

Generalized Fixed Point Theorems for Compatible Mapping in Fuzzy 3-Metric Space for Integral Type Mapping

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Abstract

In this paper, we give some new definitions of compatible mappings of types (I) and (II) in fuzzy 3- metric space and prove some common fixed point theorems for four mappings under the condition of compatible mappings of types (I) and (II) in complete fuzzy 3- metric space. Our results extend, generalize and improve the corresponding results given by many authors.

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1. Introduction and Preliminaries

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [37] in 1922. In the general setting of complete metric space, this theorem runs as the follows, **Theorem 1.1**(Banach's contraction principle) Let (X, d) be a complete metric space, $c \in (0, 1)$ and $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$, $d(fx, fy) \leq c d(x, y)$ Then f has a unique fixed point $a \in X$, such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$. After the classical result, R.Kannan [46] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions. In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X, d) satisfying a general contractive condition of integral type.

Theorem 1.2(Branciari) Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f: X \rightarrow X$ be a mapping such that for each $x, y \in X$, $\int_0^{d(fx, fy)} \varphi(t) dt \leq c \int_0^{d(x, y)} \varphi(t) dt$. Where $\varphi: [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$, non negative, and such that for each $\varepsilon > 0$, $\int_0^\varepsilon \varphi(t) dt > 0$, then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$. After the paper of Branciari, a lot of a research works have been carried out on generalizing contractive conditions of integral type for a different contractive mapping satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx, fy)} \varphi(t) dt \leq \int_0^{\max\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\}} \varphi(t) dt$$
 (1.3) The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings, satisfying a general contractive mappings such as R. Kannan type [46], S.K. Chatrterjee type [38], T. Zamfirescu type [48], etc.

The concept of fuzzy sets was introduced initially by Zadeh [49] in 1965. Since then, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. For example, Deng [12], Ereeg [13], Fang [18], George [19], Kaleva and Seikkala [27], Kramosil and Michalek [28] have introduced the concept of fuzzy metric spaces in different ways. In fuzzy metric spaces given by Kramosil and Michalek [28], Grabiec [20] obtained the fuzzy version of Banach's contraction principle, which has been improved, generalized and extended by some authors. Sessa [34] defined a generalization of commutativity introduced by Jungck [24], which is called the weak commutativity. Further, Jungck [25] introduced more generalized commutativity, so called compatibility. Mishra et al. [29] obtained some common fixed point theorems for compatible mappings in fuzzy metric spaces. Recently, Jungck et al. [26] introduced the concept of compatible mappings of type (A) in metric spaces, which is equivalent to the concept of compatible mappings under some conditions, and proved common fixed point theorems in metric spaces. Cho [09] introduced the concept of compatible mappings of type (a) in fuzzy metric spaces. Again, Pathak et al. [32] introduced the concept of compatible mappings of type (B) in metric spaces and Cho et al. [10] introduced the concept of compatible mappings of type (b) in fuzzy metric spaces. Pathak et al. [33] introduced the concept of compatible mappings of type (I) and (II) in metric spaces. On the other hand, George and Veeramani [19], Kramosil and Michalek [28] introduced the concept of fuzzy topological spaces induced by fuzzy metrics which have very

important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [14–17,49]. Many authors [21,30,39,40,41] have also proved some fixed point theorems in fuzzy (probabilistic) metric spaces (see [2–7,18,20,22,23,36]).

Definition 1.1: A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t-norm if it satisfies the following conditions:

- (1) $*$ is associative and commutative,
 - (2) $*$ is continuous,
 - (3) $a * 1 = a$ for all $a \in [0, 1]$,
 - (4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$,
- Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2: A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (Non-empty) set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,

- (1) $M(x, y, t) > 0$.
- (2) $M(x, y, t) = 1$ if and only if $x = y$,
- (3) $M(x, y, t) = M(y, x, t)$.
- (4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$.
- (5) $M(x, y, \cdot) : (0, \infty) \rightarrow [0,1]$ is continuous.

Let $M(x, y, t)$ be a fuzzy metric space. For any $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}$. Let $(X, M, *)$ be a fuzzy metric space. Let s be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then s is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \rightarrow 1$ as $n \rightarrow \infty$ for all $t > 0$. It is called a Cauchy sequence if, for any $0 < \epsilon < 1$ and $t > 0$, there exists

$n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for any $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1.3 [11]: Let $X = \mathbb{R}$ and denote $a * b = ab$ for all $a, b \in [0, 1]$. For any $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|} \text{ for all } x, y \in X. \text{ Then } M \text{ is a fuzzy metric in } X.$$

Lemma 1.4 [11]: Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t for all x, y in X .

Definition 1.5: Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \rightarrow \infty} M(x_n, x_n, t_n) = M(x, y, t)$$

Whenever a sequence $\{(x_n, x_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$,
 i.e. $\lim_{n \rightarrow \infty} M(x_n, x, t) = \lim_{n \rightarrow \infty} M(y_n, y, t) = 1$ and

$$\lim_{n \rightarrow \infty} M(x, y, t_n) = M(x, y, t)$$

Lemma 1.6: Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.

Proof: See Proposition 1 of [36].

Lemma 1.7 [11]: Let $(X, M, *)$ be a fuzzy metric space. If we define $E_{\lambda, M}: X^2 \rightarrow R^+ \cup \{0\}$

$$E_{\lambda, M} = \inf\{t > 0: M(x, y, t) > 1 - \lambda\} \text{ for all } \lambda \in (0,1) \text{ and } x, y \in X. \text{ Then we have}$$

(1) For all $\mu \in (0,1)$, there exists $\lambda \in (0,1)$ such that

$$E_{\mu, M}(x_1, x_n) \leq E_{\lambda, M}(x_1, x_2) + E_{\lambda, M}(x_2, x_3) + \dots + E_{\lambda, M}(x_{n-1}, x_n), \text{ for all } x_1, x_2, x_3, \dots, x_n \in X.$$

(2) The sequence $\{x_n\}_{n \in \mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda, M}(x_n, x) \rightarrow 0$. Also, the

Sequence $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence if and only if it is a Cauchy sequence with $E_{\lambda, M}$.

Lemma 1.8 [11]: Let $(X, M, *)$ be a fuzzy metric space. If a sequence $\{x_n\}$ in X is such that, for any $n \in \mathbb{N}$, $M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$ for all $k > 1$, then the sequence $\{x_n\}$ is a Cauchy sequence.

2. Some definitions of compatible mappings

In this section, we give some definitions of compatible mappings with types, some properties and examples of compatible mappings in fuzzy metric spaces.

Definition 2.1: Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 2.2 [29]: Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if, for all $t > 0$,

$$\lim_{n \rightarrow \infty} M(ASx_n, SAx_n, t) = 1$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

Proposition 2.3 [40]: If the self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible, then they are weak compatible.

The converse is not true as seen in following example.

Example 2.4 [11]: Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$ with t-norm defined $a * b = \min \{a, b\}$, for all $a, b \in [0,1]$ and $M(x, y, t) = \frac{t}{t + |x-y|}$, for all $x, y \in X$. Define the self-mappings A and S on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } 1 \leq x \leq 2, \end{cases} \quad Sx = \begin{cases} 2 & \text{if } x = 1, \\ \frac{x+3}{5} & \text{if otherwise} \end{cases}$$

Then we have $S1 = A1 = 2$ and $S2 = A2 = 1$. Also $SA1 = AS1 = 1$ and $SA2 = AS2 = 2$. Thus (A, S) is weak compatible. Again,

$$Ax_n = 1 - \frac{1}{4n}, \quad Sx_n = 1 - \frac{1}{10n},$$

Thus we have $Ax_n \rightarrow 1, Sx_n \rightarrow 1$. Further, it follows that $SAx_n = \frac{4}{5} - \frac{1}{20n}, ASx_n = 2$.

Therefore, we have

$$\lim_{n \rightarrow \infty} M(ASx_n, SAx_n, t) = \lim_{n \rightarrow \infty} M\left(2, \frac{4}{5} - \frac{1}{20n}, t\right) = \frac{t}{t + \frac{6}{5}} < 1 \quad \text{for all } t > 0.$$

Hence (A, S) is not compatible.

Definition 2.5 [9]: Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings A and S are said to be compatible of type (α) if, for all $t > 0$,

$$\lim_{n \rightarrow \infty} M(ASx_n, SSx_n, t) = 1, \quad \lim_{n \rightarrow \infty} M(SAx_n, AAx_n, t) = 1$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

Definition 2.6 [10]: Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. The mappings A and S are said to be compatible of type (β) if, for all $t > 0$,

$$\lim_{n \rightarrow \infty} M(AAx_n, SSx_n, t) = 1$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

Proposition 2.7 [09]: Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$ and A, S be continuous mappings from X into itself. Then A and S are compatible if and only if they are compatible of type (α) .

Proposition 2.8 [10]: Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$ and A, S be continuous mappings from X into itself. Then A and S are compatible if and only if they are compatible of type (β) .

Proposition 2.9 [10]: Let $(X, M, *)$ be a fuzzy metric space with $t * t = t$ for all $t \in [0,1]$ and A, S be continuous mappings from X into itself. Then A and S are compatible of type (α) if and only if they are compatible of type (β) .

Now, Turkoglu et al. [45] introduced the following definitions:

Definition 2.10: Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the pair (A, S) is called A-compatible if, for all $t > 0$,

$$\lim_{n \rightarrow \infty} M(ASx_n, SSx_n, t) = 1$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

Definition 2.11: Let A and S be mappings from a fuzzy metric space (X, M,*) into itself. Then the pair (A, S) is called S-compatible if and only if (S, A) is S-compatible.

Definition 2.12: Let A and S be mappings from a fuzzy metric space (X, M,*) into itself. Then the pair (A, S) is said to be compatible of type (I) if, for all t > 0,

$$\lim_{n \rightarrow \infty} M(ASx_n, x, t) \leq M(Sx, x, t)$$

Whenever {x_n} is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

Remark 2.13: in [47], the above Definition was introduced as follows:

$$\lim_{n \rightarrow \infty} M(ASx_n, x, \lambda t) \leq M(Sx, x, t)$$

Whenever $\lambda \in (0,1]$ and {x_n} is a sequence in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = x \in X.$$

If $\lambda = 1$, then, by Lemma 1.4, it is easily verified that, for all $\lambda \in (0,1]$ by using above Definition, we have Definition of [48].

Definition 2.14: Let A and S be mappings from a fuzzy metric space (X, M,*) into itself. Then the pair (A, S) is said to be compatible of type (II) if and only if (S, A) is compatible of type (I).

Remark 2.15: In [25, 26, 33, 34], we can find the equivalent formulations of above Definitions and their examples in metric spaces (see [25, 26, 33, 34]). Such mappings are independent of each other and more general than commuting and weakly commuting mappings (see [25, 35]).

Proposition 2.16: Let A and S be mappings from a fuzzy metric space (X, M,*) into itself. Suppose that the pair (A, S) is compatible of type (I) (respectively, (II)) and Az = Sz for some z ∈ X. Then, for all t > 0,

$$M(Az, SSz, t) \geq M(Az, ASz, t) \text{ (Respectively, } M(Sz, AAz, t) \geq M(Sz, SAz, t) \text{)}.$$

Proof: See Proposition 7 of [47].

Definition 2.17[31]: The 3- tuple (X, M,*) is called a fuzzy-3 metric space if X is an arbitrary set, * is a continuous

t-norm and M is a fuzzy set in $X^4 \times [0, \infty)$ satisfying the following conditions, for all x, y, z, w, u ∈ X and t₁, t₂, t₃, t₄ > 0.

- (1) M(x, y, z, w, 0) = 0,
- (2) M(x, y, z, w, t) = 1 for all t > 0, [only when the three simplex (x, y, z, w) degenerate]
- (3) M(x, y, z, t) = M(x, w, z, y, t) = M(y, z, w, x, t) = M(z, w, x, y, t) =
- (4) $M(x, y, z, w, t_1 + t_2 + t_3 + t_4) \geq M(x, y, z, w, t_1) * M(x, y, z, w, t_2) * M(x, y, z, w, t_3) * M(x, y, z, w, t_4)$
- (5) M(x, y, z, w): [0, 1] → [0, 1] is left continuous.

Definition 2.18[31]: Let (X, M,*) be a fuzzy 3-metric space, then

- (1) A sequence {x_n} in fuzzy 3-metric space X is said to be convergent to a point x ∈ X if

$$\lim_{n \rightarrow \infty} M(x_n, x, a, b, t) = 1 \text{ for all } a, b \in X \text{ and } t > 0,$$

- (2) A sequence {x_n} in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n \rightarrow \infty} M(x_{n+p}, x_n, a, b, t) = 1 \text{ for all } a, b \in X \text{ and } t > 0, p > 0.$$

- (3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 2.19[31]: A function M is continuous in fuzzy 3-metric space iff whenever x_n → x, y_n → y, then

$$\lim_{n \rightarrow \infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t) \text{ for all } a, b \in X \text{ and } t > 0.$$

Definition 2.20[31]: Two mappings A and S on fuzzy 3-metric space are weakly commuting iff M(ASu, SAu, a, b, t) ≥ M(Au, Su, a, b, t) for all u, a, b ∈ X and t > 0.

3. The main results

In this section, we prove a fixed point theorem for four mappings under the condition of compatible mappings of types (I) and (II).

Let φ be the set of all continuous and increasing functions φ: [0,1]⁷ → [0,1] in any coordinate and φ(t, t, t, t, t, t, t) > t for all t ∈ [0,1).

Theorem 3.1: Let (X, M,*) be a complete fuzzy-3 metric space with t * t = t for all t ∈ [0,1]. Let A, B, S and T be mappings from X into itself such that

- (i) A(X) ⊆ T(X), B(X) ⊆ S(X),

$$\int_0^{M(A^2x, B^2y, a, b, t)} \zeta(t) dt \geq \int_0^{\phi \left\{ \min \left\{ \begin{matrix} M(S^2x, T^2y, a, b, t), M(S^2x, A^2x, a, b, t), M(S^2x, B^2y, a, b, t), \\ M(T^2y, B^2y, a, b, t), M(A^2x, T^2y, a, b, t), M(S^2y, B^2y, a, b, t), \\ M(A^2x, B^2y, a, b, t) \end{matrix} \right\} \right\}} \zeta(t) dt$$

for all $x, y, a \in X, t > 0$ and $\phi \in \phi$.

If the mappings A, B, S and T satisfy any one of the following conditions:

(ii) The pairs (A, S) and (B, T) are weakly commuting of type (II) and A or B is continuous,

(iii) The pairs (A, S) and (B, T) are weakly commuting of type (I) and S or T is continuous, then A, B, S and T have a unique common fixed point in X.

Proof: Let $x_0 \in X$ be an arbitrary point. Since $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively, construct the sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{2n} = A^2x_{2n} = S^2x_{2n+1}$ and $y_{2n+1} = A^2x_{2n} = S^2x_{2n+1}$ and $y_{2n+1} = B^2x_{2n+1} = T^2x_{2n+2}$ for $n = 1, 2, 3, \dots$. Now we shall prove that $\{y_n\}$ is a Cauchy Sequence. Let $d_m(t) = M(y_m, y_{m+1}, a, b, t) < 1$ for $t > 0$, then we prove that $\{d_m(t)\}$ is increasing with respect to m. setting $m = 2n$, then we have

$$\begin{aligned} \int_0^{d_{2n}(t)} \zeta(t) dt &= \int_0^{M(y_{2n}, y_{2n+1}, a, t)} \zeta(t) dt \\ &= \int_0^{M(A^2x_{2n}, B^2x_{2n+1}, a, t)} \zeta(t) dt \\ &\geq \int_0^{\phi \left\{ \min \left\{ \begin{matrix} M(S^2x_{2n+1}, T^2x_{2n}, a, b, t), M(S^2x_{2n+1}, B^2x_{2n+1}, a, b, t), \\ M(S^2x_{2n+1}, A^2x_{2n}, a, b, t), M(T^2x_{2n}, A^2x_{2n}, a, b, t), \\ M(B^2x_{2n+1}, T^2x_{2n}, a, b, t), M(S^2x_{2n}, A^2x_{2n}, a, b, t), \\ M(A^2x_{2n}, B^2x_{2n+1}, a, b, t) \end{matrix} \right\} \right\}} \zeta(t) dt \\ &= \int_0^{\phi \left\{ \min \left\{ \begin{matrix} M(y_{2n}, y_{2n-1}, a, b, t), M(y_{2n}, y_{2n+1}, a, b, t), \\ M(y_{2n}, y_{2n}, a, b, t), M(y_{2n-1}, y_{2n}, a, b, t), \\ M(y_{2n+1}, y_{2n-1}, a, b, t), M(y_{2n-1}, y_{2n}, a, b, t) \end{matrix} \right\} \right\}} \zeta(t) dt \\ &= \int_0^{\phi \left\{ \min \{d_{2n-1}(t), d_{2n-1}(t), d_{2n}(t), 1, d_{2n-1}(t), d_{2n-1}(t), 1\} \right\}} \zeta(t) dt. \end{aligned}$$

$$\text{If } \int_0^{d_{2n-1}(t)} \zeta(t) dt \geq \int_0^{d_{2n}(t)} \zeta(t) dt \text{ then } \int_0^{d_{2n}(t)} \zeta(t) dt \geq \int_0^{\phi(d_{2n-1}(t))} \zeta(t) dt > \int_0^{d_{2n-1}(t)} \zeta(t) dt,$$

$$\text{a contradiction therefore } \int_0^{d_{2n}(t)} \zeta(t) dt \geq \int_0^{d_{2n-1}(t)} \zeta(t) dt \text{ for all } n \in \mathbb{N} \text{ and } t > 0.$$

$\{d_m(t)\}$ is an increasing in $[0, 1]$.

For any integer m,

$$\int_0^{M(y_n, y_{n+m}, a, b, t)} \zeta(t) dt \geq \int_0^{M(y_n, y_{n+1}, a, b, \frac{t}{m})} \zeta(t) dt \geq \dots \geq \int_0^{M(y_0, y_1, a, b, \frac{t}{m})} \zeta(t) dt.$$

Hence $\{y_n\}$ is a Cauchy sequence and, by the completeness of X, $\{y_n\}$ Converges to a point u in X. So its

Subsequences $\{B^2x_{2n+1}\}, \{T^2x_{2n}\}, \{S^2x_{2n+1}\}$ also converges to same point u .

Since (A, S) and (B, T) are weak commuting so

$$\int_0^M(A^2S^2x_{2n+1}, S^2A^2x_{2n+1}, a, b, t) \zeta(t) dt \geq \int_0^M(A^2x_{2n+1}, S^2x_{2n+1}, a, b, t) \zeta(t) dt$$

and

$$\int_0^M(B^2T^2x_{2n+1}, T^2B^2x_{2n+1}, a, b, t) \zeta(t) dt \geq \int_0^M(B^2x_{2n+1}, T^2x_{2n+1}, a, b, t) \zeta(t) dt$$

On taking limit as $\rightarrow \infty, A^2S^2x_{2n+1} = S^2A^2x_{2n+1} = S^2u$ & $B^2T^2x_{2n+1} = T^2B^2x_{2n+1} = T^2u$ now we will prove that $S^2u = u$ & $T^2u = u$.

First suppose that $S^2u \neq u$ & $T^2u \neq u$ then there exist $t > 0$ such that

$$M(S^2u, u, a, b, t) < 1, M(T^2u, u, a, b, t) < 1.$$

Now

$$\int_0^M(A^2S^2x_{2n+1}, A^2x_{2n}, a, b, t) \zeta(t) dt \geq \int_0^{\phi \left\{ \min \left[\begin{array}{l} M(S^2x_{2n+1}, T^2x_{2n}, a, b, t), M(S^2x_{2n+1}, A^2S^2x_{2n+1}, a, b, t), \\ M(S^2x_{2n+1}, A^2x_{2n}, a, b, t), M(T^2x_{2n}, A^2x_{2n}, a, b, t), \\ M(A^2S^2x_{2n+1}, T^2x_{2n}, a, b, t), M(S^2x_{2n}, A^2x_{2n}, a, b, t), \\ M(A^2x_{2n}, A^2S^2x_{2n+1}, a, b, t) \end{array} \right] \right\}} \zeta(t) dt$$

Therefore

$$\int_0^M(S^2u, u, a, b, t) \zeta(t) dt \geq \int_0^{\phi \left\{ \min \left[\begin{array}{l} M(S^2u, u, a, b, t), M(S^2u, S^2u, a, b, t), M(S^2u, u, a, b, t), \\ M(u, u, a, b, t), M(S^2u, u, a, b, t), M(S^2u, u, a, b, t), \\ M(u, S^2u, a, b, t) \end{array} \right] \right\}} \zeta(t) dt$$

This implies that

$$\int_0^M(S^2u, u, a, b, t) \zeta(t) dt \geq \int_0^{\phi [M(S^2u, u, a, b, t)]} \zeta(t) dt > \int_0^M(S^2u, u, a, b, t) \zeta(t) dt$$

Similarly we can show that u is also a fixed point of A .

$$\int_0^M(A^2u, B^2T^2x_{2n}, a, b, t) \zeta(t) dt \geq \int_0^{\phi \left\{ \min \left[\begin{array}{l} M(S^2u, T^2x_{2n}, a, b, t), M(S^2u, B^2u, a, b, t), \\ M(S^2u, B^2T^2x_{2n}, a, b, t), M(T^2x_{2n}, B^2T^2x_{2n}, a, b, t), \\ M(B^2u, T^2x_{2n}, a, b, t), M(S^2T^2x_{2n}, B^2T^2x_{2n}, a, b, t), \\ M(B^2T^2x_{2n}, B^2u, a, b, t) \end{array} \right] \right\}} \zeta(t) dt$$

Therefore

$$\int_0^M(u, T^2u, a, b, t) \zeta(t) dt \geq \int_0 \phi \left\{ \min \left\{ \begin{matrix} M(u, T^2u, a, b, t), M(u, u, a, b, t), M(u, T^2u, a, b, t), \\ M(T^2u, T^2u, a, b, t), M(u, T^2u, a, b, t), M(u, T^2u, a, b, t), \\ M(u, T^2u, a, b, t) \end{matrix} \right\} \right\} \zeta(t) dt$$

This implies that

$$\int_0^M(u, T^2u, a, b, t) \zeta(t) dt \geq \int_0 \phi [M(u, T^2u, a, b, t)] \zeta(t) dt > \int_0^M(u, T^2u, a, b, t) \zeta(t) dt$$

Which is contradiction, therefore $S^2u = u$ & $T^2u = u$. Thus u is a fixed point of S and T.

I.e. u is a common fixed point of T, S, A and B.

Uniqueness: Suppose there is another fixed point $v \neq u$ then

$$\int_0^M(A^2u, B^2v, a, b, t) \zeta(t) dt \geq \int_0 \phi \left\{ \min \left\{ \begin{matrix} M(S^2u, T^2v, a, b, t), M(S^2u, A^2u, a, b, t), \\ M(S^2u, B^2v, a, b, t), M(T^2v, B^2v, a, b, t), \\ M(A^2u, T^2v, a, b, t), M(S^2v, B^2v, a, b, t), \\ M(B^2v, A^2u, a, b, t) \end{matrix} \right\} \right\} \zeta(t) dt$$

This implies that

$$\int_0^M(u, v, a, b, t) \zeta(t) dt \geq \int_0 \phi \left\{ \min \left\{ \begin{matrix} M(u, v, a, b, t), M(u, u, a, b, t), M(u, v, a, b, t), \\ M(v, v, a, b, t), M(u, v, a, b, t), M(v, v, a, b, t), \\ M(v, u, a, b, t) \end{matrix} \right\} \right\} \zeta(t) dt$$

Therefore

$$\int_0^M(u, v, a, b, t) \zeta(t) dt \geq \int_0 \phi [M(u, v, a, b, t)] \zeta(t) dt > \int_0^M(u, v, a, b, t) \zeta(t) dt$$

A contradiction so $v = u$. Hence A, S, B and T have unique common fixed point.

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