On Fixed Point theorems in Fuzzy 2-Metric Spaces and Fuzzy 3-Metric Spaces

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Abstract

In this paper, we give some new definitions of compatible mappings of types (I) and (II) in fuzzy-2 metric space and fuzzy-3 metric space prove some common fixed point theorems for mappings under the condition of compatible mappings of types (I) and (II) in complete fuzzy-2 metric space and fuzzy-3 metric space. Our results extend, generalize and improve the corresponding results given by many authors.

Keywords: Fuzzy metric space, Fuzzy 2-metric space, fuzzy-3 metric space ,Compatible mappings, Common fixed point.

1. Introduction

Impact of fixed point theory in different branches of mathematics and its applications is immense. The first result on fixed points for contractive type mapping was the much celebrated Banach's contraction principle by S. Banach [10] in 1922. In the general setting of complete metric space, this theorem runs as the follows, Theorem 1.1(Banach's contraction principle) Let (X, d) be a complete metric space, c∈ (0, 1) and f: X→X be a mapping such that for each x, y∈X, d $(fx, fy) \le$ c d(x, y) Then f has a unique fixed point a∈X, such that for each x∈X,$\lim_{n\to\infty} f^n x = a$. After the classical result, R.Kannan [11] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions. In 2002, A. Branciari [1] analyzed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

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2 Preliminary Notes

Definition 2.1:(a) A binary operation $*$: $[0,1]\times[0,1]\times[0,1] \to [0,1]$ is a continuous t-norms if ([0,1] ,$*$) is an abeelian topological nonoid with unit 1 such that

$a_1 * b_1 * c_1 \le a_2 * b_2 * c_2$ whenever $a_1 \le a_2$, $b_1 \le b_2$ and $c_1 \le c_2$

for all a_1, a_2, b_1,b_2 ,c_1, c_2 are in [0,1].

Definition 2.2:(a) A 3-tuple (X,M,$*$) is said to be a fuzzy 2- metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^3\times(0,\infty)$ satisfying the following conditions:

for all x,y,z,t ϵ X and $t_1,t_2,t_3 > 0$,

(1)M(x,y,z,t) > 0;

(2)M(x,y,z,t) = 1, t > 0 when at least two of the three points are equal

(3) M(x,y,z,t) = M(x,z,y,t) = M(y,z,x,t)

(4)$M(x,y,z,t_1) * M(x,u,z,t_2) * M(u,y,z,t_3) \le M(x,y,z,t_1+t_2+t_3)$

The function value M(x,y,z,t) may be interpreted as the probability that the area of triangle is less than t.

(5)M(x,y,z,.): [0,1)→[0,1] is left continuous.

Definition2.3: (a)[08] Let (X, M,*) be a fuzzy- 2 metric space.

(1) A sequence $\{x_n\}$ in fuzzy -2 metric space X is said to be convergent to a point x ∈ X (denoted by

$$\lim_{n\to\infty} x_n = x \quad or \quad x_n \to x$$

if for any $\lambda \in (0,1)$ and t > 0, there exists n_0 ∈ N such that for all n ≥ n_0 and a ∈ X, M $(x_n, x, \text{a}, \text{t}) > 1 - \lambda$.

That is

$$\lim_{n\to\infty} \text{M}(x_n, x, \text{a}, \text{t}) = 1 \text{ for all a} \in \text{X and t} > 0.$$

(2) A sequence $\{x_n\}$ in fuzzy- 2 metric space X is called a Cauchy sequence, if for any $\lambda \in (0,1)$ and t > 0, there exists n_0 ∈ N such that for all m, n ≥ n_0 and a ∈ X, M $(x_n, x_m, \text{a}, \text{t}) > 1 - \lambda$.

(3) A fuzzy- 2 metric space in which every Cauchy sequence is convergent is said to be complete.

Definition2.4: (a)[08] Self mappings A and B of a fuzzy- 2 metric space (X, M,*) is said to be compatible, if

$$\lim_{n\to\infty} \text{M}(\text{AB}x_n, \text{BA}x_n, \text{a}, \text{t}) = 1 \; for\ all\ a \in X\ and\ t > 0,$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = z \quad for\ some\ z \in X.\ Then \quad \lim_{n\to\infty} ABx_n = Bz.$$

Definition 2.5:(a) Let (X, M, $*$) is a fuzzy-2 metric space.Then

(a) A sequence $\{x_n\}$ in X is said to convers to x in X if for each ϵ>o and each t>o, $\exists n_o \in N$ such That $M(x_n,x,t)>1-\epsilon$ for all $n\geq n_o$.

(b)a sequence $\{x_n\}$ in X is said to cauchy to if for each ϵ>o and each t>o, $\exists n_o \in N$ such That $M(x_n,x_m,t)>1-\epsilon$ for all $n,m\geq n_o$.

(c) A fuzzy metric space in which euery Cauchy sequence is convergent is said to be complete.

Definition 2.6:(a)[3] Two self mappings f and g of a fuzzy metric space (X,M,$*$) are called compatible if $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequencein X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ For some x in X.

Definition 2.7:(a)[1]Twoself mappings f and g of a fuzzy metric space (X,M,$*$) are called reciprocally continuous on X if $\lim_{n\to\infty} fgx_n = fx$ and $\lim_{n\to\infty} gfx_n = gx$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = x$ for some x in X.

Lemma2.2.1: [08] Let (X, M,*) be a fuzzy- 2 metric space. If there exists q ∈ (0, 1) such that $M(x, y, z, qt + 0) \geq M(x, y, z, t)$ for all x, y, z ∈ X with z ≠ x, z ≠ y and t > 0, then x = y,

Lemma 2.2.2:[4] Let X be a set, f,g owc self maps of X. If f and g have a unique point of coincidence, w = fx = gx, then w is the unique common fixed point of f and g.

3 Main Results

Theorem 3.1:(a) Let (X, M, $*$) be a complete fuzzy 2-metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

$$M(Px,Ry,a,qt)\geq \min\{ M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t)\} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

For all x,y ϵ X and for all t > o, then there exists a unique point w ϵ X such that Pw = Sw = w and a unique point z ϵ X such that Rz = Tz = z. Moreover z = w so that there is a unique common fixed point of P,R,S and T.

Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px=Sx andRy=Ty. We claim thatPx=Ry. If not, by inequality (1)

$$\begin{aligned} M(Px,Ry,a,qt) &\geq \min\{ M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t)\} \\ &\geq \min\{ M(Px,Ry,a,t), M(Px,Px,a,t), M(Ty,Ty,a,t), M(Px,Ry,a,t), M(Ry,Px,a,t), M(Px,Ry,a,t), M(Px,Ry,a,t)* M(Px,Px,a,t)\} \\ &\geq \min\{ M(Px,Ry,a,t), M(Px,Px,a,t), M(Ty,Ty,a,t), M(Px,Ry,a,t), M(Px,Ry,a,t),M(Px,Ry,a,t), M(Px,Ry,a,t)*1\} \\ &=M(Px,Ry,a,t). \end{aligned}$$

Therefore Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (1) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.By Lemma 2.8 w is the only common fixed point of P and S.Similarly there is a unique point zϵX such that z = Rz = Tz.

Assume that w ≠ z. we have

$$\begin{aligned} M(w,z,a,qt) &= M(Pw,Rz,a,qt) \\ &\geq\min\{ M(Sw,Tz,a,t), M(Sw,Pw,a,t), M(Rz,Tz,a,t), M(Pw,Tz,a,t), M(Rz,Sw,a,t), M(Pw,Rz,a,t), M(Sw,Tz,a,t)* M(Pw,Pw,a,t)\} \\ &\geq \min\{ M(w,z,a,t), M(w,w,a,t), M(z,z,a,t), M(w,z,a,t), M(z,w,a,t), M(w,z,a,t), M(w,z,a,t)* M(w,w,a,t)\} \end{aligned}$$

$=M(w,z,a,t)$.

Therefore we have z = w and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.2:(a) Let (X, M, $*$) be a complete fuzzy 2- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

$$M(Px,Ry,a,qt) \geq \emptyset\,(\min\{ M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t)) \ldots\ldots\ldots\ldots\ldots(2)$$

For all x,y ϵ Xand $\emptyset : [0,1]\rightarrow [0,1]$ such that $\emptyset(t) > t$ for all $0< t < 1$, then there existsa unique common fixed point of P,R,S and T.

Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (2)

$$M(Px,Ry,a,qt)\geq \emptyset\,(\min\{ M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t)\})$$
$$>\emptyset\,(M(Px,Ry,a,t)). \qquad \text{From Theorem 3.1}$$
$$=M(Px,Ry,a,t).$$

Assume that w $\neq$ z. we have

$$M(w,z,a,qt) = M(Pw,Rz,a,qt)$$

$$\geq \emptyset\,(\min\{ M(Sw,Tz,a,t), M(Sw,Pw,a,t), M(Rz,Tz,a,t), M(Pw,Tz,a,t), M(Rz,Sw,a,t), M(Pw,Rz,a,t), M(Sw,Tz,a,t)* M(Pw,Pw,a,t)\})$$

$$=M(w,z,a,t). \qquad \text{From Theorem 3.1}$$

Therefore we have z = w and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.3:(a) Let (X, M, $*$) be a complete fuzzy 2- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

$$M(Px,Ry,a,qt) \geq \emptyset\,(M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t)) \ldots\ldots\ldots\ldots\ldots(3)$$

For all x,y ϵ X and $\emptyset: [0,1]^7\rightarrow[0,1]$ such that $\emptyset(t,1,1,t,t,1,t) > t$ for all $0 < t < 1$, then there exists a unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (3)

$$M(Px,Ry,a,qt) \geq \emptyset\,(M(Sx,Ty,a,t), M(Sx,Px,a,t), M(Ry,Ty,a,t), M(Px,Ty,a,t), M(Ry,Sx,a,t), M(Px,Ry,a,t), M(Sx,Ty,a,t)* M(Px,Px,a,t))$$
$$\geq \emptyset(M(Px,Ry,a,t), M(Px,Px,a,t), M(Ty,Ty,a,t), M(Px,Ry,a,t), M(Ry,Px,a,t), M(Px,Ry,a,t), M(Px,Ry,a,t)* M(Px,Px,a,t))$$
$$\geq \emptyset(M(Px,Ry,a,t), M(Px,Px,a,t), M(Ty,Ty,a,t), M(Px,Ry,a,t), M(Px,Ry,a,t),M(Px,Ry,a,t), M(Px,Ry,a,t)*1)$$
$$= \emptyset(M(Px,Ry,a,t), 1, 1, M(Px,Ry,a,t), M(Px,Ry,a,t),M(Px,Ry,a,t), M(Px,Ry,a,t))$$
$$=M(Px,Ry,a,t).$$

A contradiction, therefore Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (3) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.By Lemma 2.8 w is the only common fixed point of P and S.Similarly there is a unique point z ϵ X such that z = Rz = Tz.Thus z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds from (3).

Theorem 3.4:(a) Let (X, M, $*$) be a complete fuzzy 2- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists qϵ(0,1) for all x,y ϵ X and $t > 0$

$M(Px,Ry,a,qt) \geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Ry,Sx,a,t)_*$
$M(Px,Ry,a,t)_* \ M(Sx,Ty,a,t)$ ………………… (4)

Then there existsa unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points x,yϵX such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (4)

We have

$M(Px,Ry,a,qt) \geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Ry,Sx,a,t)_*$
$M(Px,Ry,a,t)_* \ M(Sx,Ty,a,t)$
$= M(Px,Ry,a,t)_* \ M(Px,Px,a,t)_* \ M(Ty,Ty,a,t)_* \ M(Px,Ry,a,t)_* \ M(Ry,Px,a,t)_*$
$M(Px,Ry,a,t)_* \ M(Px,Ry,a,t)$
$= M(Px,Ry,a,t)_* \ 1_* \ 1_* \ M(Px,Ry,a,t)_* \ M(Ry,Px,a,t)_* M(Px,Ry,a,t)_* \ M(Px,Ry,a,t)$

$> M(Px,Ry,a,t).$

Thus we have Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (4) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.Similarly there is a unique point zϵX such that z = Rz = Tz.Thus w is a common fixed point of P,R,S and T.

Corollary 3.5:(a) Let (X, M, ∗) be a complete fuzzy 2- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) for all x,y ϵ X and t > 0

$M(Px,Ry,a,qt) \geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Ry,Sx,a,2t)_*$
$M(Px,Ry,a,t)_* \ M(Sx,Ty,a,t)$ …………………(5)

Then there existsa unique common fixed point of P,R,S and T.

Proof: We have

$M(Px,Ry,a,qt) \geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Ry,Sx,a,2t)_*$
$M(Px,Ry,a,t)_* \ M(Sx,Ty,a,t)$
$\geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Sx,Ty,a,t)_*$
$M(Ty,Ry,a,t)_* \ M(Px,Ry,a,t)_* \ M(Sx,Ty,a,t)$
$\geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Ry,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Px,Ry,a,t)_*$
$M(Sx,Ty,a,t)$
$= M(Px,Ry,a,t)_* \ M(Px,Px,a,t)_* \ M(Ty,Ty,a,t)_* \ M(Px,Ry,a,t)_* \ M(Ry,Px,a,t)_*$
$M(Px,Ry,a,t)_* \ M(Px,Ry,a,t)$
$= M(Px,Ry,a,t)_* \ 1_* \ 1_* \ M(Px,Ry,a,t)_* \ M(Ry,Px,a,t)_*$
$M(Px,Ry,a,t)_* \ M(Px,Ry,a,t)$

$> M(Px,Ry,a,t).$

And therefore from theorem 3.4, P,R,S and T have a common fixed point.

Corollary 3.6:(a) Let (X, M, ∗) be a complete fuzzy 2-metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists qϵ(0,1) for all x,yϵ X and t > 0

$M(Px,Ry,a,qt) \geq M(Sx,Ty,a,t)$ …………………(6)

Then there existsa unique common fixed point of P,R,S and T.

Proof: The Proof follows from Corollary 3.5

Theorem 3.7:(a) Let (X, M, ∗) be a complete fuzzy 2- metric space.Then continuous self-mappings S and T of X have a common fixed point in X if and only if there exites a self mapping P of X such that the following conditions are satisfied

(i) $PX \subset TX \cap SX$

(ii) The pairs {P,S} and {P,T} are weakly compatible,

(iii) There exists a point qϵ(0,1) such that for all x,yϵX and t > 0

$M(Px,Py,a,qt) \geq M(Sx,Ty,a,t)_* \ M(Sx,Px,a,t)_* \ M(Py,Ty,a,t)_* \ M(Px,Ty,a,t)_* \ M(Py,Sx,a,t)$ …………………(7)

Then P,S and T havea unique common fixed point.

Proof: Since compatible implies ows, the result follows from Theorem 3.4

Theorem 3.8:(a) Let (X, M, $*$) be a complete fuzzy 2- metric space and let P and R be self-mappings of X. Let the P and R areowc.If there exists q ϵ (0,1) for all x,y ϵ X and t > 0

M(Sx,Sy,a,qt) ≥αM(Px,Py,a,t)+β min{M(Px,Py,a,t), M(Sx,Px,a,t), M(Sy,Py,a,t), M(Sx,Py,a,t)} …………………(8)

For all x,y ϵ X where α,β > 0, α+β > 1. Then P and S have a unique common fixed point.

Proof: Let the pairs {P,S} be owc, so there are points x ϵX such that Px = Sx. Suppose that exist another point y ϵX for whichPy = Sy. We claim that Sx = Sy. By inequality (8)
We have
M(Sx,Sy,a,qt) ≥ α M(Px,Py,a,t) + β min{M(Px,Py,a,t) , M(Sx,Px,a,t), M(Sy,Py,a, M(Sx,Py,a,t)}
=α M(Sx,Sy,a,t) + β min{M(Sx,Sy,a,t) , M(Sx,Sx,a,t), M(Sy,Sy,a,t), M(Sx,Sy,a,t)}
=(α+β)M(Sx,Sy,a,t)

A contradiction, since (α+β) > 1.Therefore Sx = Sy. Therefore Px = Py and Px is unique.
From lemma 2.2.2 , P and S have a unique fixed point.

Definition 3.1:(b)[09]: The 3- tuple (X, M,*) is called a fuzzy-3 metric space if X is an arbitrary set, * is a continuous
t-norm and M is a fuzzy set in $X^4 \times [0,\infty)$ satisfying the following conditions, for all x, y, z, w, u ∈ X and $t_1, t_2, t_3, t_4 > 0$.
(1) M(x, y, z, w, 0) = 0,
(2) M(x, y, z, w, t) =1 for all t > 0, [only when the three simplex (x, y, z, w) degenerate]
(3) M(x, y, z, t) = M(x,w, z, y, t) = M(y, z,w, x, t) = M(z,w, x, y, t) = ...
(4) M(x, y, z, w,$t_1 + t_2 + t_3 + t_4$) ≥ M(x, y, z, w, t_1) * M(x, y, z, w,t_2) * M(x, y, z, w,t_3) * M(x, y, z, w,t_4)
(5) M(x, y, z, w): [0, 1) → [0, 1] is left continuous.

Definition 3.2:(b)[09]: Let (X, M,*) be a fuzzy 3-metric space, then
(1) A sequence $\{x_n\}$ in fuzzy 3-metric space X is said to be convergent to a point x∈ X if

$$\lim_{n\to\infty} \mathrm{M}\ (x_n, x, \mathrm{a}, \mathrm{b}, \mathrm{t}) = 1\ for\ all\ a, b \in X\ and\ t > 0,$$

(2) A sequence $\{x_n\}$ in fuzzy 3-metric space X is called a Cauchy sequence, if

$$\lim_{n\to\infty} \mathrm{M}\ (x_{n+p}, x_n, \mathrm{a}, \mathrm{b}, \mathrm{t}) = 1\ for\ all\ a, b \in X\ and\ t > 0, p > 0.$$

(3) A fuzzy 3-metric space in which every Cauchy sequence is convergent is said to be complete.

Definition 3.3:(b)[09]: A function M is continuous in fuzzy 3-metric space iff whenever $x_n \to x$, $y_n \to y$, then

$$\lim_{n\to\infty} \mathrm{M}\ (x_n, y_n, \mathrm{a}, \mathrm{b}, \mathrm{t}) = \mathrm{M}\ (x, y, \mathrm{a}, \mathrm{b}, \mathrm{t})\ for\ all\ a, b \in X\ and\ t > 0.$$

Definition 3.4:(b)[09]: Two mappings A and S on fuzzy 3-metric space are weakly commuting iff M(ASu, SAu,a,b,t) ≥ M(Au, Su,a,b,t) for all u, a, b∈ X and t > 0 .

3 Main Results

Theorem 3.1(b) Let (X, M, $*$) be a complete fuzzy 3-metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that
M(Px,Ry,a,b,qt)≥ min{ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t), M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)$*$ M(Px,Px,a,b,t)} ……………(1)
For all x,y ϵ X and for all t > o, then there exists a unique point w ϵ X such that Pw = Sw = w and a unique point z ϵ X such that Rz = Tz = z. Moreover z = w so that there is a unique common fixed point of P,R,S and T.
Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx andRy = Ty. We claim that Px = Ry. If not, by inequality (1)

M(Px,Ry,a,b,qt)≥ min{ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t),M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)$*$M(Px,Px,a,b,t)}

= min{ M(Px,Ry,a,b,t), M(Px,Px,a,b,t), M(Ty,Ty,a,b,t), M(Px,Ry,a,b,t), M(Ry,Px,a,b,t), M(Px,Ry,a,b,t), M(Px,Ry,a,b,t)$*$ M(Px,Px,a,b,t)}

= min{ M(Px,Ry,a,b,t), M(Px,Px,a,b,t), M(Ty,Ty,a,b,t), M(Px,Ry,a,b,t), M(Ry,Px,a,b,t), M(Px,Ry,a,b,t), M(Px,Ry,a,b,t)$*$1}

=M(Px,Ry,a,b,t).

Therefore Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (1) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.By Lemma 2.2.2 w is the only common fixed point of P and S.Similarly there is a unique point z ϵ X such that z = Rz = Tz.

Assume that w $\neq$ z. we have

M(w,z,a,b,qt) = M(Pw,Rz,a,b,qt)

≥min{ M(Sw,Tz,a,b,t), M(Sw,Pw,a,b,t), M(Rz,Tz,a,b,t), M(Pw,Tz,a,b,t), M(Rz,Sw,a,b,t), M(Pw,Rz,a,b,t), M(Sw,Tz,a,b,t)$*$ M(Pw,Pw,a,b,t)}

= min{ M(w,z,a,b,t), M(w,w,a,b,t), M(z,z,a,b,t), M(w,z,a,b,t), M(z,w,a,b,t), M(w,z,a,b,t), M(w,z,a,b,t)$*$ M(w,w,a,b,t)}

=M(w,z,a,b,t).

Therefore we have z = w and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.2(b) Let (X, M, $*$) be a complete fuzzy 3- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

M(Px,Ry,a,b,qt) ≥ Ø (min{ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t), M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)$*$ M(Px,Px,a,b,t)})……………(2)

For all x,y ϵ X and Ø : [0,1]→ [0,1] such that Ø(t) > t for all 0< t < 1, then there existsa unique common fixed point of P,R,S and T.

Proof : Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (2)

M(Px,Ry,a,b,qt) ≥ Ø (min{ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t), M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)$*$ M(Px,Px,a,b,t)})

>Ø (M(Px,Ry,a,b,t)). From Theorem 3.1

=M(Px,Ry,a,b,t).

Assume that w $\neq$ z. we have

M(w,z,a,b,qt) = M(Pw,Rz,a,b,qt)

≥ Ø (min{ M(Sw,Tz,a,b,t), M(Sw,Pw,a,b,t), M(Rz,Tz,a,b,t), M(Pw,Tz,a,b,t), M(Rz,Sw,a,b,t), M(Pw,Rz,a,b,t), M(Sw,Tz,a,b,t)$*$ M(Pw,Pw,a,b,t)})

=M(w,z,a,b,t). From Theorem 3.1

Therefore we have z = w and z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds.

Theorem 3.3(b) Let (X, M, $*$) be a complete fuzzy 3- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) such that

M(Px,Ry,a,b,qt) ≥ Ø ({ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t), M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)$*$ M(Px,Px,a,b,t)}) ……………(3)

For all x,y ϵ X and Ø: $[0,1]^7 \to [0,1]$ such that Ø(t,1,1,t,t,1,t) > t for all 0 < t < 1, then there exists a unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (3)

M(Px,Ry,a,b,qt) ≥ Ø ({ M(Sx,Ty,a,b,t), M(Sx,Px,a,b,t), M(Ry,Ty,a,b,t), M(Px,Ty,a,b,t), M(Ry,Sx,a,b,t), M(Px,Ry,a,b,t), M(Sx,Ty,a,b,t)∗ M(Px,Px,a,b,t)})

= Ø ({M(Px,Ry,a,b,t), M(Px,Px,a,b,t), M(Ty,Ty,a,b,t), M(Px,Ry,a,b,t), M(Ry,Px,a,b,t), M(Px,Ry,a,b,t), M(Px,Ry,a,b,t)∗ M(Px,Px,a,b,t)})

= Ø ({M(Px,Ry,a,b,t), M(Px,Px,a,b,t), M(Ty,Ty,a,b,t), M(Px,Ry,a,b,t), M(Ry,Px,a,b,t), M(Px,Ry,a,b,t), M(Px,Ry,a,b,t)∗1})

=M(Px,Ry,a,b,t).

A contradiction, therefore Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (3) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.By Lemma 2.8 w is the only common fixed point of P and S.Similarly there is a unique point z ϵ X such that z = Rz = Tz.Thus z is a common fixed point of P,R,S and T. The uniqueness of the fixed point holds from (3).

Theorem 3.4(b) Let (X, M, ∗) be a complete fuzzy 3- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists qϵ(0,1) for all x,y ϵ X and t > 0

M(Px,Ry,a,b,qt) ≥ M(Sx,Ty,a,b,t)∗M(Sx,Px,a,b,t)∗M(Ry,Ty,a,b,t)∗M(Px,Ty,a,b,t)∗ M(Ry,Sx,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Sx,Ty,a,b,t) ………………… (4)

Then there existsa unique common fixed point of P,R,S and T.

Proof: Let the pairs {P,S} and {R,T} be owc, so there are points x,y ϵ X such that Px = Sx and Ry = Ty. We claim that Px = Ry. If not, by inequality (4)

We have

M(Px,Ry,a,b,qt) ≥ M(Sx,Ty,a,b,t)∗M(Sx,Px,a,b,t)∗M(Ry,Ty,a,b,t)∗M(Px,Ty,a,b,t)∗ M(Ry,Sx,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Sx,Ty,a,b,t)

= M(Px,Ry,a,b,t)∗ M(Px,Px,a,b,t)∗ M(Ty,Ty,a,b,t)∗ M(Px,Ry,a,b,t)∗ M(Ry,Px,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Px,Ry,a,b,t)

= M(Px,Ry,a,b,t)∗ 1∗ 1∗ M(Px,Ry,a,b,t)∗ M(Ry,Px,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Px,Ry,a,b,t)

> M(Px,Ry,a,b,t).

Thus we have Px = Ry, i.e. Px = Sx = Ry = Ty. Suppose that there is a another point z such that Pz = Sz then by (4) we have Pz = Sz = Ry = Ty, so Px=Pz and w = Px = Sx is the unique point of coincidence of P and S.Similarly there is a unique point z ϵ X such that z = Rz = Tz.Thus w is a common fixed point of P,R,S and T.

Corollary 3.5(b) Let (X, M, ∗) be a complete fuzzy 3- metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists q ϵ (0,1) for all x,y ϵ X and t > 0

M(Px,Ry,a,b,qt) ≥ M(Sx,Ty,a,b,t)∗M(Sx,Px,a,b,t)∗M(Ry,Ty,a,b,t)∗M(Px,Ty,a,b,t)∗ M(Ry,Sx,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Sx,Ty,a,b,t)…………………(5)

Then there exists a unique common fixed point of P,R,S and T.

Proof: We have

M(Px,Ry,a,b,qt) ≥ M(Sx,Ty,a,b,t)∗M(Sx,Px,a,b,t)∗M(Ry,Ty,a,b,t)∗M(Px,Ty,a,b,t)∗ M(Ry,Sx,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Sx,Ty,a,b,t)

= M(Px,Ry,a,b,t)∗ M(Px,Px,a,b,t)∗ M(Ty,Ty,a,b,t)∗ M(Px,Ry,a,b,t)∗ M(Ry,Px,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Px,Ry,a,b,t)

= M(Px,Ry,a,b,t)∗ 1∗ 1∗ M(Px,Ry,a,b,t)∗ M(Ry,Px,a,b,t)∗M(Px,Ry,a,b,t)∗ M(Px,Ry,a,b,t)

> M(Px,Ry,a,b,t).

And therefore from theorem 3.4, P,R,S and T have a common fixed point.

Corollary 3.6(b) Let (X, M, ∗) be a complete fuzzy 3-metric space and let P,R,S and T be self-mappings of X. Let the pairs {P,S} and {R,T} be owc.If there exists qϵ(0,1) for all x,y ϵ X and t > 0

M(Px,Ry,a,qt) ≥ M(Sx,Ty,a,t) …………………(6)

Then there existsa unique common fixed point of P,R,S and T.
Proof: The Proof follows from Corollary 3.5
Theorem 3.7(b) Let (X, M, $*$) be a complete fuzzy 3- metric space.Then continuous self-mappings S and T of X have a common fixed point in X if and only if there exites a self mapping P of X such that the following conditions are satisfied
(i) $PX \subset TX \cap SX$
(ii) The pairs {P,S} and {P,T} are weakly compatible,
(iii) There exists a point q ϵ (0,1) such that for all x,y ϵ X and t > 0
$M(Px,Py,a,b,qt) \geq M(Sx,Ty,a,b,t) * M(Sx,Px,a,b,t) * M(Py,Ty,a,b,t) * M(Px,Ty,a,b,t) * M(Py,Sx,a,b,t)$…………………(7)

Then P,S and T have a unique common fixed point.
Proof: Since compatible implies ows, the result follows from Theorem 3.4

Theorem 3.8(b) Let (X, M, $*$) be a complete fuzzy 3- metric space and let P and R be self-mappings of X. Let the P and R are owc. If there exists q ϵ (0,1) for all x,y ϵ X and t > 0

$M(Sx,Sy,a,b,qt) \geq \alpha\, M(Px,Py,a,b,t)+\beta \min\{M(Px,Py,a,b,t), M(Sx,Px,a,b,t), M(Sy,Py,a,b,t), M(Sx,Py,a,b,t)\}$ …………………(8)
For all x,y ϵ X where $\alpha,\beta > 0$, $\alpha+\beta > 1$. Then P and S have a unique common fixed point.

Proof: Let the pairs {P,S} be owc, so there are points x ϵX such that Px = Sx. Suppose that exist another point y ϵX for whichPy = Sy. We claim that Sx = Sy. By inequality (8)
We have

$$M(Sx,Sy,a,b,qt) \geq \alpha\, M(Px,Py,a,b,t)+\beta \min\{M(Px,Py,a,b,t), M(Sx,Px,a,b,t), M(Sy,Py,a,b,t), M(Sx,Py,a,b,t)\}$$
$$=\alpha\, M(Sx,Sy,a,b,t) + \beta \min\{M(Sx,Sy,a,b,t), M(Sx,Sx,a,b,t), M(Sy,Sy,a,b,t), M(Sx,Sy,a,b,t)\}$$
$$=(\alpha+\beta)M(Sx,Sy,a,b,t)$$

A contradiction, since $(\alpha+\beta) > 1$.Therefore Sx = Sy. Therefore Px = Py and Px is unique.
From lemma 2.2.2 , P and S have a unique fixed point.

Acknowledgement: One of the author (Dr. R.K. B.) is thankful to MPCOST Bhopal for the project No 2556

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