Fixed Point and Common Fixed Point Theorems in Complete Metric Spaces

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Abstract
In this paper we established a fixed point and a unique common fixed point theorems in four pair of weakly compatible self-mappings in complete metric spaces satisfy weakly compatibility of contractive modulus.
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Introduction
The concept of the commutativity has generalised in several ways .In 1998, Jungck Rhoades [3] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but conversely. Brian Fisher [1] proved an important common fixed point theorem. Seesa S [9] has introduced the concept of weakly commuting and Gerald Jungck [2] initiated the concept of compatibility. It can be easily verified that when the two mappings are commuting then they are compatible but not conversely. Thus the study of common fixed point of mappings satisfying contractive type condition have been a very active field of research activity during the last three decades.
In 1922 the polished mathematician, Banach, proved a theorem ensures, under appropriate conditions, the existence and uniqueness of a fixed point. His result is called Banach fixed point theorem or the Banach contraction principle. This theorem provides a technique for solving a variety of a applied problems in a mathematical science and engineering. Many authors have extended, generalized and improved Banach fixed point theorem in a different ways. Jungck [2] introduced more generalised commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings.
The main purpose of this paper is to present fixed point results for two pair of four self maps satisfying a new contractive modulus condition by using the concept of weakly compatible maps in complete metric spaces.

Preliminaries
The definition of complete metric spaces and other results that will be needed are:
Definition 1: Let f and g two self-maps on a set X. Maps f and g are said to be commuting if
$$fgx = gfx \text{ } for \text{ } all \text{ } x \epsilon X.$$
Definition 2: Let f and g two self-maps on a set X. If $fx = gx, for\ some\ x\ in\ X\ then\ x\ is$ called coincidence point of f and g.
Definition 3: Let f and g two self-maps defined on a set X, then f and g are said to be compatible if they commute at coincidence points. That is, if $fu = gu\ for\ some\ u\ \epsilon\ X\ , then\ fgu = gfu.$
Definition 4: Let f and g be two weakly compatible self-maps defined on a set X, If f and g have a unique point of coincidence, that is, $w = fx = gx\ , then\ w\ is\ a\ unique\ common\ fixed\ point\ of\ f\ and\ g.$
Definition 5: A sequence $\{x_n\}$ in a metric space $(X,d) is\ said\ to\ be\ convergent\ to\ a\ point\ x\epsilon\ X,$
denoted by $\lim_{n\to\infty} x_n = x\ , if\ \lim_{n\to\infty} d(x_n, x) = 0.$
Definition 6: A sequence $\{x_n\}$ in a metric space $(X,d)\ is\ said\ to\ be\ Cauchy\ sequence\ if$
$$\lim_{t\to\infty} d(x_n, x_m) = 0\ for\ all\ n, m\ > t.$$
Definition 7: A metric space $(X,d)\ is\ said\ to\ be\ Complete\ if$ every Cauchy sequence in X is convergent.
Definition 8: A function $\phi: [0,\infty) \to [0,\infty)\ is\ said\ to\ be\ a\ contractive\ modulus\ if$
$$\phi: [0,\infty) \to [0,\infty)\ and\ \phi(t) < t\ for t\ > 0\ .$$
Definition 9: A real valued function ϕ defined on $X \subseteq R$ is said to be upper semi continuous
$$if \lim_{n\to\infty} sup\ \phi(t_n) \le \phi(t)\ , for\ every\ sequence\ \{t_n\}\ \epsilon\ X\ with\ t_n \to t\ as\ n \to \infty\ .$$
Hence it is clear that every continuous function is upper semi continuous but converse may not true.

MAIN RESULT

In this section we established a common fixed point theorem for two pairs of weakly compatible mappings in complete metric spaces using a contractive modulus.

Theorem 1: Let (X,d) be a complete metric space. Suppose that the mapping E, F, G and H are four self maps of X satisfying the following condition:

(i) $H(X) \subseteq E(X)\ and\ G(X) \subseteq F(X)$

(ii) $d(G_x, H_y) \le \phi(\lambda(x,y))$

Where ϕ is a upper semi continuous, contractive modulous and

$$\lambda(x,y) = \max \left\{ \begin{array}{c} d(E_x,F_y), d(E_x,G_x), d(F_y,H_y), \frac{1}{2}[d(E_x,H_y) + d(F_y,G_x)], \\ \frac{1}{4}\left\{\frac{d(E_x,F_y)+d(E_x,G_x)+d(F_y,G_x)}{1+d(E_x,F_y)d(E_x,G_x)d(F_y,G_x)}\right\}, \\ \frac{3}{2}\left\{\frac{d(F_y,H_y)+d(F_y,G_x)+d(F_y,E_x)}{1+d(F_y,H_y)d(F_y,G_x)d(F_y,E_x)}\right\} \end{array} \right\}$$

(iii) The pair (G, E) and (H, F) are weakly compatible then E, F, G& H have a unique common fixed point.

Proof : Suppose $C\ is\ an\ arbitrary\ point\ of\ X\ and\ define\ the\ sequence\ \{y_n\}\ in\ X\ such\ that$

$$y_n = Gx_n = Fx_{n+1}$$
$$y_{n+1} = Hx_{n+1} = Ex_{n+2}$$

By (ii) we have

$$d(y_n, y_{n+1}) = d(Gx_n, Hx_{n+1}) \le \phi(\lambda(x_n, x_{n+1}))$$

$$\text{Where } \lambda(x_n,x_{n+1}) = max \left\{ \begin{array}{c} d(Ex_n,Fx_{n+1}), d(Ex_n,Gx_n), d(Fx_{n+1},Hx_{n+1}), \\ \frac{1}{2}[d(Ex_n,Hx_{n+1}) + d(Fx_{n+1},Gx_n)], \\ \frac{1}{4}\left[\frac{d(Ex_n,Fx_{n+1})+d(Ex_n,Gx_n)+d(Fx_{n+1},Gx_n)}{1+d(Ex_n,Fx_{n+1})d(Ex_n,Gx_n)d(Fx_{n+1},Gx_n)}\right], \\ \frac{3}{2}\left[\frac{d(Fx_{n+1},Hx_{n+1})+d(Fx_{n+1},Gx_n)+d(Fx_{n+1},Ex_n)}{1+d(Fx_{n+1},Hx_{n+1})d(Fx_{n+1},Gx_n)d(Fx_{n+1},Ex_n)}\right] \end{array} \right\}$$

$$= max \left\{ \begin{array}{c} d(Hx_{n-1},Gx_n), d(Hx_{n-1},Gx_n), d(Gx_n,Hx_{n+1}), \\ \frac{1}{2}[d(Hx_{n-1},Hx_{n+1}) + d(Gx_n,Gx_n)], \\ \frac{1}{4}\left[\frac{d(Hx_{n-1},Gx_n)+d(Hx_{n-1},Gx_n)+d(Gx_n,Gx_n)}{1+d(Hx_{n-1},Gx_n)d(Hx_{n-1},Gx_n)d(Gx_n,Gx_n)}\right], \\ \frac{3}{2}\left[\frac{d(Gx_n,Hx_{n+1})+d(Gx_n,Gx_n)+d(Gx_n,Hx_{n-1})}{1+d(Gx_n,Hx_{n+1})d(Gx_n,Gx_n)d(Gx_n,Hx_{n-1})}\right] \end{array} \right\}$$

$$\le \max\left\{d(Hx_{n-1},Gx_n), (Gx_n,Hx_{n+1}), \frac{1}{2}[d(Hx_{n-1},Hx_{n+1})]\frac{1}{4}\left\{\frac{d(Hx_{n-1},Gx_n)}{1}\right\}\frac{3}{2}\left\{\frac{d(Hx_{n-1},Hx_{n+1})}{1}\right\}\right\}$$

$$\le \max\left\{d(y_{n-1},y_n), d(y_n,y_{n+1}), \frac{1}{2}[d(y_{n-1},y_{n+1})], \frac{1}{4}d(y_{n-1},y_n), \frac{3}{2}[d(y_{n-1},y_{n+1})]\right\}$$

$$\le \max\{d(y_{n-1},y_n), d(y_n,y_{n+1})\}$$

Since ϕ is a contractive modulus; $\lambda(x_n,x_{n+1}) = d(y_n,y_{n+1})\ is\ not\ possible, Thus$

$$d(y_n,y_{n+1}) \le \phi\, d(y_{n-1},y_n) \qquad (1)$$

Since ϕ is an upper semi continuous contractive modulus, Equation (1) implies that the sequence $\{d(y_n,y_{n+1})\}$ is monotonic decreasing and continuous.

Hence there exists a real number, say $r \ge 0\ such\ that\ \lim_{n\to\infty} d(y_n,y_{n+1}) = r$

$$\therefore\ as\ n \to \infty, Eq^n(1)\ implies\ that$$

$$r \le \phi(r)$$

Which is possible only If r = 0 because ϕ is a contractive modulus, Thus

$$\lim_{n\to\infty} d(y_n,y_{n+1}) = 0$$

Now we show that $\{y_n\}$is a Cauchy sequence.

Let If possible we assume that $\{y_n\}$is not a cauchy sequence , Then there exist an $\in > 0$ and subsequence $\{n_i\}\ and\ \{m_i\}\ such\ that\ m_i < n_i < m_{i+1}\ and$

$d(y_m, y_n) \geq \in$ and $d(y_m, y_{n_{i-1}}) < \in$

$$d(y_m, y_n) \geq \in \text{ and } d(y_m, y_{n_{i-1}}) < \in \quad (2)$$

So that $\in \leq d(y_m, y_{n_i}) \leq d(y_{m_i}, y_{n_{i-1}}) + d(y_{n_{i-1}}, y_{n_i}) < \in + d(y_{n_{i-1}}, y_{n_i})$

Therefore $\lim_{n\to\infty} d(y_{m_i}, y_{n_i}) = d(Gx_{m_i}, Hx_{n_i}) \leq \phi(\lambda(x_{m_i}, x_{n_i}))$

$$\text{i.e. } \in \leq \phi(\lambda(x_{m_i}, x_{n_i})) \quad (3)$$

Where,

$$\lambda(x_{m_i}, x_{n_i}) = \max \left\{ \begin{array}{l} d(Ex_{m_i}, Fx_{n_i}), d(Ex_{m_i}, Gx_{m_i}), d(Fx_{n_i}, Hx_{n_i}), \\ \frac{1}{2}\left[d(Ex_{m_i}, Hx_{n_i}) + d(Fx_{n_i}, Gx_{m_i})\right] \\ \frac{1}{4}\left[\frac{d(Ex_{m_i}, Fx_{n_i}) + d(Ex_{m_i}, Gx_{m_i}) + d(Fx_{n_i}, Gx_{m_i})}{1 + d(Ex_{m_i}, Fx_{n_i}) + d(Ex_{m_i}, Gx_{m_i}) d(Fx_{n_i}, Gx_{m_i})}\right] \\ \frac{3}{2}\left[\frac{d(Fx_{n_i}, Hx_{n_i}) + d(Fx_{n_i}, Gx_{m_i}) + d(Fx_{n_i}, Ex_{m_i})}{1 + d(Fx_{n_i}, Hx_{n_i}) d(Fx_{n_i}, Gx_{m_i}) d(Fx_{n_i}, Ex_{m_i})}\right] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} d(Hx_{m_{i-1}}, Gx_{n_{i-1}}), d(Hx_{m_{i-1}}, Gx_{m_i}), d(Gx_{n_{i-1}}, Hx_{n_i}), \\ \frac{1}{2}\left[d(Hx_{m_{i-1}}, Hx_{n_i}) + d(Gx_{n_{i-1}}, Gx_{m_i})\right], \\ \frac{1}{4}\left[\frac{d(Hx_{m_{i-1}}, Gx_{n_{i-1}}) + d(Hx_{m_{i-1}}, Gx_{m_i}) + d(Gx_{n_{i-1}}, Gx_{m_i})}{1 + d(Hx_{m_{i-1}}, Gx_{n_{i-1}}) d(Hx_{m_{i-1}}, Gx_{m_i}) d(Gx_{n_{i-1}}, Gx_{m_i})}\right] \\ \frac{3}{2}\left[\frac{d(Gx_{n_{i-1}}, Hx_{n_i}) + d(Gx_{n_{i-1}}, Gx_{m_i}) + d(Gx_{n_{i-1}}, Hx_{m_{i-1}})}{1 + d(Gx_{n_{i-1}}, Hx_{n_i}) d(Gx_{n_{i-1}}, Gx_{m_i}) d(Gx_{n_{i-1}}, Hx_{m_{i-1}})}\right] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{l} d(y_{m_{i-1}}, y_{n_{i-1}}), d(y_{m_{i-1}}, y_{m_i}), d(y_{n_{i-1}}, y_{n_i}), \\ \frac{1}{2}\left[d(y_{m_{i-1}}, y_{n_i}) + d(y_{n_{i-1}}, y_{m_i})\right], \\ \frac{1}{4}\left[\frac{d(y_{m_{i-1}}, y_{n_{i-1}}) + d(y_{m_{i-1}}, y_{m_i}) + d(G_{n_{i-1}}, y_{m_i})}{1 + d(y_{m_{i-1}}, y_{n_{i-1}}) d(y_{m_{i-1}}, y_{m_i}) d(y_{n_{i-1}}, y_{m_i})}\right] \\ \frac{3}{2}\left[\frac{d(y_{n_{i-1}}, y_{n_i}) + d(y_{n_{i-1}}, y_{m_i}) + d(y_{n_{i-1}}, y_{m_{i-1}})}{1 + d(y_{n_{i-1}}, y_{n_i}) d(y_{n_{i-1}}, y_{m_i}) d(y_{n_{i-1}}, y_{m_{i-1}})}\right] \end{array} \right\}$$

By taking limit as $i \to \infty$, *we get*

$$\lim_{i\to\infty} \lambda(x_{mi}, x_{ni}) = max\left\{\in, 0, 0, \frac{1}{2}(\in + \in), \frac{1}{4}\left[\frac{\in + 0 + 0}{1+0}\right], \frac{3}{2}\left[\frac{0 + 0 + \in}{1+0}\right]\right\}$$

Thus we have ,$\lim_{t\to\infty} \lambda(x_m, x_n) = \in$
Therefore from (3) $\in \leq \phi(E)$
This is a contraction because $0 < \in$ *and* ϕ *is contractive modulus.*
Thus $\{y_n\}$ *is a cauchy sequence in* X,
since X *is complete , there exist a point* z *in* X *such that* $\lim_{n\to\infty} y_n = z$.
Thus $\lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Fx_{n+1} = z$ *and* $\lim_{n\to\infty} Hx_{n+1} = \lim_{n\to\infty} Ex_{n+2} = z$
i. e. $\lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Fx_{n+1} = \lim_{n\to\infty} Hx_{n+1} = \lim_{n\to\infty} Ex_{n+2} = z$
since $G(X) \subseteq F(X)$ *, there exist apoint* $u \in X$ *such that* $z = Eu$
Then by (ii), we have

$$d(Gu, z) \leq d(Gu, Hx_{n+1}) + d(Hx_{n+1}, z) \leq \phi(\lambda(u, x_{n+1})) + d(Hx_{n+1}, z)$$

Where, $$\lambda(u, x_{n+1}) = max \begin{Bmatrix} d(Eu, Fx_{n+1}), d(Eu, Gu), d(Fx_{n+1}, Hx_{n+1}), \\ \frac{1}{2}[d(Eu, Hx_{n+1}) + d(Fx_{n+1}, Gu)], \\ \frac{1}{4}\left[\frac{d(Eu,Fx_{n+1})+d(Eu,Gu)+d(Fx_{n+1},Gu)}{1+d(Eu,Fx_{n+1})\, d(Eu,Gu)\, d(Fx_{n+1},Gu)}\right], \\ \frac{3}{2}\left[\frac{d(Fx_{n+1},Hx_{n+1})+d(Fx_{n+1},Gu)+d(Fx_{n+1},Eu)}{1+d(Fx_{n+1},Hx_{n+1})\, d(Fx_{n+1},Gu)\, d(Fx_{n+1},Eu)}\right] \end{Bmatrix}$$

$$= max \begin{Bmatrix} d(z, Gx_n), d(z, Gu), d(Gx_n, Hx_{n+1}), \\ \frac{1}{2}[d(z, Hx_{n+1}) + d(Gx_n, Gu)], \\ \frac{1}{4}\left[\frac{d(z, Gx_n) + d(z, Gu) + d(Gx_n, Gu)}{1 + d(z, Gx_n)\, d(z, Gu)\, d(Gx_n, Gu)}\right], \\ \frac{3}{2}\left[\frac{d(Gx_n, Hx_{n+1}) + d(Gx_n, Gu) + d(Gx_n, z)}{1 + d(Gx_n, Hx_{n+1})\, d(Gx_n, Gu)\, d(Gx_n, z)}\right] \end{Bmatrix}$$

Taking the limit as $n \to \infty$ *yields*

$$\lambda(u, x_{n+1}) = max \begin{Bmatrix} d(z, z), d(z, Gu), d(z, z), \frac{1}{2}[d(z, z) + d(z, Gu)], \\ \frac{1}{4}\left[\frac{d(z, z) + d(z, Gu) + d(z, Gu)}{1 + d(z, z)\, d(z, Gu)\, d(z, Gu)}\right], \\ \frac{3}{2}\left[\frac{d(z, z) + d(z, Gu) + d(z, z)}{1 + d(z, z)\, d(z, Gu)\, d(z, z)}\right] \end{Bmatrix} = d(Gu, z)$$

Thus as $n \to \infty, d(Gu, z) \leq \phi(d(Gu, z)) + d(z, z) = \phi(d(Gu, z))$

If $Gu \neq z$ *then* $d(Gu, z) > 0$ *and hence as* ϕ *is contractive modulus* $\phi(d(Gu, z)) < d(Gu, z)$.
Therefore $d(Gu, z) < d(Gu, z)$ *which is a contradiction.*
Thus $Gu = z$ *, so* $Eu = Gu = z$.
So u *is a coincidence point of* E *and* G *.*
Since the pair of maps G *and* E *are weakly compatible ,*
$GEu = EGu$, *ie.* $Gz = Ez$ *.*
Again since $G(X) \subseteq F(X)$*, there exist a point* $v \in X$ *such that* $z = Fv$.
Then by (ii), we have

$$d(z, Hv) = d(Gv, Hv) \leq \phi\,(\lambda(u, v))$$

where, $$\lambda(u, v) = max \begin{Bmatrix} d(Eu, Fv), d(Eu, Gu), d(Fv, Hv), \frac{1}{2}[d(Eu, Hv) + d(Fv, Gu)], \\ \frac{1}{4}\left[\frac{d(Eu, Fv) + d(Eu, Gu) + d(Fv, Gu)}{1 + d(Eu, Fv)\, d(Eu, Gu)\, d(Fv, Gu)}\right], \\ \frac{3}{2}\left[\frac{d(Fv, Hv) + d(Fv, Gu) + d(Fv, Eu)}{1 + d(Fv, Hv)\, d(Fv, Gu)\, d(Fv, Eu)}\right] \end{Bmatrix}$$

$$= max \begin{Bmatrix} d(z,z), d(z,z), d(z,Hv), \frac{1}{2}[d(z,Hv) + d(z,z)], \\ \frac{1}{4}\left[\frac{d(z,z)+d(z,z)+d(z,z)}{1+d(z,z)\ d(z,z)\ d(z,z)}\right], \\ \frac{3}{2}\left[\frac{d(z,Hv)+d(z,z)+d(z,z)}{1+d(z,Hv)\ d(z,z)\ d(z,z)}\right] \end{Bmatrix}$$

$$= d(z,Hv)$$

$$Thus\ d(z,Hv) \le \ \phi\ d(z,Hv)$$

$$If\ Hv \ne z\ then\ d(z,Hv) > 0\ and\ hence\ as\ \phi\ is\ contractive\ \ modulus\ ,$$

$$\phi\ d(z,Hv) < d(z,Hv)\ , which\ is\ a\ contradiction.$$

Therefore $Hv = Fv = z$.

So v *is a coincidence point of* F *and* H.

Since the pair of maps F and H are weakly compatible,

$$FHv = HFv\ , i.e\ \ Fz = Hz.$$

Now we show that z is a fixed point of G.

$$Then\ by\ (ii), we\ have\ d(Gz,z) = d(Gz,Hv)$$

$$\le\ \phi\ \big(\lambda(z,v)\big)$$

Where,

$$\lambda(z,v) = max \begin{Bmatrix} d(Ez,Fv), d(Ez,Gz), d(Fv,Hv), \frac{1}{2}[d(Ez,Hv) + d(Fv,Gz)], \\ \frac{1}{4}\left[\frac{d(Ez,Fv) + d(Ez,Gz) + d(Fv,Gz)}{1 + d(Ez,Fv)\ d(Ez,Gz)\ d(Fv,Gz)}\right], \\ \frac{3}{2}\left[\frac{d(Fv,Hv) + d(Fv,Gz) + d(Fv,Ez)}{1 + d(Fv,Hv)\ d(Fv,Gz)\ d(Fv,Ez)}\right] \end{Bmatrix}$$

$$= max \begin{Bmatrix} d(Gz,z), d(Gz,Gz), d(z,z), \frac{1}{2}[d(Gz,z) + d(z,Gz)], \\ \frac{1}{4}\left[\frac{d(Gz,z)+d(Gz,Gz)+d(z,Gz)}{1+d(Gz,z)\ d(Gz,Gz)\ d(z,Gz)}\right], \\ \frac{3}{2}\left[\frac{d(z,z)+d(z,Gz)+d(z,Gz)}{1+d(z,z)\ d(z,Gz)\ d(z,Gz)}\right] \end{Bmatrix}$$

$$= d(Gz,z)$$

$$Thus\ d(Gz,z) \le\ \phi\ \big(d(Gz,z)\big)$$

$$If\ Gz \ne z\ then\ d(Gz,z) > 0\ and\ hence\ as\ \phi\ is\ contractive\ modulus$$

$$\phi\ \big(d(Gz,z)\big) < d(Gz,z).$$

Therefore $Gz = z$

Hence $Gz = Ez = z$

By (ii), we have,

$d(z,Hz) = d(Ez,Hz)$

$\le \phi\big(\lambda(z,z)\big)$

Where,

$$\lambda(z,z) = max \begin{Bmatrix} d(Ez,Fz), d(Ez,Gz), d(Fz,Hz), \frac{1}{2}[d(Ez,Hz) + d(Fz,Gz)], \\ \frac{1}{4}\left[\frac{d(Ez,Fz) + d(Ez,Gz) + d(Fz,Gz)}{1 + d(Ez,Fz)\ d(Ez,Gz)\ d(Fz,Gz)}\right], \\ \frac{3}{2}\left[\frac{d(Fz,Hz) + d(Fz,Gz) + d(Fz,Ez)}{1 + d(Fz,Hz)\ d(Fz,Gz)\ d(Fz,Ez)}\right] \end{Bmatrix}$$

$$= max \begin{Bmatrix} d(z,Hz), d(z,z), d(Hz,Hz), \frac{1}{2}[d(z,Hz) + d(Hz,z)], \\ \frac{1}{4}\left[\frac{d(z,Hz)+d(z,z)+d(Hz,z)}{1+d(z,Hz)\ d(z,z)\ d(Hz,z)}\right], \\ \frac{3}{2}\left[\frac{d(Hz,Hz)+d(Hz,z)+d(Hz,z)}{1+d(Hz,Hz)\ d(Hz,z)\ d(Hz,z)}\right] \end{Bmatrix}$$

$$= d(z,Hz)$$

$$Thus\ d(z,Hz) \le\ \phi\ \big(d(z,Hz)\big)$$

If $z \ne Hz\ then\ d(z,Hz) > 0\ and\ hence\ as\ \phi\ is\ acontractive\ modulus\ ,$

$$\phi\ \big(d(z,Hz)\big) < d(z,Hz)$$

Therefore $d(z,Hz) < d(z,Hz)\ , which\ is\ a\ contradiction.$

Hence$z = Hz$

Therefore $Hz = Fz = z$.
Therefore $Gz = Ez = Hz = Fz = z$, $ie. z$ $is\ a\ common\ fixed\ point\ of\ E, F, G\ and\ H$.
Uniqueness : For uniqueness, Let we assume that $z\ and\ w$, $(z \neq w)$ $are\ common\ fixed\ point\ of\ E, F, G\ and\ H$.
By (ii), we have,

$$d(z,w) = d(Gz, Hw)$$

$$\leq \phi(\lambda(z,w))$$

Where,

$$(z,w) = max\begin{Bmatrix} d(Ez,Fw), d(Ez,Gz), d(Fw,Hw), \frac{1}{2}[d(Ez,Hw) + d(Fw,Gz)], \\ \frac{1}{4}\left[\frac{d(Ez,Fw)+d(Ez,Gz)+d(Fw,Gz)}{1+d(Ez,Fw)\ d(Ez,Gz)\ d(Fw,Gz)}\right], \\ \frac{3}{2}\left[\frac{d(Fw,Hw)+d(Fw,Gz)+d(Fw,Ez)}{1+d(Fw,Hw)\ d(Fw,Gz)\ d(Fw,Ez)}\right] \end{Bmatrix}$$

$$= max\begin{Bmatrix} d(z,w), d(z,z), d(w,w), \frac{1}{2}[d(z,w) + d(w,z)], \\ \frac{1}{4}\left[\frac{d(z,w)+d(z,z)+d(w,z)}{1+d(z,w)\ d(z,z)\ d(w,z)}\right], \\ \frac{3}{2}\left[\frac{d(w,w)+d(w,z)+d(w,z)}{1+d(w,w)\ d(w,z)\ d(w,z)}\right] \end{Bmatrix}$$

$$= d(z,w)$$

Thus $d(z,w) \leq \phi\left(d(z,w)\right)$
Since $z \neq w\ then\ d(z,w) > 0\ and\ hence\ as\ \phi\ is\ a\ contractive\ modulus$,

$$\phi\left(d(z,w)\right) < d(z,w)$$

$$d(z,w) < d(z,w)\ which\ is\ a\ contradiction.$$

Therefore $z = w$
Thus z is the unique common fixed point of E, F, G & H .
Hence the theorem.

Corollary 1 : Let (X, d) $be\ a\ complete\ metric\ space\ suppose\ that\ the\ mappings\ E, G\ and\ H\ are\ self\ mapsof\ X, Satisfying\ the\ following\ conditions$

$$(i)\ H(X) \subseteq E(X)\ and\ G(X) \subseteq E(X)$$

$$(ii)\ d(Gx, Hy) \leq \phi(\lambda(x,y))$$

Where ϕ $is\ an\ upper\ semi\ continuous, contractive\ modulus\ and$

$$\lambda(x,y) = max\begin{Bmatrix} d(E_x,E_y), d(E_x,G_x), d(E_y,H_y), \frac{1}{2}[d(E_x,H_y) + d(E_y,G_x)], \\ \frac{1}{4}\left\{\frac{d(E_x,E_y)+d(E_x,G_x)+d(E_y,G_x)}{1+d(E_x,E_y)d(E_x,G_x)d(E_y,G_x)}\right\}, \\ \frac{3}{2}\left\{\frac{d(E_y,H_y)+d(E_y,G_x)+d(E_y,E_x)}{1+d(E_y,H_y)d(E_y,G_x)d(E_y,E_x)}\right\} \end{Bmatrix}$$

$$(iii)\ The\ pair\ (G,E)\ and\ (H,E)\ are\ weakly\ compatible$$

Then E, G and H have a unique common fixed point.
Proof : By taking E = F in main theorem we get the proof.

Corollary 2 : Let (X,d) be a complete metric space, suppose that the mappings E and G are self maps of X, satisfying the following conditions :
$(i)\ G(X) \subseteq E(X)$

$$(ii)\ d(Gx, Gy) \leq \phi(\lambda(x,y))$$

Where ϕ $is\ an\ upper\ semi\ continuous, contractive\ modulus\ and$

$$\lambda(x,y) = max\begin{Bmatrix} d(E_x,E_y), d(E_x,G_x), d(E_y,G_y), \frac{1}{2}[d(E_x,G_y) + d(E_y,G_x)], \\ \frac{1}{4}\left\{\frac{d(E_x,E_y) + d(E_x,G_x) + d(E_y,G_x)}{1 + d(E_x,E_y)d(E_x,G_x)d(E_y,G_x)}\right\}, \\ \frac{3}{2}\left\{\frac{d(E_y,G_y) + d(E_y,G_x) + d(E_y,E_x)}{1 + d(E_y,G_y)d(E_y,G_x)d(E_y,E_x)}\right\} \end{Bmatrix}$$

$$(iii)\ The\ pair\ (G,E)\ are\ weakly\ compatible$$

Then E and G have a unique common fixed point.
Proof : By taking E = F and G = H in theorem 3.1 we get the proof.

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