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Fixed Point Result in Probabilistic Metric Space

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ABSTRACT
In this paper we prove common fixed point theorem for four mapping with weak compatibility in probabilistic metric space.
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1. INTRODUCTION:

Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a very dynamic area of mathematical research. The notion of probabilistic metric space is introduced by Menger in 1942 [9] and the first result about the existence of a fixed point of a mapping which is defined on a Menger space is obtained by Sehgel and Barucha-Reid.

Recently, a number of fixed point theorems for single valued and multivalued mappings in menger probabilistic metric space have been considered by many authors [1],[2],[3],[4],[5],[6]. In 1998, Jungck [7] introduced the concept weakly compatible maps and proved many theorems in metric space. In this paper we prove common fixed point theorem for four mapping with weak compatibility and rational contraction without appeal to continuity in probabilistic metric space. Also we illustrate example in support of our theorem.

2. PRELIMINARIES:

Now we begin with some definition

Definition 2.1: Let R denote the set of reals and R^+ the non-negative reals. A mapping $F: R \to R^+$ is called a distribution function if it is non decreasing left continuous with $\inf_{t\in R} F(t) = 0 \quad and \quad \sup_{t\in R} F(t) = 1$

Definition 2.2: A probabilistic metric space is an ordered pair (X, F) where X is a nonempty set, L be set of all distribution function and $F: X \times X \to L$. We shall denote the distribution function by $F\,(p,q)$ or $F_{p,q}$; $p, q \in X$ and $F_{p,q}\,(x)$ will represents the value of $F\,(p,q)$ at $x \in R$. The function $F(p,q)$ is assumed to satisfy the following conditions:

1. $F_{p,q}(x) = 1\ for\ all\ x > 0\ if\ and\ only\ if\ p = q$
2. $F_{p,q}(0) = 0\ for\ every\ p, q \in X$
3. $F_{p,q} = F_{q,p}\ for\ every\ p, q \in X$
4. $F_{p,q}(x) = 1\ \ and\ F_{q,r}(y) = 1\ \ then\ F_{p,r}(x+y) = 1\ for\ every\ p, q, r \in X.$

In metric space (X, d) , the metric d induces a mapping $F: X \times X \to L$ such that $F_{p,q}(x) = F_{p,q} = H\,(x - d(p,q))$ for every $p, q \in X$ and $x \in R$, where H is the distribution function defined as

$$H(x) = \begin{cases} 0, \text{if x } \le 0 \\ 1, \text{if x} > 0 \end{cases}$$

Definition 2.3: A mapping $*$: $[0,1] \times [0,1] \to [0,1]$ is called t-norm if

1. $(a * 1) = a\ \forall\, a \in [0,1]$
2. $(0 * 0) = 0, \forall\, a, b \in [0,1]$
3. $(a * b) = (b * a)$,
4. $(c * d) \ge (a * b)\ for\ c \ge a, d \ge b$, and
5. $((a * b) * c) = (a * (b * c))$

Example: (i) $(a * b) = ab$, (ii) $(a * b) = min\,(a, b)$
(iii) $(a * b) = max\,(a + b - 1; 0)$

Definition 2.4: A Menger space is a triplet $(X, F, *)$ where (X, F)a PM-space and Δ is is a t-norm with the following condition

$$F_{u,w}(x + y) \ge F_{u,v}(x) * F_{v,w}(y)$$

The above inequality is called Menger's triangle inequality.

EXAMPLE: Let $X = R$, $(a * b) = min(a, b) \forall\, a, b \in (0,1)$ and

$$F_{u,v}(x) = \begin{cases} H(x) & for\ u \neq v \\ 1 & for\ u = v \end{cases}$$

where $H(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 \leq x \leq 1 \\ 1 & x \geq 1 \end{cases}$

Then $(X, F, *)$ is a Menger space.

Definition 2.5: Let $(X, F, *)$ be a Menger space. If $u \in X, \varepsilon > 0, \lambda \in (0,1)$, then an (ε, λ) neighbourhood of u, denoted by $U_u(\varepsilon, \lambda)$ is defined as

$$U_u(\varepsilon, \lambda) = \{v \in X; F_{u,v}(\varepsilon) > 1 - \lambda\}.$$

If $(X, F, *)$ be a Menger space with the continuous t-norm t, then the family$U_u(\varepsilon, \lambda); u \in X; \varepsilon > 0, \lambda \in (0,1)$ of neighbourhood induces a hausdorff topology on X and if $\sup_{a<1}(a * a) = 1$, it is metrizable.

Definition 2.6: A sequence $\{p_n\}$ in $(X, F, *)$ is said to be convergent to a point $p \in X$ if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $\boldsymbol{N} = \boldsymbol{N}(\varepsilon, \lambda)$ such that $p_n \in U_p(\varepsilon, \lambda)$ for all $n \geq \boldsymbol{N}$ or equivalently $F_{x_n,x}(\epsilon) > 1 - \lambda$ for all $n \geq \boldsymbol{N}$.

Definition 2.7: A sequence $\{p_n\}$ in $(X, F, *)$ is said to be Cauchy sequence if for every $\varepsilon > 0$ and $\lambda > 0$, there exists an integer $\boldsymbol{N} = \boldsymbol{N}(\varepsilon, \lambda)$ such that $F_{p_n,p_m}(\epsilon) > 1 - \lambda$ for all $n, m \geq \boldsymbol{N}$.

Definition 2.8: A Menger space $(X, F, *)$ with the continuous t-norm Δ is said to be complete if every Cauchy sequence in X converges to a point in X.

Definition 2.9: A coincidence point (or simply coincidence) of two mappings is a point in their domain having the same image point under both mappings.

Formally, given two mappings $f, g : X \to Y$ we say that a point x in X is a coincidence point of f and g if $f(x) = g(x)$.

Definition 2.10: Let $(X, F, *)$ be a Menger space. Two mappings $f, g : X \to X$ are said to be weakly compatible if they commute at the coincidence point, i.e., the pair $\{f, g\}$ is weakly compatible pair if and only if $fx = gx$ implies that $fgx = gfx$.

Example: Define the pair $A, S: [0,3] \to [0,3]$ by

$$A(x) = \begin{cases} x, & x \in [0,1) \\ 3, & x \in [1,3] \end{cases}, \qquad S(x) = \begin{cases} 3 - x, & x \in [0,1) \\ 3, & x \in [1,3]. \end{cases}$$

Then for any $x \in [1,3]$, $ASx = SAx$, showing that A, S are weakly compatible maps on $[0,3]$.

Definition 2.11: Let $(X, F, *)$ be a Menger space. Two mappings $A, S : X \to X$ are said to be semi compatible if $F_{ASx_n,Sx}(t) \to 1$for all $t > 0$ whenever $\{x_n\}$ is a sequence in X such that $Ax_n, Sx_n \to p$ for some p in X as$n \to \infty$. It follows that (A, S) is semi compatible and $Ay = Sy$ imply $ASy = SAy$ by taking $\{x_n\} = y\ and\ x = Ay = Sy$.

Lemma 2.12[15]: Let $\{p_n\}$ be a sequence in Menger space $(X, F, *)$ where $*$ is continuous and $(x * x) \geq x$ for all $x \in [0,1]$. If there exists a constant $k \in (0,1)$ such that $x > 0$ and $n \in \boldsymbol{N}$ $F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x)$, then $\{p_n\}$ is a Cauchy sequence.

Lemma 2.13[13]: If (X, d) is a metric space, then the metric d induces a mapping $F: X \times X \to L$, defined by $F(p,q) = H(x - d(p,q))$, $p, q \in X\ and\ x \in R$. Further more if $*: [0,1] \times [0,1] \to [0,1]$ is defined by $(a * b) = min(a, b)$, then $(X, F, *)$ is a Menger space. It is complete if (X, d) is complete. The space $(X, F, *)$ so obtained is called the induced Menger space.

Lemma 2.14[10]: Let $(X, F, *)$ be a Menger space. If there exists a constant $k \in (0,1)$ such that $F_{x,y}(kt) \geq F_{x,y}(t)$, for all $x, y \in X$ and $t > 0$ then $x = y$.

3. MAIN RESULT:

Theorem 3.1: Let $(X, F, *)$ be a complete Menger space where $*$ is continuous and $(t * t) \geq t$ for all $t \in [0,1]$. Let A, B, T and S be mappings from X into itself such that

3.1.1. $A(X) \subset S(X)\ and\ B(X) \subset T(X)$

3.1.2. $S\ or\ T$ are continuous

3.1.3. The pair (S, A) and (T, B) are Semi compatible

3.1.4. There exists a number $k \in (0,1)$ such that

$$F_{Ax,By}(kt) \geq F_{Sx,Ty}(t) * F_{Sx,Ax}(t) * F_{Ax,Ty}(t) * F_{Ty,By}(t) * F_{Sx,By}((2-\alpha)t)$$
$$for\ all\ x, y \in X, \alpha \in (0,2)\ and\ t > 0.$$

Then, A, B, S and T have a unique common fixed point in X.

Proof: Since $A(X) \subset S(X)$ for any $x_0 \in X$ there exists a point $x_1 \in X$ such that$Ax_0 = Sx_1$. Since $B(X) \subset T(X)$ for this point x_1 we can choose a point $x_2 \in X$ such that $Tx_1 = Bx_2$.

Inductively we can find a sequence $\{y_n\}$ as follows

$$y_{2n} = Ax_{2n} = Sx_{2n+1}$$
$$and\ y_{2n+1} = Bx_{2n+1} = Tx_{2n+2}$$

For $n = 0, 1, 2, 3 \dots \dots \dots$ by (3.1.4.), for all $t > 0$ and $\alpha = 1 - q$ with $q \in (0,1)$, we have

$$F_{y_{2n},y_{2n+1}}(kt) = F_{Ax_{2n},Bx_{2n+1}}(kt)$$
$$\geq F_{Sx_{2n},Tx_{2n+1}}(t) * F_{Sx_{2n+1},Ax_{2n+1}}(t) * F_{Ax_{2n},Tx_{2n+1}}(t) * F_{Tx_{2n+1},Bx_{2n+1}}(t) * F_{Sx_{2n},Bx_{2n+1}}((1+q)t)$$
$$= F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n-1},y_{2n+1}}((1+q)t)$$
$$\geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(qt)$$
$$= F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n},y_{2n+1}}(qt)$$

Since t-norm is continuous, letting $q \to 1$, we have

$$F_{y_{2n},y_{2n+1}}(kt) \geq F_{y_{2n-1},y_{2n}}(t) * F_{y_{2n},y_{2n+1}}(t)$$

Similarly

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t) * F_{y_{2n+1},y_{2n+2}}(t)$$

Similarly

$$F_{y_{2n+2},y_{2n+3}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t) * F_{y_{2n+2},y_{2n+3}}(t)$$

Therefore

$$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t) * F_{y_n,y_{n+1}}(t) \; for \; all \; n \in N$$

Consequently

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(k^{-1}t) *, F_{y_n,y_{n+1}}(k^{-1}t) \; for \; all \; n \in N$$

Repeated application of this inequality will imply that

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(k^{-1}t) * F_{y_n,y_{n+1}}(k^{-1}t) \geq \cdots \dots \dots \dots . \geq F_{y_{n-1},y_n}(k^{-1}t) * F_{y_n,y_{n+1}}(k^{-i}t), i \in N$$

Since $F_{y_n,y_{n+1}}(k^{-i}t) \to 1$ as $i \to \infty$, it follows that

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(k^{-1}t) \; for \; all \; n \in N$$

Consequently

$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t)$ for all $n \in N$

Therefore by Lemma [2.12], $\{y_n\}$ is a Cauchy sequence in X. Since X is complete,$\{y_n\}$converges to a point $z \in X$. Since $\{Ax_{2n}\}, \{Bx_{2n+1}\}, \{Sx_{2n+1}\}$ and $\{Tx_{2n+2}\}$ are subsequences of $\{y_n\}$, they also converge to the point z,

$i.e.$ as $n \to \infty$, $Ax_{2n}, Bx_{2n+1}, Sx_{2n+1} Tx_{2n+2} \to z.$

Case I: Since S is continuous. In this case we have

$$SAx_n \to Sz, \qquad SSx_n \to Sz$$

Also $(A, S)$is semi-compatible, we have $ASx_n \to Sz$

Step I: Let $x = Sx_n, y = x_n$ $with$ $\alpha = 1$ in (3.1.4) we get

$$F_{ASx_n,Bx_n}(kt) \geq F_{SSx_n,Tx_n}(t) * F_{SSx_n,ASx_n}(t) * F_{ASx_n,Tx_n}(t) * F_{Tx_n,Bx_n}(t) * F_{SSx_n,Bx_n}(t)$$
$$F_{Sz,z}(kt) \geq F_{Sz,z}(t) * F_{Sz,Sz}(t) * F_{Sz,z}(t) * F_{z,z}(t) * F_{Sz,z}(t)$$
$$F_{Sz,z}(kt) \geq F_{Sz,z}(t)$$

So we get $Sz = z.$

Step II: By putting $x = z, y = x_n$ $with$ $\alpha = 1$ in (3.1.4) we get

$$F_{Az,Bx_n}(kt) \geq F_{Sz,Tx_n}(t) * F_{Sz,Az}(t) * F_{Az,Tx_n}(t) * F_{Tx_n,Bx_n}(t) * F_{Sz,Bx_n}(t)$$
$$F_{Az,z}(kt) \geq F_{z,z}(t) * F_{z,Az}(t) * F_{Az,z}(t) * F_{z,z}(t) * F_{z,z}(t)$$
$$F_{Az,z}(kt) \geq F_{Az,z}(t)$$

So we get $Az = z.$

Case II: Since T is continuous. In this case we have

$$TBx_n \to Tz, \qquad TTx_n \to Tz$$

Also $(B, T)$is semi-compatible, we have $BTx_n \to Tz$

Step I: Let $x = x_n, y = Tx_n$ $with$ $\alpha = 1$ in (3.1.4) we get

$$F_{Ax_n,BTx_n}(kt) \geq F_{Sx_n,TTx_n}(t) * F_{Sx_n,Ax_n}(t) * F_{Ax_n,TTx_n}(t) * F_{TTx_n,BTx_n}(t) * F_{Sx_n,BTx_n}(t)$$
$$F_{z,Tz}(kt) \geq F_{z,Tz}(t) * F_{z,z}(t) * F_{z,Tz}(t) * F_{Tz,Tz}(t) * F_{z,Tz}(t)$$
$$F_{z,Tz}(kt) \geq F_{z,Tz}(t)$$

So we get $Tz = z.$

Step II: By putting $x = x_n, y = z$ $with$ $\alpha = 1$ in (3.1.4) we get

$$F_{Ax_n,Bz}(kt) \geq F_{Sx_n,Tz}(t) * F_{Sx_n,Ax_n}(t) * F_{Ax_n,Tz}(t) * F_{Tz,Bz}(t) * F_{Sx_n,Bz}(t)$$
$$F_{z,Bz}(kt) \geq F_{z,z}(t) * F_{z,z}(t) * F_{z,z}(t) * F_{z,Bz}(t) * F_{z,Bz}(t)$$
$$F_{Bz,z}(kt) \geq F_{Bz,z}(t)$$

So we get $Bz = z.$

Thus, we have $Az = Sz = Tz = Bz = z.$

That is z is a common fixed point of S, T, A and B.
For uniqueness, let w $(w \neq z)$ be another common fixed point of S, T, A and B .Then $Aw = Sw == Bw = Tw = w$.
Put $x = z, y = w$ and $\beta = 1$, in (3.1.4.), we get

$$F_{Az,Bw}(kt) \geq F_{Sz,Tw}(t) * F_{Sz,Aw}(t) * F_{Az,Tw}(t) * F_{Tw,Bw}(t) * F_{Sz,Bw}(t)$$
$$F_{z,w}(kt) \geq F_{z,w}(t) * F_{z,w}(t) * F_{z,w}(t) * F_{w,w}(t) * F_{z,w}(t)$$
$$F_{z,w}(kt) \geq F_{z,w}(t) * F_{z,w}(t) * F_{z,w}(t) * 1 * F_{z,w}(t)$$
$$F_{z,w}(kt) \geq F_{z,w}(t)$$

Thus we have$z = w$. Therefore z is a unique fixed point of A,S, B and T.
This completes the proof of the theorem.

COROLLARY 3.2: Let $(X, F, *)$ be a complete Menger space where $*$ is continuous and $(t * t) \geq t$ for all $t \in [0,1]$. Let A, and S be mappings from X into itself such that
3.2.1. $A(X) \subset S(X)$
3.2.2. S is continuous
3.2.3. The pair (S, A) is semi compatible
3.2.4. There exists a number $k \in (0,1)$ such that

$$F_{Ax,Sy}(kt) \geq F_{Sx,Sy}(t) * F_{Sx,Ax}(t) * F_{Ax,Sy}(t) * F_{Sy,Ay}(t) * F_{Sx,Ay}((2-\alpha)t)$$
$$for\ all\ x, y \in X, \alpha \in (0,2)\ and\ t > 0.$$

Then, $A,$ and S have a unique common fixed point in X.

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