Coupled Fixed Point Theorems In Cone D*-Metric Space

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Abstract

Abstract: In this paper, we have proved some coupled fixed point theorems in partially ordered cone D*-Metric spaces. Related important results are also discussed.

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1. Introduction:

In recent years, nonlinear analysis have attracted much attention .The study of non contraction mapping concerning the existence of fixed points draw attention of various authors in non linear analysis.

Huang and Zhang [4] generalized the notion of metric spaces, replacing the real numbers by an ordered Banach space and defined cone metric spaces. Dhage [1, 2] et al. introduced the concept of *D*-metric spaces as generalization of ordinary metric spaces and went on to present several fixed point results for single and multivalued mappings. Mustafa and Sims [5] introduced more appropriate notion of generalized metric space which called a G-metric space and obtained some topological properties. Later in 2007 Shaban Sedghi et.al [7] modified the *D*-metric space and defined *D**-metric spaces and then C.T.Aage and J.N.Salunke [8] generalized the *D**-metric spaces by replacing the real numbers by an ordered Banach space and defined *D**-cone metric spaces and prove the topological properties. Recently Sachin Bedre et al. [17] extended this result and obtain a fixed and common fixed point theorems in D*-metric space.

Bhashkar and Lakshmikantham [14] introduced the concept of a coupled fixed point of a mapping $F : X \times X \to X$ (a non-empty set) and established some coupled fixed point theorems in partially ordered complete metric spaces which can be used to discuss the existence and uniqueness of solution for periodic boundary value problems. Later, Lakshmikantham and ´Ciri´c [15] proved coupled coincidence and coupled common fixed point results for nonlinear mappings $F : X \times X \to X$ and $g : X \to X$ satisfying certain contractive conditions in partially ordered complete metric spaces. Using the concepts of coupled fixed point and coupled coincidence point, we will proved coupled (coincidence) fixed point theorems in *D**-metric spaces.

First, we present some known definitions and propositions in *D**-cone metric spaces. Let *E* be a real Banach space and *P* is a subset of *E*. *P* is called cone if and only if:

(i) *P* is closed, non-empty and $p \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers *a, b*,

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\le$ on *E* with respect to *P* by $x \le y$ if and only if $y - x \in P$. We shall write $x < y$ if $x < y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in \text{int} P$, where $\text{int} P$ denotes the interior of *P*. The cone *P* is called normal if there is a number $K > 0$ such that for all $x, y \in E$, $0 \le x \le y$ implies $\|x\| \le K\|y\|$.

The least positive number *K* satisfying above is called normal constant of *P* [4]. The cone *P* is called regular if every increasing sequence which is bounded above is convergent. That is, if $\{x_n\}_{n \ge 1}$ is a sequence such that $x_1 \le x_2 \le ..... \le y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n \to \infty} \|x_n - x\| = 0$.

The cone *P* is regular if and if every decreasing sequence which bounded below is convergent. Rezapour and Hamlbarani [6] proved every regular cone is normal and there are normal cone with normal constant $M \ge 1$.

In the following we always suppose that *E* is a Banach space, *P* is a cone in *E* with $\text{int} p \neq \{0\}$ and $\le$ is a partial ordering with respect to *P*.

The concept of generalized *D**-metric space is defined as follows

2. Preliminaries:

Definition 2.1 [8]: Let X is a non empty set. A generalized D^*-metric on X is a function, $D^*: X^3 \to E$, that satisfies the following conditions for all $x,y,z,a \in X$:

1) $D^*(x,y,z) \geq 0$,
2) $D^*(x,y,z) = 0 \Leftrightarrow x = y = z$,
3) $D^*(x,y,z) = D^*(p\{x,y,z\})$ (symmetry) where p is a permutation function,
4) $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$.

Then the function D^* is called a generalized D^*-metric and the pair (X,D^*) is called a generalized D^*-metric space

Example 2.2 [8]: Let $E = R^2, P = \{(x,y) \in E : x,y \geq 0\}, X = R$ and $D^*: X \times X \times X \to E$ defined by $D^*(x,y,z) = (|x-y| + |y-z| + |x-z|, a(|x-y| + |y-z| + |x-z|))$, where $a \geq 0$ is a constant. Then (X,D^*) is a generalized D^*- metric space.

Proposition 2.3 [8]: If (X,D^*) be generalized D^*- metric space, then for all $x,y,z \in X$, we have $D^*(x,x,y) = D^*(x,y,y)$

Definition 2.4 [8]: Let (X,D^*) be a generalized D^*- metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $m,n \geq N, D^*(x_m,x_n,x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $x_n \to x\ (n \to \infty)$.

Lemma 2.5 [8]: Let (X,D^*) be a generalized D^*- metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if

$$D^*(x_m,x_n,x) \to 0\ (m,n \to \infty).$$

Lemma 2.6 [8]: Let (X,D^*) be a generalized D^*-metric space then the following are equivalent.

(I) $\{x_n\}$ is D^*-convergent to x.

(ii) $D^*(x_n,x_n,x) \to 0$, as $n \to \infty$

(iii) $D^*(x_n,x,x) \to 0$, as $n \to \infty$

Lemma 2.7 [8]: Let (X,D^*) be a generalized D*- metric space, P be a normal cone with normal constant K and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ Converges to y, then $x = y$. That is the limit of $\{x_n\}$, if exists, is unique.

Definition 2.8 [8]: Let (X,D^*) be a generalized D^*- metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there exist N such that for all $m,n,l > N, D^*(x_m,x_n,x_l) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 2.9 [8]: Let (X,D^*) be a generalized D^*-metric space. If every Cauchy Sequence in X is convergent in X, then X is called a complete generalized D^*-metric space.

Lemma 2.10 [8]: Let (X,D^*) be a generalized D*-metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then $\{x_n\}$ is a Cauchy sequence.

Lemma 2.11 [8]: Let (X,D^*) be a generalized D*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X then $\{x_n\}$ is a Cauchy sequence if and only if $D^*(x_m,x_n,x_l) \to 0\ (m,n,l \to \infty)$.

Definition 2.12 [8]: Let $(X,D^*), (X',D'^*)$ be generalized D^*-metric spaces, then a function $f: X \to X'$ is said to be D^*-continuous at a point $x \in X$ if and only if it is D^*-sequentially continuous at x, that is, whenever $\{x_n\}$ is D^*-convergent to x we have $\{fx_n\}$ is D^*-convergent to fx.

Lemma 2.13 [8]: Let (X,D^*) be a generalized D*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be three sequences in X and $x_n \to x, y_n \to y, z_n \to z\ (n \to \infty)$. Then

$D^*(x_n, y_n, z_n) \quad D^*(x, y, z) (n \quad ?)$.

Definition 2.14 [3]: Let *f* and *g* be self maps of a set *X*. If $w = fx = gx$ for some *x* in *X*, then *x* is called a coincidence point of *f* and *g*, and *w* is called a point of coincidence of *f* and *g*.

Proposition 2.15 [8]: Let *f* and *g* be weakly compatible self maps of a set *X*. If *f* and *g* have a unique point of coincidence $w = fx = gx$, then *w* is the unique common fixed point of *f* and *g*.

Definition 2.17: Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$. The mapping *F* is said to have the mixed monotone property if *F* is monotone property in *x* and *F* is monotone non-increasing in *y*, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$$

Definition 2.18: An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $x = f(x, y) \text{ and } y = f(y, x)$.

Definition 2.19: Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping *F* is said to have the mixed g-monotone property if *F* is monotone g-non-decreasing in its first argument and *F* is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X,\ gx_1 \leq gx_2 \Rightarrow F(x_1, y) \leq F(x_2, y)$$

$$y_1, y_2 \in X,\ gy_1 \leq gy_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$$

Definition 2.20: An element $(x, y) \in X \times X$ is called

(1) a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$.

(2) a coupled common fixed point of the mapping $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 2.21: The mappings *F* and *g* where $F : X \times X \to X$, $g : X \to X$ are said to commute if $F(gx, gy) = g(Fx, Fy)$ for all $x, y \in X$

Definition 2.22: The mappings *F* and *g* where $F : X \times X \to X$, $g : X \to X$ are said to be *W*-compatible if $gF(x, y) = F(gy, gx)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 2.23: Let (X, D^*) be a cone *D**-Metric space. The mapping *F* and *g* where $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n \to \infty} D^*(gF(x_n, y_n), F(gx_n, gy_n)) = 0 \text{ and } \lim_{n \to \infty} D^*(gF(y_n, x_n), F(gy_n, gx_n)) = 0,$$

Where $\{x_n\} \text{ and } \{y_n\}$ are sequences in X such that $\lim_{n \to \infty} F(x_n, y_n) = \lim_{n \to \infty} gx_n = x \text{ and } \lim_{n \to \infty} F(y_n, x_n) = \lim_{n \to \infty} gy_n = y$ for all $x, y \in X$ are satisfied.

It is easy to see that if *F* and *g* commute then they are compatible.

In the present paper we are proving some new results

3. Main Result:

Theorem: Let $(X, £)$ be partially set and suppose there is a metric *D** such that (X, D^*) is complete cone *D**-metric space. Let $F : X 串X \quad X$ and $g : X ® \quad X$ be such that *F* has mixed *G*-monotone property and there exist non-negative constants a, b, g, d, h, l satisfying $a + b + g + d + 2h < 1$ such that

$$D^*(F(x, y), F(u, v), f(z, w)) ? \ aD^*(gx, gu, gz) \quad bD^*(F(x, y), gx, gu) + gD^*(gy, gv, gw)$$
$$+ dD^*(F(u, v), gu, gz) + hD^*(F(y, x), gv, gw) + lD^*(F(u, v), gx, gz) \quad \ldots (1)$$

$\forall x,y,z,u,v,w \in X$ with $gx \geq gu$ and $gy \leq gv$. Further suppose that $F(X \times X) \subseteq g(X)$, g is continuous and g, f is compatible. Suppose either

a) F is continuous or

b) X has the following property

i) If $\{x_n\}$ is non-decreasing sequence and $\lim_n x_n = x$ then $gx_n \leq gx$ for all n.

ii) If $\{y_n\}$ is a non-increasing sequence $\lim_n y_n = y$ then $gy \leq gy_n$ for all n.

If there exist $x_0, y_0 \in X$ such that $gx_0 \geq F(x_0,y_0)$ and $gy_0 \leq F(y_0,x_0)$ then F and g have coupled coincident point.

Proof:

Let $x_0, y_0 \in X$ be such that $gx_0 \geq F(x_0,y_0)$ and $gy_0 \leq F(y_0,x_0)$. Since $F(X \times X) \subseteq g(X)$, we construct sequence $\{x_n\}$, $\{y_n\}$ in X as follows

$gx_{n+1} = F(x_n,y_n)$ and $gy_{n+1} = F(y_n,x_n)$ For all $n \geq 0$. ……. (2)

We now show that

$gx_n \leq gx_{n+1}$, For all $n \geq 0$ …… (3)

$gy_n \geq gy_{n+1}$, for all $n \geq 0$ …… (4)

Since $gx_0 \geq F(x_0,y_0)$ and $gy_0 \leq F(y_0,x_0)$, $gx_1 = F(x_0,y_0)$ and $gy_1 = F(y_0,x_0)$ we have $gx_0 \geq gx_1$ and $gy_0 \leq gy_1$. Thus (3) and (4) hold for $n = 0$.

Suppose that (3) and (4) hold for some $n \geq 0$ then since $gx_n \geq gx_{n+1}$ and $gy_n \leq gy_{n+1}$ and by the g-mixed monotone property of F, we have

$$gx_{n+2} = F(x_{n+1},y_{n+1}) \geq F(x_n,y_{n+1}) \geq F(x_n,y_n) = gx_{n+1} \quad \text{…… (5)}$$

And

$$gy_{n+2} = F(y_{n+1},x_{n+1}) \leq F(y_n,x_{n+1}) \leq F(y_n,x_n) = gy_{n+1} \quad \text{…. (6)}$$

Now, from (5) and (6) we obtain

$$gx_{n+1} \geq gx_{n+2} \text{ and } gy_{n+1} \leq gy_{n+2}$$

Thus by mathematical induction we conclude that (3) and (4) hold for all $n \geq 0$

Since $gx_{n-1} \geq gx_n$ and $gy_{n-1} \leq gy_n$, from (1) and (2) we have

$$D^*(gx_n, gx_{n+1}, gx_{n+1})$$
$$= D^*(F(x_{n-1},y_{n-1}), F(x_n,y_n), F(x_n,y_n))$$
$$\leq \alpha D^*(gx_{n-1},gx_n,gx_n) + \beta D^*(F(x_{n-1},y_{n-1}),gx_{n-1},gx_n)$$
$$+ \gamma D^*(gy_{n-1},gy_n,gy_n) + \delta D^*(F(x_n,y_n),gx_n,gx_n)$$
$$+ \eta D^*(F(y_{n-1},x_{n-1}),gy_n,gy_n) + \lambda D^*(F(x_n,y_n),gx_{n-1},gx_n)$$
$$\leq \alpha D^*(gx_{n-1},gx_n,gx_n) + \beta D^*(gx_n,gx_{n-1},gx_n)$$
$$+ \gamma D^*(gy_{n-1},gy_n,gy_n) + \delta D^*(gx_{n+1},gx_n,gx_n)$$
$$+ \eta D^*(gy_n,gy_n,gy_n) + \lambda D^*(gx_{n+1},gx_{n-1},gx_n)$$

$$\leq \alpha D^*(gx_{n-1},gx_n,gx_n) + \beta D^*(gx_{n-1},gx_n,gx_n)$$
$$+ \gamma D^*(gy_{n-1},gy_n,gy_n) + \delta D^*(gx_n,gx_{n+1},gx_{n+1})$$
$$+ \lambda \,[\text{illegible}]\, D^*(gx_{n-1},gx_n,gx_n) + D^*(gx_n,gx_{n+1},gx_{n+1})$$

…… (7)

Therefore,

$$D*(gx_n, gx_{n+1}, gx_{n+1}) \le \frac{a+b+l}{1-d-l} D*(gx_{n-1}, gx_n, gx_n)$$
$$+ \frac{g}{1-d-l} D*(gy_{n-1}, gy_n, gy_n)$$

……… (8)

Similarly $gy_n \ 3\, gy_{n-1}$ *and* $gx_n \quad gx_{n-1}$ from (1) and (2) we have

$$D*(gy_n, gy_{n+1}, gy_{n+1})$$
$$= D*(F(y_{n-1}, x_{n-1}), F(y_n, x_n), F(y_n, x_n))$$
$$? \ aD*(gy_{n-1}, gy_n, gy_n) \quad bD*(F(y_{n-1}, x_{n-1}), gy_{n-1}, gy_n)$$
$$+ gD*(gx_{n-1}, gx_n, gx_n) + dD*(F(y_n, x_n), gy_n, gy_n)$$
$$+ hD*(F(x_{n-1}, y_{n-1}), gx_n, gx_n) + lD*(F(y_n, x_n), gy_{n-1}, gy_n)$$

$$? \ aD*(gy_{n-1}, gy_n, gy_n) \quad bD*(gy_n, gy_{n-1}, gy_n)$$
$$+ gD*(gx_{n-1}, gx_n, gx_n) + dD*(gy_{n+1}, gy_n, gy_n)$$
$$+ hD*(gx_n, gx_n, gx_n) + lD*(gy_{n+1}, gy_{n-1}, gy_n)$$

$$? \ aD*(gy_{n-1}, gy_n, gy_n) \quad bD*(gy_{n-1}, gy_n, gy_n)$$
$$+ gD*(gx_{n-1}, gx_n, gx_n) + dD*(gy_{n+1}, gy_n, gy_n)$$
$$+ l \,?\, D*(gy_{n+1}, gy_n, gy_n) + (gy_n, gy_{n-1}, gy_{n-1})$$

…….. (9)

Therefore,

$$? \ \frac{a+b+l}{1-d-l} D*(gy_{n-1}, gy_n, gy_n) \quad \frac{g}{1-d-l} D*(gx_{n-1}, gx_n, gx_n)$$ ……... (10)

From (13) and (15) we have

$$D*(gx_n, gx_{n+1}, gx_{n+1}) + D*(gy_n, gy_{n+1}, gy_{n+1})$$
$$? \ \frac{a+b+g+l}{1-d-l} [D*(gx_{n-1}, gx_n, gx_n) \quad D*(gy_{n-1}, gy_n, gy_n)]$$ ……. (11)

For all *n*. Set $K = \frac{a+b+g+l}{1-d-l} < 1$; from (16) we have

$$D*(gx_n, gx_{n+1}, gx_{n+1}) + D*(gy_n, gy_{n+1}, gy_{n+1}) ? \ K[D*(gx_{n-1}, gx_n, gx_n) \quad D*(gy_{n-1}, gy_n, gy_n)]$$
$$? \ K^2[D*(gx_{n-2}, gx_{n-1}, gx_{n-1}) \quad D*(gy_{n-2}, gy_{n-1}, gy_{n-1})]$$
$$.$$
$$.$$
$$.$$
$$? \ K^n[D*(gx_0, gx_1, gx_1) \quad D*(gy_0, gy_1, gy_1)]$$

This implies

$$D*(gx_n, gx_{n+1}, gx_{n+1}) ? \ K^n[D*(gx_0, gx_1, gx_1) \quad D*(gy_0, gy_1, gy_1)]$$

and

$$D*(gy_n, gy_{n+1}, gy_{n+1}) ? \ K^n[D*(gx_0, gx_1, gx_1) \quad D*(gy_0, gy_1, gy_1)]$$

We shall show that $\{gx_n\}$ *and* $\{gy_n\}$ are Cauchy sequences

For $m > n$, we have

$$D^*(gx_n, gx_m, gx_m) ?\ D^*(gx_n, gx_n, gx_{n+1}) \quad D^*(gx_{n+1}, gx_{n+2}, gx_{n+2})$$
$$+ D^*(gx_{n+2}, gx_{n+3}, gx_{n+3}) + \ldots\ldots\ldots\ldots\ldots$$
$$+ D^*(gx_{m-1}, gx_{m-1}, gx_m)$$

$$= D^*(gx_n, gx_{n+1}, gx_{n+1}) + D^*(gx_{n+1}, gx_{n+2}, gx_{n+2})$$
$$+ D^*(gx_{n+2}, gx_{n+3}, gx_{n+3}) + \ldots + D^*(gx_{m-1}, gx_m, gx_m)$$
$$?\ (K^n \quad K^{n+1} + K^{n+2} + \ldots\ldots + K^{m-1})\left[D^*(gx_0, gx_1, gx_1) + D^*(gy_0, gy_1, gy_1)\right]$$
$$?\ \frac{K^n}{1-k}\left[D^*(gx_0, gx_1, gx_1) \quad D^*(gy_0, gy_1, gy_1)\right]$$

This implies that

$$D^*(gx_n, gx_m, gx_m) \to 0 \quad as \quad n, m \to \infty$$

Since $\frac{K^n}{1-k}\left\| D^*(gx_0, gx_1, gx_1) + D^*(gy_0, gy_1, gy_1)\right\| \quad 0\ as\ n, m \quad ?$

For $n, m, l \in N$ and

$$D^*(gx_n, gx_m, gx_l) \le D^*(gx_n, gx_m, gx_m) + D^*(gx_m, gx_l, gx_l)$$

From (1.1)

$$\left\|D^*(gx_n, gx_m, gx_l)\right\| \le \left\|D^*(gx_n, gx_m, gx_m)\right\| + \left\|D^*(gx_m, gx_l, gx_l)\right\|$$

Taking limit as $n, m, l \to \infty$, we get $D^*(gx_n, gx_m, gx_l) \to 0$. So $\{D^*(gx_n)\}$ is a D*-Cauchy sequence. Similarly one can show that $\{gy_n\}$ is also a Cauchy sequence. Since X is D*-complete, there exist $x, y \in X$ such that

$$\{gx_n\} \to x \quad and \quad \{gy_n\} \to y \quad as \quad n \to \infty$$

i.e. $\lim_{n\to\infty} gx_n = x \quad and \quad \lim_{n\to\infty} gy_n = y$ ……….. (12)

Thus

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad and \quad \lim_{n\to\infty} F(y_n, x_n) \lim_{n\to\infty} gy_n = y \qquad \text{……….. (13)}$$

Since F and G are compactable, from (18) we have

$$\lim_{n\to\infty} D^*(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) = 0 \qquad \text{……… . (14)}$$

and

$$\lim_{n\to\infty} D^*(gF(y_n, x_n), F(gy_n, gx_n), F(gy_n, gx_n)) = 0 \qquad \text{……….. (15)}$$

Now, suppose that assumption (a) holds since F, g is continuous, by (18)

$$gF(x_n, y_n) \to gx \ and \ F(gx_n, gy_n) \to F(x, y)) = 0 \ as \ n \to \infty$$

Let $0 < C$ be given; there exist $K \in R$, such that, for all $n > K$

$$D^*(gx, gx, gF(x_n, y_n)) \le \frac{C}{3}, D^*(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n)) \le \frac{C}{3}$$

and

$$D^*(F(gx_n, gy_n)), F(x, y), F(x, y)) \le \frac{C}{3}$$

Therefore

$$D^*(gx, F(x, y), F(x, y)) \le D^*(gx, gx, gF(x_n, y_n)) + D^*(gF(x_n, y_n), F(gx_n, gy_n), F(gx_n, gy_n))$$
$$+ D^*(F(gx_n, gy_n), F(x, y), F(x, y))$$

$$\leq C$$

For all $n > K$. Since C is arbitrary, we get

$$D^*(gx, F(x,y), F(x,y)) = \frac{C}{m} \quad ″ m ? N$$

Notice that $\frac{C}{m} \quad 0 \; as \; m \quad ?$ and we conclude that

$$\frac{C}{m} - D^*(gx, F(x,y), F(x,y)) ? \quad D^*(gx, F(x,y), F(x,y)) \; as \; m$$

Since P is closed we get $-D^*(gx, F(x,y), F(x,y)) ? P$. Thus

$$D^*(gx, F(x,y), F(x,y)) \text{吻} P \quad (-P)$$

Hence

$$D^*(gx, F(x,y), F(x,y)) = 0$$

$$\backslash \; gx = F(x,y)$$

Similarly, we can show that $gy = F(y,x)$

Finally suppose that (b) holds. Since $\{gx_n\}$ is non-decreasing sequence and $\{gx_n\} ® \; x$. Also $\{gy_n\}$ is non-increasing sequence and $\{gy_n\} ® \; y$. We have $ggx_n \; 3 \; gx \; and \; ggy_n \quad gy$ for all *n*. Since *F* and *g* are compatible and continuous from (12), (14) and (15) we have

$$\lim_n ggx_n = gx = \lim_n gF(x_n, y_n) = \lim_n F(gx_n, gy_n) \quad \text{………. (16)}$$

And

$$\lim_n ggy_n = gy = \lim_n gF(y_n, x_n) = \lim_n F(gy_n, gx_n) \quad \text{……….. (17)}$$

We have

$$\begin{aligned} D^*(gx, F(x,y), F(x,y)) &? \; D^*(gx, gx, ggx_{n+1}) \quad D^*(ggx_{n+1}, F(x,y), F(x,y)) \\ &= D^*(gx, gx, ggx_{n+1}) + D^*(gF(x_n, y_n), F(x,y), F(x,y)) \\ &= D^*(gx, gx, ggx_{n+1}) + D^*(F(gx_n, gy_n), F(x,y), F(x,y)) \\ &? \; D^*(gx, gx, ggx_{n+1}) \quad aD^*(ggx_n, gx, gx) + bD^*(F(gx_n, gy_n), ggx_n, gx_n) \\ &\quad + gD^*(ggy_n, gy, gy) + dD^*(F(x,y), gx, gx) + hD^*(F(gy_n, gx_n), gy, gy) \\ &\quad + lD^*(F(x,y), ggx_n, gx) \\ &? \; D^*(gx, gx, ggx_{n+1}) \quad aD^*(ggx_n, gx, gx) + bD^*(F(gx_n, gy_n), ggx_n, gx_n) \\ &\quad + gD^*(ggy_n, gy, gy) + dD^*(F(x,y), gx, gx) + hD^*(F(gy_n, gx_n), gy, gy) \\ &\quad + l\,[D^*(F(x,y), gx, gx) + D^*(gx, ggx_n, ggx_n)] \end{aligned}$$

This implies

$$D^*(gx, F(x,y), F(x,y)) ? \; \frac{1}{1-d-l} \left\{ \begin{aligned} &D^*(gx, gx, ggx_{n+1}) + aD^*(ggx_n, gx, gx) \\ &bD^*(F(gx_n, gy_n), ggx_n, gx) + gD^*(ggy_n, gy, gy) \\ &hD^*(F(gy_n, gx_n), gy, gy) + l\,D^*(gx, ggx_n, ggx_n) \end{aligned} \right.$$

…………. (18) Let $0 = C$ by (16), (17) there exist $n_0 \; \hat{I} \; ¥$ such that

$$D*(ggx_n, gx, gx) = \frac{C(1- d- l)}{1+ a + b + g + h + l},$$

$$D*(F(gx_n, gy_n), ggx_n, gx) = \frac{C(1- d- l)}{1+ a + b + g + h + l},$$

$$D*(ggy_n, gy, gy) = \frac{C(1- d- l)}{1+ a + b + g + h + l},$$

$$D*(F(gy_n, gx_n), gy, gy) = \frac{C(1- d- l)}{1+ a + b + g + h + l} \quad " n > n_0$$

Thus (18), we have $D*(gx, F(x,y), F(x,y)) = ?\ for\,all\ n \quad n_0$.

Therefore $gx = F(x,y)$

Similarly, one can show that *F* and *g* have a coupled coincidence point.

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