Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.6, 2013-Selected from International Conference on Recent Trends in Applied Sciences with Engineering Applications

Fixed Point and Common Fixed Point Theorems in Vector Metric Spaces

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Abstract
In the present paper we prove fixed point and common fixed point theorems in two self mappings satisfy quasi type contraction.
Keywords : Fixed point, Common Fixed point ,self mapping, vector Metric space , E-complete.
Mathematical Subject Classification: 47B60, 54H25.

Introduction and Preliminaries
Vector metric space, which is introduced in [6] by motivated this paper [7] , is generalisation of metric space ,where the metric is Riesz space valued .Actually in both of them , the metric map is vector space valued. One of the difference between our metric definition and Huang-Zhang's metric definition is that there exist a cone due to the natural existence of ordering on Riesz space .The other difference is that our definition eliminates the requirement for the vector space to have a topological structure.
A Riesz space (or a vector lattice) is an ordered vector space and alattice. Let E be a Riesz space with the positive $E_+ = \{x \in E : x \geq 0\}$. If $\{a_n\}$ is a decreasing sequence in E such that $\inf a_n = a$, we write $a_n \downarrow a$

Definition 1. The Riesz space E is said to be Archimedean if $\frac{1}{n}a \downarrow 0$ holds for every $a \in E_+$.

Definition 2. A sequence (b_n) is said to order convergent (or o-convergent) to b, if there is a sequence (a_n) in E satisfying $a_n \downarrow 0$ and $|b_n - b| \leq a_n$ for all n, and written $b_n \overset{o}{\to} b$ or o-lim$b_n = b$, where $|a| = \sup\{a, -a\}$ for any $a \in E$.

Definition3. A sequence (b_n) is said to order Cauchy (or o-cauchy) if there exists a sequence (a_n) in E such that $a_n \downarrow 0$ and $|b_n - b_{n+1}| \leq a_n$ holds for n and p.

Definition4. The Riesz space E is said to be o-cauchy complete if every o-cauchy sequence is o-convergent.

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Definition 4. Let X be anon empty set and E be a Riesz space .The function $d : X \times X \to E$ is said to be avector metric (or E-metric) if it is satisfying the following properties :
(i) $d(x, y) = 0$ if and only if $x = y$.
(ii)$d(x, y) \leq d(x, z) + d(y, z)$
For all $x, y, z \in X$. Also the triple (X , d ,E) (briefly X with the default parameters omitted) is said to be vector metric space. .

Main Results
Recently ,many authors have studied on common fixed point theorems for weakly compatible pairs (see [1], [3], [4], [8], [9]). Let T and S be self maps of a set X. if $y = Tx = Sx$ for some , $x \in X$, then y is said to be a point of coincidence and xis said to be coincidence point of T and S. If T and S are weakly compatible ,that is they are commuting at their coincidence point on X, then the point of coincidence y is the unique common fixed point of these maps [1].

Theorem 1 : Let X be a vector space with E is Archimedean. Suppose the mappings $S, T : X \times X \to X$ satisfies the following conditions
(i)for all $x, y \in X$, $d(Tx, Ty) \leq ku(x, y)$ (1)
where $k \in [0,1)$is a constant and

$$u(x, y) \in \left\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx), \frac{1}{2}[d(Sx, Tx) + d(Sy, Tx)]\right\}$$

(ii) $T(x) \subseteq S(x)$
(iii) $S(x)$ or $T(x)$is a E-Complete subspace of x.
Then T and S have a unique fixed point of coincidence in X . Moreover, if S and Tare weakly compatible ,then they have a unique common fixed point in X.

Proof : Let $x_0, x_1 \in X$. Define the sequence (x_n) by $Sx_{n+1} = Tx_n = y_n$ for $n \in N$.
We first show that

$$d(y_n, y_{n+1}) \leq k\, d(y_{n-1}, y_n) \quad (2)$$

For all n. we have that

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}) \leq ku(x_n, x_{n+1})$$

Fir all n. Now we have to consider the following three cases:

If $u(x_n, x_{n+1}) = d(y_{n-1}, y_n)$ then clearly (2)holds . If $u(x_n, x_{n+1}) = d(y_n, y_{n+1})$
Then according to Remark 1 $d(y_n, y_{n+1}) = 0$ and (2)is immediate. Finally , suppose that $u(x_n, x_{n+1}) = \frac{1}{2} d(y_{n-1}, y_{n+1})$. Then,

$$d(y_n, y_{n+1}) \leq \frac{k}{2} d(y_{n-1}, y_{n+1}) \leq \frac{k}{2} d(y_{n-1}, y_n) + \frac{1}{2} d(y_n, y_{n+1})$$

Holds , and we prove (2).
We have

$$d(y_n, y_{n+1}) \leq k^n d(y_0, y_1)$$

For all n and p,

$$d(y_n, y_{n+p}) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots + d(y_{n+p-1}, y_{n+p})$$
$$\leq (k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+p-1})\, d(y_0, y_1)$$
$$d(y_n, y_{n+p}) \leq \frac{k^n}{1-k} d(y_0, y_1)$$

Holds .Now since E is Archimedean then (y_0) is an E-cauchy sequence. Since the range of S contains the range of T and the range of at least one is E-Complete , there exists a $z \in S(X)$ such that $Sx_n \xrightarrow{d.E} z$. Hence there exists a sequence (a_n)in E such that $a_n \downarrow 0$ and $d(Sx_n, z) \leq a_n$. On the other hand , we can find $w \in X$ such that $Sw = z$.

Let us show that $Tw = z$ we have

$d(Tw, z) \leq d(Tw, Tx_n) + d(Tx_n, z) \leq ku(x_n, w) + a_{n+1}$

for all n . Since

$$u(x_n, w) \in \left\{ \begin{array}{c} d(Sx_n, Sw), d(Sx_n, Tx_n), d(Sw, Tw), d(Sx_n, Tw), d(Sw, Tx_n) \\ , \frac{1}{2}[d(Sx_n, Tw) + d(Sw, Tx_n)] \end{array} \right\}$$

At least one of the following four cases holds for all n.

Case 1 :

$$d(Tw, z) \leq d(Sx_n, Sw) + a_{n+1} \leq a_n + a_{n+1} \leq 2a_n$$

Case 2 :

$$d(Tw, z) \leq d(Sx_n, Tx_n) + a_{n+1} \leq d(Sx_n, z) + 2a_{n+1} \leq 3a_n$$

Case 3:

$$d(Tw, z) \leq kd(Sw, Tw) + a_{n+1} \leq kd(Tw, z) + a_n ,$$

that is $d(Tw, z) \leq \frac{1}{1-k} a_n$.

Case 4 :

$$d(Tw, z) \leq d(Sx_n, Tw) + a_{n+1} \leq d(Sx_n, z) + d(Tw, z) + 3a_{n+1} \leq 4a_n$$

Case 5 :

$d(Tw, z) \leq kd(Sw, Tx_n) + a_{n+1} \leq k\, d(Tx_n, z) + 2a_{n+1} \leq 3a_n$

Case 6 :

$$d(Tw, z) \leq \frac{1}{2}[d(Sx_n, Tw) + d(Sw, Tx_n)] + a_{n+1}$$
$$\leq \frac{1}{2} d(Sx_n, Tw) + \frac{3}{2} a_{n+1}$$
$$\leq \frac{1}{2} d(Sx_n, z) + \frac{1}{2} d(Tw, z) + \frac{3}{2} a_n$$
$$\leq \frac{1}{2} d(Tw, z) + 2a_n$$

That is , $d(Tw, z) \leq 4a_n$

Since the infimum of the sequence on the right side of last inequality are zero, then $d(Tw, z) = 0$, i. e. $Tw = z$. Therefore , z is a point of coincidence of T and S.

If z_1 is another point of coincidence then there is $w_1 \in X$ with $z_1 = Tw_1 = Sw_1$.

Now from (1), it follows that

$d(z, z_1) = d(Tw, Tw_1) \leq ku(w, w_1)$.

Where

$$u(w, w_1) \in \left\{ \begin{array}{c} d(Sw, Sw_1), d(Sw, Tw), d(Sw_1, Tw_1), d(Sw, Tw_1) d(Sw_1, Tw), \\ \frac{1}{2}[d(Sw, Tw_1) + d(Sw_1, Tw)] \end{array} \right\}$$
$$= \{0, d(z, z_1)\}.$$

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Hence $d(z, z_1) = 0$ that is $z = z_1$.
If S and T are weakly compatible then it is obvious that z is unique common fixed point of T and S by [1].

Theorem 2 : Let X be an vector metric space with E is Archimedean. Suppose the mappings $S, T : X \to X$ satisfies the following conditions
(i) for all $x, y \in X$,

$$d(Tx, Ty) \leq ku(x, y) \tag{3}$$

Where $k \in [0,1)$ is a constant and

$$u(x, y) \in \left\{ \begin{matrix} d(Sx, Sy), \frac{1}{2}[d(Sx, Tx) + d(Sy, Ty)], \frac{1}{2}[d(Sx, Ty) + d(Sy, Tx)], \\ \frac{1}{2}[d(Sx, Tx) + d(Sx, Ty)], \frac{1}{2}[d(Sy, Ty) + d(Sy, Tx)] \end{matrix} \right\}$$

(ii) $T(X) \sqsubseteq S(X)$,

(iii) S(X) or T(X) is E- Complete subspace of X.
Then T and S have a unique point of coincidence in X. Moreover, if S and T are weakly compatible, then they have a unique common fixed point in X.
Proof : Let us define the sequence (x_n) and (y_n) as in the proof of Theorem 1 , we first
Show that

$$d(y_n, y_{n+1}) \leq k\, d(y_{n-1}, y_n) \tag{4}$$

For all n. Notice that

$$d(y_n, y_{n+1}) = d(Tx_n, Tx_{n+1}) \leq ku(x_n, x_{n+1}),$$

For all n.
As in Theorem 1, we have to consider three cases: $u(x_n, x_{n+1}) = d(y_{n-1}, y_n)$,
$u(x_n, x_{n+1}) = \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$ and $u(x_n, x_{n+1}) = \frac{1}{2}d(y_{n-1}, y_{n+1})$.
First and third have been shown in the proof of Theorem 1. Consider only the second case.
If $u(x_n, x_{n+1}) = \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})]$, then from (3) we have

$$d(y_n, y_{n+1}) \leq \frac{1}{2}[d(y_{n-1}, y_n) + d(y_n, y_{n+1})] \leq \frac{k}{2}d(y_{n-1}, y_n) + \frac{1}{2}d(y_n, y_{n+1}).$$

Hence. (4) Holds.
In the proof of this Theorem 1 we illustrate that (y_n) is an E-Cauchy sequence. Then there exist $z \in S(X), w \in X$ and (a_n) in E such that $Sw = z, d(Sx_n, z) \leq a_n$ and $a_n \downarrow 0$
Now, we have to show that Tw = z. We have

$$d(Tw, z) \leq d(Tw, , Tx_n) + d(Tx_n, z) \leq u(x_n, w) + a_{n+1}$$

For all n. since

$$u(x_n, w) \in \left\{ \begin{matrix} d(Sx_n, Sw), \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)], \frac{1}{2}[d(Sx_n, Tw) + d(Sw, Tx_n)], \\ \frac{1}{2}[d(Sx_n, Tx_n) + d(Sx_n, Tw)], \frac{1}{2}[d(Sw, Tx_n) + d(Sw, Tw)] \end{matrix} \right\}$$

At least three of the five holds for all n. Consider only the cases of

$$u(x_n, w) = \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)], \frac{1}{2}[d(Sx_n, Tx_n) + d(Sx_n, Tw)], \frac{1}{2}[d(Sw, Tx_n) + d(Sw, Tw)]$$

Because the other four cases have shown that the proof of Theorem 1 . It is satisfied that

$$\begin{aligned} d(Tw, z) &\leq \frac{1}{2}[d(Sx_n, Tx_n) + d(Sw, Tw)] + \frac{1}{2}[d(Sx_n, Tx_n) + d(Sx_n, Tw)] \\ &\quad + \frac{1}{2}[d(Sw, Tx_n) + d(Sw, Tw)] + a_{n+1} \\ &\leq \frac{1}{2}[d(Sx_n, z) + d(Tx_n, z) + d(z, Tw)] + \frac{1}{2}[d(Sx_n, z) + d(Tx_n, z) + d(Sx_n, z) + d(z, Tw)] \\ &\quad + \frac{1}{2}[d(Tx_n, z) + d(z, Tw)] + a_{n+1} \\ &\leq \frac{1}{2}d(Sx_n, z) + \frac{1}{2}d(Tx_n, z) + \frac{1}{2}d(z, Tw) + a_{n+1} \\ &\leq \frac{1}{2}a_n + \frac{1}{2}\, d(z, Tw) + \frac{3}{2}a_{n+1} \end{aligned}$$

$$d(Tw,z) \leq \frac{1}{2}\, d(z,Tw) + 2a_n$$

That is $d(z,Tw) \leq 4a_n$. Since $4a_n \downarrow 0$ then $Tw = z$. Hence z is a point of coincidence of T and S. The uniqueness of z as in the proof of Theorem 1. Also, If S and T are weakly compatible, then it is obvious that z is unique common fixed point of Tand S by [1].

Theorem 3 : Let X be an vector space with E is Archimedean. Suppose the mappings $S,T: X \to X$ satisfies the following conditions

(i) for all x , y ∈ X

$$d(Tx,Ty) \leq \alpha\, d(Sx,Tx) + \beta\, d(Sy,Ty) + \gamma\, d(Sx,Ty) + \delta\, d(Sy,Tx) + \eta\, d(Sx,Sy) + \frac{1}{2}\, \mu\{d(Sx,Tx) + d(Sy,Ty)\}$$

(ii) $T(X) \subseteq S(X)$,

(iii) S(X) or T(X) is E-complete subspace of X

Then T and S have a unique point of coincidence in X .Moreover, If S and T are weakly compatible, then they have a unique common fixed point in X .

Proof : Let us define the sequence $(x_n)$and (y_n) as in the proof of Theorem 1 , we have to show that

$$d(y_n, y_{n+1}) \leq k\, d(y_{n-1}, y_n) \qquad (5)$$

For some $k \in [0,1)$ and for all n. Consider $Sx_{n+1} = Tx_n = y_n$ for all n. Then

$$d(y_n, y_{n+1}) \leq (\alpha+\eta)d(y_{n-1},y_n) + \beta\, d(y_n, y_{n+1}) + \gamma d(y_{n-1},y_{n+1}) + \frac{1}{2}\mu[d(y_{n-1},y_n) + d(y_n, y_{n+1})]$$

And

$$d(y_n, y_{n+1}) \leq \alpha\, d(y_n, y_{n+1}) + (\beta+\eta)d(y_{n-1},y_n) + \delta d(y_{n-1},y_{n+1})$$

For all n. Hence,

$$d(y_n, y_{n+1}) \leq \frac{\alpha+\beta+\gamma+\delta+2\eta+\mu}{2-\alpha+\beta+\gamma+\delta} d(y_{n-1},y_n)$$

If we choose $k = \frac{\alpha+\beta+\gamma+\delta+2\eta+\mu}{2-\alpha+\beta+\gamma+\delta}$, then $k \in [0,1)$and (5)is hold.

In the proof of Theorem 1 we illustrate that (y_n) is an E-Cauchy sequence .then there exist $z \in s(X), w \in X$ and (a_n) in E such that $Sw = z$, $d(Sx_n, z) \leq a_n$ and $a_n \downarrow 0$.

Let us show that Tw = z .we have

$$\begin{aligned} d(Tw,z) &\leq d(Tw,Tx_n) + d(Tx_n,z) \\ &\leq (\alpha+\delta+\mu)d(Tw,z) + (\beta+\delta+\eta)d(Sx_n,z) + (\beta+\gamma+1)d(Tx_n,z) \\ &\leq (\alpha+\delta+\mu)\, d(Tw,z) + (\beta+\delta+\eta)a_n + (\beta+\gamma+1) + a_{n+1} \\ &\leq (\alpha+\delta+\mu)\, d(Tw,z) + (2\beta+\gamma+\delta+\eta+1)a_n \end{aligned}$$

That is $d(Tw,z) \leq \frac{(2\beta+\gamma+\delta+\eta+1)}{1-(\alpha+\delta+\mu)} a_n$ for all n.

Then $d(Tw,z) = 0$, i.e. $Tw = z$. Hence ,

Z is a point of coincidence of T and S. The uniqueness of z is easily seen. Also ,If S and T are weakly compatible , then it is obvious that z is unique common fixed point of T and S by [1].

Corrollary 1: Let X be an vector space with E is Archimedean. Suppose the mappings $S,T: X \to X$ satisfies the following conditions

(i) For all x , y ∈ X ,

$$d(Tx,Ty) \leq k\, d(Sx,Sy) \qquad (6)$$

Where k< 1

(ii) $T(X) \sqsubseteq S(X)$,

(iii) S(X) or T(X) is E- Complete subspace of X.

Then T and S have a unique point of coincidence in X .Moreover, If S and T are weakly compatible, then they have a unique common fixed point in X .

Now we give an illustrative example

Example : Let $E = \mathbb{R}^2$ with coordinatwise ordering (since $\mathbb{R}^2$ is not Archmedean with lexicografical ordering ,then we can not use this ordering), X= $\mathbb{R}^2$, $d(x,y) = (|x-y|, \alpha|x-y|), \alpha > 0$, $Tx = 2x^2 + 1$ and $Sx = 4x^2$. Then , for all $x, y \in X$ we have

$$d(Tx,Ty) = \frac{1}{2} d(Sx,Sy) \leq k\, d(Sx,Sy)$$

For $k \in \left[\frac{1}{2}, 1\right)$,

$$T(X) = [1,\infty) \subseteq [0,\infty) = S(X)$$

And T(X) is E-complete subspace of X. Therefore all conditions of corollary 1 are satisfied. Thus T and S have a unique point of coincidence in X .

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