation ~ on X as follows:

$<A,B> \sim <C,D>$ if and only if $\Phi_{f\,\varepsilon}(<A,B>) = \Phi_{f\,\varepsilon}(<C,D>)$.

Then ~ is an equivalence relation on X, which partitions X into disjoint equivalence classes denoted by $\overline{<A,B>}, \overline{<C,D>}$.

Proof : Obvious.

2.13. Theorem

Let $\{A = A_0, A_1, A_2, \dots, A_n = B\}$ be a simple chain [3] then $<A,B>$ is function$-f-2\varepsilon$ −chainable where $\varepsilon > max\,\{\delta_f(A),\delta_f(A_1),\delta_f(A_2),\dots,\delta_f(B)\}$.

Proof : Let $x \in A$, $y \in (A\cap A_1)$, $z \in A_1$

then $|f(x) - f(y)| < \varepsilon$ and $|f(y) - f(z)| < \varepsilon$ or $|f(x) - f(z)| < 2\varepsilon$ or $|f(x) - f(A_1)| < 2\varepsilon$

or $x \in V_{f\,2\varepsilon}(A_1)$ or $A \subseteq V_{f\,2\varepsilon}(A_1)$.

then $\inf_{x\in A}|f(x) - f(z)| < 2\varepsilon$ or $|f(z) - f(A)| < 2\varepsilon$

Or $z \in V_{f\,2\varepsilon}(A)$ or $A \subset V_{f\,2\varepsilon}(A_1)$ and $A_1 \subset V_{f\,2\varepsilon}(A)$.

Similarly $A_1 \subset V_{f\,\varepsilon}(A_2)$, $A_2 \subset V_{f\,\varepsilon}(A_1)$, …, $A_{n-1} \subset V_{f\,\varepsilon}(A_n)$ and $A_n \subset V_{f\,\varepsilon}(A_{n-1})$.

Thus the sets $A_1, A_2, \dots, A_{n-1}$ forms a function$-f-\varepsilon$ −chain from A to B that is $<A,B>$ is function$-f-2\varepsilon$ −chainable.

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