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Fractional Integration and Fractional Differentiation of the Product of M-Series and H-Function

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ABSTRACT
In this paper, we have derived formulae for the Riemann-Liouville fractional integral and fractional derivative of the product of the Manoj Sharma's M-series and the Fox H-function. Also the fractional integrals defined by Saxena and Kumbhat of the M-series is found with the help of integral of H-function. The M- series is a particular case of the $\bar{H}$-function of Inayat-Hussain. Certain special cases of the formulae have also been discussed.
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1. INTRODUCTION

The purpose of this paper is to establish theorems on the fractional integrals and fractional derivatives of the product of M-series and H-function. The theorems derived in this paper provide an extension of the work [6].

The Riemann-Liouville Fractional Integral of order α [3] is defined and represented as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, x > a \qquad (1.1)$$

where $\alpha \in C$, $R(\alpha) > 0, f(x) \in L(a,b)$ which is the Space of Lebesgue measurable function.

The Riemann-Liouville Fractional differential of order $\alpha \in C$ [3] is defined and represented as

$$(D_{a+}^{\alpha}\varphi)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x \frac{\varphi(t)dt}{(x-t)^{\alpha-n+1}}, \ n = [R(\alpha)] + 1; x > a \qquad (1.2)$$

Where $[R(\alpha)]$ means the integral part of $R(\alpha) \geq 0$.

Various definitions of fractional integration have been given from time to time by many authors, viz. Kober (1940), Erdélyi (1950-51), Saxena (1967), Kalla (1969) and many others

The fractional integral operator involving the H-function have been defined and denoted by Saxena and Khumbat [7] in the following manner:

For $r = 1$

$$R_x^{\eta,\alpha}[f(x)] = x^{-\eta-\alpha-1} \int_0^x t^{\eta}(x-t)^{\alpha} f(t) H_{P,Q}^{M,N}\left[k\left(\frac{t}{x}\right)^m \left(1-\frac{t}{x}\right)^n \middle| \begin{matrix} (e_j,\rho_j)_{1,P} \\ (f_j,\omega_j)_{1,Q} \end{matrix}\right] dt \qquad (1.3)$$

$$K_x^{\delta,\alpha}[f(x)] = x^{\delta} \int_x^{\infty} t^{-\delta-\alpha-1}(t-x)^{\alpha} f(t) H_{P,Q}^{M,N}\left[k\left(\frac{x}{t}\right)^m \left(1-\frac{x}{t}\right)^n \middle| \begin{matrix} (e_j,\rho_j)_{1,P} \\ (f_j,\omega_j)_{1,Q} \end{matrix}\right] dt \qquad (1.4)$$

The conditions of validity of these operators are as follows:

(i) $1 \leq P, Q < \infty, P^{-1} + Q^{-1} = 1$

(ii) $Re(\eta) + m \min_{1\leq j\leq M}\left[Re\left(f_j/\omega_j\right)\right] > -Q^{-1}$

(iii) $Re(\alpha) + n \min_{1\leq j\leq M}\left[Re\left(f_j/\omega_j\right)\right] > -Q^{-1}$

(iv) $Re(\delta+\alpha) + m \min_{1\leq j\leq M}\left[Re\left(f_j/\omega_j\right)\right] > -P^{-1}$

(v) $f(x) \in L_p(0,\infty)$.

Under these conditions $R_x^{\eta,\alpha}[f(x)]$ and $K_x^{\delta,\alpha}[f(x)]$ exist and both belong to $L_p(0,\infty)$.

2. DEFINITIONS

FOX'S H-FUNCTION:

The H-function, defined by Fox[1], in terms of Mellin-Barnes type contour integral as follows:

$$H_{P,Q}^{M,N}\left[x\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_L \theta(s)x^s\,ds \tag{2.1}$$

Where

$$\theta(s) = \frac{\prod_{j=1}^{M}\Gamma(f_j-\omega_j s)\prod_{j=1}^{N}\Gamma(1-e_j+\rho_j s)}{\prod_{j=M+1}^{Q}\Gamma(1-f_j+\omega_j s)\prod_{j=N+1}^{P}\Gamma(e_j-\rho_j s)} \tag{2.2}$$

$x \neq 0$, and an empty product is interpreted as unity. The integers M,N,P,Q are such that $0 \leq N \leq P, 0 \leq M \leq Q$; the coefficients $\rho_j (j = 1, \ldots, P)$, $\omega_j (j = 1, \ldots, Q)$ are all positive; $e_j (j = 1, \ldots, P), f_j (j = 1, \ldots, Q)$ are all complex numbers. L is a suitably chosen contour such that all the poles of $\theta(s)$ are simple.

Braksma has shown that the integral in the right hand side of (2.1) is absolutely convergent when $A > 0, |\arg z| < \frac{1}{2}A\pi$, where

$$A = \sum_{j=1}^{N}\rho_j - \sum_{j=N+1}^{P}\rho_j + \sum_{j=1}^{M}\omega_j - \sum_{j=M+1}^{Q}\omega_j \tag{2.3}$$

THE M-SERIES:

This series is a special case of the $\overline{H}$-function of Inayat-Hussain. The Manoj sharma's M-series [6] is interesting because the ${}_pF_q$-hypergeometric function and the Mittag-Leffler function follows as its particular cases, and these functions have found essential applications in solving problems in physics, biology, engineering and applied sciences.

It is denoted and defined as:

$${}_pM_q^{\upsilon}(x) = {}_pM_q^{\upsilon}(a_1,\ldots,a_p;b_1,\ldots,b_q;x) = \sum_{k=0}^{\infty}\frac{(a_1)_k\ldots(a_p)_k}{(b_1)_k\ldots(b_p)_k}\frac{x^k}{\Gamma(\upsilon k+1)} \tag{2.4}$$

Here, $\upsilon \in C, R(\upsilon) > 0$ and $(a_j)_k, (b_j)_k$ are the Pochammer symbols. The series (2.3) is defined when none of the parameters $b_j s, j = 1,2, \ldots q$, is a negative integer or zero. If any numerator parameter is a negative integer or zero, then the series terminatesto a polynomial in x.

3. MATHEMATICAL PREREQUISITES

The following results are needed to establish the theorems:

The Beta function is defined as:

$$\int_0^1 u^{m-1}(1-u)^{n-1}\,du = B(m,n) \tag{3.1}$$

The modified Beta function is as follows:

$$\int_a^b (t-a)^{m-1}(b-t)^{n-1}dt = (b-a)^{m+n-1}B(m,n), \text{ for } R(m) > 0, R(n) > 0 \tag{3.2}$$

The following integrals of the H-function [7] is also used:

$$\int_0^x t^{\eta-1}(x-t)^{\sigma-1}H_{P,Q}^{M,N}\left[yt^{\mu}(x-t)^{\vartheta}\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]dt$$

$$= x^{\eta+\sigma-1}H_{P+2,Q+1}^{M,N+2}\left[yx^{\mu+\vartheta}\left|\begin{matrix}(1-\eta,\mu),(1-\sigma,\vartheta),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(1-\eta-\sigma,\mu+\vartheta)\end{matrix}\right.\right] \tag{3.3}$$

The conditions of validity of (3.3) are:

(i) $\mu \geq 0, \vartheta \geq 0$ (not both zero simultaneously), η, σ are complex numbers,

$(ii)\ Re(\eta)+\mu \min_{1\le j\le M}\left[Re\left({f_j}/{\omega_j}\right)\right]>0 \quad \& \quad Re(\sigma)+\vartheta \min_{1\le j\le M}\left[Re\left({f_j}/{\omega_j}\right)\right]>0.$

$$\int_x^\infty t^{\eta-1}(t-x)^{\sigma-1}H_{P,Q}^{M,N}\left[yt^\mu(t-x)^\vartheta \left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]dt$$
$$=x^{\eta+\sigma-1}H_{P+2,Q+1}^{M+1,N+1}\left[yx^{\mu+\vartheta}\left|\begin{matrix}(1-\sigma,\vartheta),(e_j,\rho_j)_{1,P},(1-\eta,\mu)\\(1-\eta-\sigma,\mu+\vartheta),(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right] \tag{3.4}$$

The conditions of validity of (3.4) are:

(i) $\mu\ge 0,\vartheta\ge 0$ (not both zero simultaneously), η,σ are complex numbers,

(ii) $min\left[Re\left(\frac{1-\eta-\sigma}{\mu+\vartheta}\right),\min_{1\le j\le M} Re\left({f_j}/{\omega_j}\right)\right]>max\left[-Re\left(\frac{\sigma}{\vartheta}\right),\max_{1\le j\le N}\left[Re\left(\frac{e_j-1}{\rho_j}\right)\right]\right].$

4. THEOREMS ON THE PRODUCT OF THE H-FUNCTION AND M-SERIES

The fractional Riemann-Liouville (R-L) integral operator (for lower limit $a=0$, with respect to variable x), of the product of the H-function and M-series:

Theorem1:

$$I_x^\alpha\left\{ {}_pM_q^{\,v}(x)H_{P,Q}^{M,N}[cx^\sigma]\right\}= x^\alpha\ {}_pM_q^{\,v}(x)H_{P+1,Q+1}^{M,N+1}\left[cx^\sigma\left|\begin{matrix}(-k,\sigma),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(-\alpha-k,\sigma)\end{matrix}\right.\right] \tag{4.1}$$

Here $v\epsilon C,R(v)>0,\alpha\epsilon C,R(\alpha)>0,\sigma>0$ and M, N, P and Q are non-negative integers satisfying the condition (2.3). Also the uniform convergence of the M-series is discussed above.

Proof . Expressing the H-function and the M-series with the help of (2.1) and (2.4) respectively, we get

$$I_x^\alpha\left\{ {}_pM_q^{\,v}(x)H_{P,Q}^{M,N}[cx^\sigma]\right\}=\frac{1}{\Gamma(\alpha)}\int_0^x(x-t)^{\alpha-1}\sum_{k=0}^\infty\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{t^k}{\Gamma(vk+1)}\frac{1}{2\pi i}\int_L c^s\,\theta(s)t^{\sigma s}ds dt$$

Then, using the term by term integration, we obtain

$$I_x^\alpha\left\{ {}_pM_q^{\,v}(x)H_{P,Q}^{M,N}[cx^\sigma]\right\}$$
$$=\frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^\infty\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(vk+1)}\int_0^x(x-t)^{\alpha-1}t^{k+\sigma s}dt\,ds$$
$$=\frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^\infty\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(vk+1)}x^{\alpha-1}\int_0^x\left(1-\frac{t}{x}\right)^{\alpha-1}t^{k+\sigma s}dt\,ds$$

Using the substitution $\frac{t}{x}=u$, the above equation takes the form,

$$=\frac{1}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^\infty\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(vk+1)}x^{k+\sigma s+\alpha}\int_0^1(1-u)^{\alpha-1}u^{k+\sigma s}du ds$$

Using the definition of Beta function from (3.1), we have,

$$=\frac{x^\alpha}{\Gamma(\alpha)}\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^\infty\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{x^k}{\Gamma(vk+1)}\frac{\Gamma(\alpha)\Gamma(k+\sigma s+1)}{\Gamma(\alpha+k+\sigma s+1)}x^{\sigma s}ds$$

Rearranging the terms follows the right hand side of (4.1).

Theorem 2: The Riemann-Liouville Fractional differential of order $\alpha\in C$ of the product of H-function and M-series are

$$D_x^\alpha\left\{ {}_pM_q^{\,v}(x)\,H_{P,Q}^{M,N}\left[cx^\sigma\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]\right\}$$
$$=x^{-\alpha}\ {}_pM_q^{\,v}(x)\,H_{P+1,Q+1}^{M,N+1}\left[cx^\sigma\left|\begin{matrix}(-k,\sigma),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(\alpha-k,\sigma)\end{matrix}\right.\right] \tag{4.2}$$

Here $v\epsilon C,R(v)>0,\alpha\epsilon C,R(\alpha)>0,\sigma>0$ and M, N, P and Q are non-negative integers satisfying the condition (2.3). Also the uniform convergence of the M-series is discussed above.

Proof . Expressing the H-function and the M-series with the help of (2.1) and (2.4) respectively, we get

$$D_x^{\alpha}\ {}_pM_q^{\upsilon}(x)\, H_{P,Q}^{M,N}\left[cx^{\sigma}\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_0^x (x-t)^{n-\alpha-1}$$
$$\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{t^k}{\Gamma(\upsilon k+1)}\frac{1}{2\pi i}\int_L c^s\theta(s)t^{\sigma s}ds\ dt$$

Term by term integration leads to

$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}\int_L (x-t)^{n-\alpha-1}t^{k+\sigma s}dt\ ds$$

Using the modified Beta function (3.2), we get

$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\frac{1}{2\pi i}\int_L c^s\theta(s)\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}B(n-\alpha,k+\sigma s+1)x^{n-\alpha+k+\sigma s}\,ds$$

Differentiating n-times ,the term $x^{n-\alpha+k+\sigma s}$, we get

$$=x^{-\alpha}\ {}_pM_q^{\upsilon}(x)\, H_{P+1,Q+1}^{M,N+1}\left[cx^{\sigma}\left|\begin{matrix}(-k,\sigma),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(\alpha-k,\sigma)\end{matrix}\right.\right]$$

FRACTIONAL INTEGRALS OF THE M-SERIES:

The fractional integral operator of the M-series involving the H-function defined by Saxena and Khumbat is derived in Theorem3 and Theorem4 as follows:

THEOREM 3:

$$R_x^{\eta,\sigma}\left[\ {}_pM_q^{\upsilon}(x^{\lambda})\right]$$
$$=\ {}_pM_q^{\upsilon}(x^{\lambda})\, H_{P+2,Q+2}^{M,N+2}\left[k\left|\begin{matrix}(1-\lambda k-\eta,\mu),(-\sigma,\vartheta),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(-\lambda k-\eta-\sigma-1,\mu+\vartheta)\end{matrix}\right.\right] \quad (4.3)$$

$\upsilon\in C, R(\upsilon)>0$,The conditions of validity of this operator is given with (1.3) and the the convergence of M-series is provided with (2.4).

Proof: Applying the fractional integral (1.3) and expressing the M-series from (2.4), we get

$$R_x^{\eta,\sigma}\left[\ {}_pM_q^{\upsilon}(x^{\lambda})\right]=x^{-\eta-\sigma-1}\int_0^x t^{\eta}(x-t)^{\sigma}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}t^{\lambda k}.$$
$$H_{P,Q}^{M,N}\left[kx^{-(\mu+\vartheta)}t^{\mu}(x-t)^{\vartheta}\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]$$

Changing the order of summation and integration and applying the integral (3.3), we obtain

$$=x^{-\eta-\sigma-1}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}x^{(\lambda k+\eta+\sigma+2)-1}.$$
$$H_{P,Q}^{M,N}\left[kx^{-(\mu+\vartheta)}x^{\mu+\vartheta}\left|\begin{matrix}(-\lambda k-\eta,\mu),(-\sigma,\vartheta)(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(-\lambda k-\eta-\sigma-1,\mu+\vartheta)\end{matrix}\right.\right]$$

Rearranging the terms of the above equation we obtain the right hand side of (4.3).

THEOREM 4:

$$K_x^{\delta,\sigma}\left[\ {}_pM_q^{\upsilon}(x^{\lambda})\right]=\ {}_pM_q^{\upsilon}(x^{\lambda})\, H_{P,Q}^{M,N}\left[k\left|\begin{matrix}(-\sigma,\vartheta),(e_j,\rho_j)_{1,P},(1+\delta+\sigma-\lambda k,\mu)\\(\delta-\lambda k,\mu+\vartheta),(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right] \quad (4.4)$$

$\upsilon\in C, R(\upsilon)>0$,The conditions of validity of this operator is given with (1.3) and the the convergence of M-series is provided with (2.4).

Proof: Applying the fractional integral (1.4) and expressing the M-series from (2.4), we get

$$K_x^{\delta,\sigma}\left[\ {}_pM_q^{\upsilon}(x^{\lambda})\right]=x^{\delta}\int_x^{\infty} t^{-\delta-\sigma-1}(t-x)^{\sigma}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}t^{\lambda k}.$$

$$H_{P,Q}^{M,N}\left[kx^{\mu}t^{-(\mu+\vartheta)}(t-x)^{\vartheta}\left|\begin{matrix}(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]$$

Changing the order of summation and integration and applying the integral of H-function (3.4) , we obtain

$$=x^{\delta}\sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k}\frac{1}{\Gamma(\upsilon k+1)}x^{-\delta-\sigma+\lambda k+\sigma+1-1}.$$

$$H_{P,Q}^{M,N}\left[kx^{\mu}x^{-(\mu+\vartheta)+\vartheta}\left|\begin{matrix}(-\sigma,\vartheta),(e_j,\rho_j)_{1,P},(1+\delta+\sigma-\lambda k,\mu)\\(\delta-\lambda k,\mu+\vartheta),(f_j,\omega_j)_{1,Q}\end{matrix}\right.\right]$$

Rearranging the terms of the above equation we obtain the right hand side of (4.4).

5. Special Cases

If in the integral (4.1) we put $a_j=0, b_j=0, j=1,\dots,q$ the M-series reduces to Mittag-Leffler function [4] we arrive at the following result after a little simplification.

$$I_x^{\alpha}\left\{E_{\upsilon}(x)\,H_{P,Q}^{M,N}[cx^{\sigma}]\right\}=\;x^{\alpha}E_{\upsilon}(x)H_{P+1,Q+1}^{M,N+1}\left[cx^{\sigma}\left|\begin{matrix}(-k,\sigma),(e_j,\rho_j)_{1,P}\\(f_j,\omega_j)_{1,Q},(-\alpha-k,\sigma)\end{matrix}\right.\right]\qquad(5.1)$$

Here $\upsilon\epsilon C, R(\upsilon)>0, \alpha\epsilon C, R(\alpha)>0, \sigma>0$.

A number of special cases involving functions that are special cases of M-series and Fox H-function can be obtained from the above five results but we do not record them here.

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