A Common Random Fixed Point Theorem for Rational Inequality in Hilbert Space

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ABSTRACT
The object of this paper is to obtain a common random fixed point theorem for four continuous random operators defined on a non empty closed subset of a separable Hilbert space for rational inequality.
Keywords: Separable Hilbert space, random operators, common random fixed point, rational inequality

1. INTRODUCTION AND PRELIMINARIES

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for Contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [9] in 1922. In the general setting of complete metric space, this theorem runs as the follows,

Theorem 1.1 (Banach's contraction principle) Let (X,d) be a complete metric space, $c \in (0,1)$ and f: X→X be a mapping such that for each x, y ∈X, $d\,(fx,fy) \le cd\;x,y$ Then f has a unique fixed point a ∈X, such that for each x ∈ X, $\lim_{n\to\infty} f^n x = a$.

After the classical result, Kannan [7] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of rational type.

After the paper of Brianciari, a lot of a research works have been carried out on generalizing contractive conditions of rational type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Brianciari by replacing the condition.

The aim of this paper is to generalize some mixed type of contractive conditions to the mapping and then a pair of mappings satisfying general contractive mappings such as Kannan type [7], Chatrterjee type [8], Zamfirescu type [11], etc.

In recent years, the study of random fixed points has attracted much attention. Some of the recent literatures in random fixed point may be noted in Rhoades [3], and Binayak S. Choudhary [4].In this paper, we construct a sequence of measurable functions and consider its convergence to the common unique random fixed point of four continuous random operators defined on a non-empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the four continuous random operators. We have used a rational inequality (from B. Fisher [5] and S.S. Pagey [10]) and the parallelogram law. Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subsets of Ω, H stands for a separable Hilbert space and C is a nonempty closed subset of H.

Definition 1.3. A function f: Ω → C is said to be measurable if
$f^{-1}(B \cap C) \in \sum$ for every Borel subset B of H.

Definition 1.4. A function $F:\Omega\times C \to C$ is said to be a random operator if $F(.,x):\Omega \to C$ is measurable for every $x \in C$

Definition 1.5. A measurable $g:\Omega \to C$ is said to be a random fixed point of the random operator $F:\Omega\times C \to C$ if $F(t,g(t)) = g(t)$ for all $t \in \Omega$

Definition 1.6. A random operator F: Ω×C → C is said to be continuous if for fixed $t \in \Omega$, $F(t,.):C \to C$ is continuous.

Condition (A). Four mappings E, F, T, S: C → C, where C is a non-empty closed subset of a Hilbert space H, is said to satisfy condition (A) if

ES = SE, FT = TF, E (H) ⊂ T (H) and F (H) ⊆ S (H) -------------------------- (1)

$$\|Ex - Fy\|^2 \le \alpha \max\{ \frac{\|Ty - Fy\|^2 + \|Ex - Ty\|^2 + \|Sx - Ex\|^2}{1 + \|Ty - Fy\|^2 \|Ex - Ty\|^2 \|Sx - Ex\|^2},$$
$$\frac{\|Ex - Ty\| \ \|Ty - Fy\| \ [\|Sx - Ex\|^2 + \|Ty - Fy\|^2]}{1 + \|Ex - Ty\| \ \|Ty - Fy\| \ + \|Sx - Ex\|^2 + \|Ty - Fy\|^2}\}$$
$$+ \beta\{\|Sx - Ex\|^2 + \|Ty - Fy\|^2\} + \gamma \|Sx - Ty\|^2$$

For all x,y ϵ C with $\|Ty - Fy\|^2 \|Ex - Ty\|^2 \|Sx - Ex\|^2 \neq -1$ and

$$\|Ty - Fy\|^2 + \|Sx - Ex\|^2 + \|Ex - Ty\| \ \|Ty - Fy\| \neq -1, \alpha, \beta, \gamma \ge 0 \text{ ----------------(2)}$$

where $\xi : \mathcal{R}^+ \to \mathcal{R}^+$ is a lesbesgue- integrable mapping which is summable on each compact subset of $\mathcal{R}^+$, non negative, and such that for each for each ϵ > o , $\xi(t) > 0$

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2. MAIN RESULTS

Theorem 2.1. Let C be a non-empty closed subset of a separable Hilbert space H . Let E, F, S and T be four continuous random operators defined on C such that for, $t \in \Omega$ $E(t,.), F(t,.), T(t,.), S(t,.) : C \to C$ satisfy condition (A).Then E, F, S and T have unique common random fixed point.

Proof: Let the function $g : \Omega \to C$ be arbitrary measurable function. By (1), there exists a function $g_1 : \Omega \to C$ such that $T(t, g_1(t)) = E(t, g_0(t))$ for $t \in \Omega$ and for this function $g_1 : \Omega \to C$, we can choose another function $g_2 : \Omega \to C$ such that $F(t, g_1(t) = S(t, g_2(t))$ for $t \in \Omega$, and so on. Inductively, we can define a sequence of functions for $t \in \Omega$, $\{y_n(t)\}$ such that

$$y_{2n}(t) = T(t, g_{2n+1}(t)) = E(t, g_{2n}(t))$$
$$y_{2n+1}(t) = S(t, g_{2n+2}(t)) = F(t, g_{2n+1}(t)) \quad \text{For } t \in \Omega, \ n = 0,1,2,3------ \ (4)$$

From condition (A), we have for, $t \in \Omega$

$$\|y_{2n}(t) - y_{2n+1}(t)\|^2 = \|E(t, g_{2n}(t) - F(t, g_{2n+1}(t)\|^2$$
$$\le \alpha \max\{ \frac{\|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2 + \|E(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 + \|S(t, g_{2n}(t)) - E(t, g_{2n}(t}{1 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2 \|E(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 \|S(t, g_{2n}(t)) - E(t, g_{2n}(t)}$$
$$\frac{\|E(t, g_{2n}(t) - T(t, g_{2n+1}(t))\| \ \|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\|}{1 + \|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\| \ \|E(t, g_{2n}(t) - T(t, g_{2n+1}(t)\|}$$
$$\frac{[\|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\|^2 + \|S(t, g_{2n}(t) - E(t, g_{2n}(t)\|^2]}{+ \|S(t, g_{2n}(t) - E(t, g_{2n}(t)\|^2 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2}\}$$
$$+ \beta\{\|S(t, g_{2n}(t) - E(t, g_{2n}(t))\|^2 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2\}$$
$$+ \gamma \|S(t, g_{2n}(t) - T(t, g_{2n+1}(t))\|^2$$
$$= \alpha \max\{ \frac{\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \|y_{2n}(t) - y_{2n}(t))\|^2 + \|y_{2n-1}(t) - y_{2n}(t))\|^2}{1 + \|y_{2n}(t) - y_{2n+1}(t)\|^2 \|y_{2n}(t) - y_{2n}(t))\|^2 \|y_{2n-1}(t) - y_{2n}(t)\|^2},$$
$$\frac{\|y_{2n}(t) - y_{2n}(t))\| \ \|y_{2n}(t) - y_{2n+1}(t)\|}{1 + \|y_{2n}(t) - y_{2n+1}(t)\| \ \|y_{2n}(t) - y_{2n}(t)\|}$$
$$\frac{[\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \|y_{2n-1}(t) - y_{2n}(t)\|^2]}{+ \|y_{2n-1}(t) - y_{2n}(t)\|^2 + \|y_{2n}(t) - y_{2n+1}(t))\|^2}\}$$
$$+ \beta\{\|y_{2n-1}(t) - y_{2n}(t))\|^2 + \|y_{2n}(t)) - y_{2n}(t))\|^2\} + \gamma \|y_{2n-1}(t) - y_{2n}(t))\|^2$$

$$= \alpha \max\{ \frac{\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \|y_{2n-1}(t) - y_{2n}(t))\|^2}{1}, 0\}$$
$$+ \beta\{\|y_{2n-1}(t) - y_{2n}(t))\|^2 + 0\} + \gamma\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$= \alpha \max\{ \frac{\|y_{2n}(t) - y_{2n+1}(t)\|^2 + \|y_{2n-1}(t) - y_{2n}(t))\|^2}{1}, 0\}$$
$$+ \beta\{\|y_{2n-1}(t) - y_{2n}(t))\|^2 + 0\} + \gamma\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$= \alpha\|y_{2n-1} - y_{2n+1}\|^2 + \beta\|y_{2n-1}(t) - y_{2n}(t))\|^2 + \gamma\|y_{2n-1}(t) - y_{2n}(t))\|^2$$

Now

$$\|y_{2n} - y_{2n+1}\|^2 \le \frac{\alpha}{2}\|y_{2n-1} - y_{2n+1}\|^2 + \beta\|y_{2n-1}(t) - y_{2n}(t))\|^2 + \gamma\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$\therefore \|y_{2n} - y_{2n+1}\|^2 \le \alpha\{\|y_{2n-1} - y_{2n}\|^2 + \|y_{2n} - y_{2n+1}\|^2\} + \beta\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$+ \gamma\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$\therefore (1-\alpha)\|y_{2n} - y_{2n+1}\|^2 \le (\alpha + \beta + \gamma)\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$\therefore \|y_{2n} - y_{2n+1}\|^2 \le \frac{\alpha + \beta + \gamma}{1-\alpha}\|y_{2n-1}(t) - y_{2n}(t))\|^2$$
$$\therefore \|y_{2n} - y_{2n+1}\| \le k^{\frac{1}{2}}\|y_{2n-1}(t) - y_{2n}(t))\| \quad \text{Where} \quad k = (\frac{\alpha + \beta + \gamma}{1-\alpha})^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots(5)$$

Now, we shall prove that for $t \in \Omega$, $y_n(t)$ is a Cauchy sequence. For this for Common random fixed point theorem every Positive integer p, we have

$$\therefore \|y_n(t) - y_{n+p}(t)\| = \|y_n(t) - y_{n+1}(t) + y_{n+1}(t) - y_{n+2}(t) + \ldots.. + y_{n+p-1}(t) - y_{n+p}(t)\|$$
$$\le \|y_n(t) - y_{n+1}(t)\| + \|y_{n+1}(t) - y_{n+2}(t)\| + \ldots\ldots + \|y_{n+p-1}(t) - y_{n+p}(t)\|$$
$$\le [k^n + k^{n+1} + \ldots\ldots + k^{n+p-1}]\|y_0(t) - y_1(t)\|$$
$$= k^n[1 + k + k^2 + \ldots\ldots + k^{p-1}]\|y_0(t) - y_1(t)\|$$
$$\le \frac{k^n}{1-k}\|y_0(t) - y_1(t)\|$$
$$\therefore \lim_{n\to\infty}\|y_n(t) - y_{n+p}(t)\| = 0 \quad \text{for } t \in \Omega \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

From equation (6), it follows that for, $\{y_n(t)\}$ is a Cauchy sequence and hence is convergent in closed subset C of Hilbert space H.

For $t \in \Omega$, let $\lim_{n\to\infty}\{y_n(t)\} = y(t)$ ………………………(7)

Again, closeness of C gives that g is a function from C to C. and consequently the subsequences $\{E(t, g_{2n}(t))\}$, $\{F(t, g_{2n+1}(t))\}$ $\{T(t, g_{2n+1}(t))\}$ and{S(t,g2n+2(t))}of $\{y_n(t)\}$ for $t \in \Omega$, also converges to the $y(t)$ ....(*) and continuity of E,F, T and S gives

$$E(t, S(t, g_n(t))) \to E(t, y(t))$$
$$S(t, E(t, g_n(t))) \to S(t, y(t))$$
$$F(t, T(t, g_n(t))) \to F(t, y(t))$$
$$T(t, F(t, g_n(t))) \to T(t, y(t))$$
$$E(t, y(t)) = S(t, y(t)) = F(t, y(t)) = T(t, y(t)) \quad \text{For } t \in \Omega \ldots\ldots\ldots(8)$$

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Existence of random fixed point: Consider for $t \in \Omega$

$$\|E(t, y(t) - y(t))\|^2 = \|E(t, y(t) - y_{2n+1} + y_{2n+1} - y(t))\|^2$$

$$\leq 2\|E(t, y(t) - F(t, g_{2n+1}(t))\|^2 + 2\|y_{2n+1} - y(t))\|^2$$

$$\leq 2\alpha \max\{ \frac{\|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2 + \|E(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 + \|S(t, g_{2n}(t)) - E(t, g_{2n}(t))\|^2}{1 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2 \|E(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 \|S(t, g_{2n}(t)) - E(t, g_{2n}(t))\|^2},$$

$$\frac{\|E(t, g_{2n}(t) - T(t, g_{2n+1}(t))\| \; \|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\|}{1 + \|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\| \; \|E(t, g_{2n}(t) - T(t, g_{2n+1}(t)\|}$$

$$\frac{[\|T(t, g_{2n+1}(t) - F(t, g_{2n+1}(t)\|^2 + \|S(t, g_{2n}(t) - E(t, g_{2n}(t)\|^2]}{+ \|S(t, g_{2n}(t) - E(t, g_{2n}(t)\|^2 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2}\}$$

$$+ 2\beta\{\|S(t, g_{2n}(t) - E(t, g_{2n}(t))\|^2 + \|T(t, g_{2n+1}(t)) - F(t, g_{2n+1}(t))\|^2\}$$

$$+ 2\gamma\|S(t, g_{2n}(t) - T(t, g_{2n+1}(t))\|^2 + 2\|y_{2n+1} - y(t))\|^2$$

$$\leq 2\alpha \max\{ \frac{\|y(t)) - y(t))\|^2 + \|E(t, y(t)) - y(t))\|^2 + \|S(t, y(t)) - E(t, y(t))\|^2}{1 + \|y(t) - y(t))\|^2 \|E(t, y(t)) - y(t))\|^2 \|S(t, y(t)) - E(t, y(t))\|^2},$$

$$\frac{\|E(t, y(t)) - y(t)\| \; \|y(t) - y(t)\|}{1 + \|y(t) - y(t)\| \; \|E(t, y(t)) - y(t)\|}$$

$$\frac{[\|y(t) - y(t)\|^2 + \|S(t, y(t)) - E(t, y(t))\|^2]}{+ \|S(t, y(t) - E(t, y(t)\|^2 + \|y(t) - y(t))\|^2}\}$$

$$+ 2\beta\{\|S(t, y(t)) - E(t, y(t))\|^2 + \|y(t) - y(t))\|^2\} \qquad + 2\gamma\|S(t, y(t)) - y(t))\|^2$$

$$+ 2\|y_{2n+1} - y(t))\|^2$$

Therefore for $t \in \Omega$

$$\|E(t, y(t) - y(t))\|^2 \leq 2(\alpha + \gamma)\|E(t, y(t) - y(t))\|^2$$

$$\therefore (1 - 2\alpha - 2\gamma)\|E(t, y(t) - y(t))\|^2 \leq 0$$

$$\therefore \|E(t, y(t) - y(t))\|^2 = 0 \qquad \text{as } (1 - 2\alpha - 2\gamma) \prec 1 \qquad \therefore (\alpha + \gamma) \succ 0$$

Common random fixed point theorem

E(t,y(t))=y(t) for $t \in \Omega$ ---(9)

From (8) and (9) we have for all $t \in \Omega$ E(t,y(t))=y(t)= S(t,y(t)) ……………………………..(10)

In an exactly similar way, we can prove that for all $t \in \Omega$ F(t,y(t))=y(t)= T(t,y(t)) ……………………………..(11)

Again, if A : Ω×C → C is a continuous random operator on a nonempty closed subset C of a separable Hilbert space H, then for any measurable function f : Ω → C, the function h(t) = A(t, f(t)) is also measurable [4].

It follows from the construction of $\{y_n(t)\}$ for $t \in \Omega$, (by (4)) and above consideration that $\{y_n(t)\}$ is a sequence of measurable function. From (7), it follows that $y(t)$ for $t \in \Omega$, is also measurable function. This fact along with (10) and (11) shows that g: Ω → C is a common random fixed point of E, F, S and T.

Uniqueness:

Let h: Ω → C be another random fixed point common to E, F, T and S, that is, for $t \in \Omega$,

$F(t,\ h(t)) = h(t), F(t,\ h(t)) = h(t),\ T(t,\ h(t)) = \ h(t)\ and\ S(t,\ h(t)) = h(t)$ …………..(12) Then for $t \in \Omega$

$$\|g(t)) - h(t))\|^2 = \|E(t, g(t)) - F(t, h(t))\|^2$$

$$\le \alpha \max\{ \frac{\|T(t,h(t)) - F(t,h(t))\|^2 + \|E(t,g(t)) - T(t,h(t))\|^2 + \|S(t,g(t)) - E(t,g(t))\|^2}{1 + \|T(t,h(t) - F(t,h(t))\|^2 \|E(t,g(t)) - T(t,h(t))\|^2 \|S(t,g(t)) - E(t,g(t))\|^2},$$

$$\frac{\|E(t,g(t)) - T(t,h(t)\| \ \|T(t,h(t) - F(t,h(t)\|}{1 + \|E(t,g(t) - T(t,h(t)\| \ \|T(t,h(t)) - F(t,h(t)\|}$$

$$\frac{[\|T(t,h(t) - F(t,h(t)\|^2 + \|S(t,g(t)) - E(t,g(t))\|^2]}{+ \|S(t,g(t) - E(t,g(t)\|^2 + \|T(t,h(t) - F(t,h(t))\|^2}\}$$

$$+ \beta\{\|S(t,g(t)) - E(t,g(t))\|^2 + \|T(t,h(t)) - F(t,h(t))\|^2\} + \gamma\|S(t,g(t)) - T(t,h(t))\|^2$$

$$\le \alpha\|g(t) - h(t)\|^2 + \gamma\|g(t) - h(t))\|^2$$

$$= (\alpha + \gamma)\|g(t) - h(t)\|^2$$

$$\therefore (1 - \alpha - \gamma)\|g(t) - h(t)\|^2 \le 0$$

$$\therefore g(t) = h(t)$$

This completes the proof of the theorem (2.1).

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