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Some Results Concerning Hilbert Space

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ABSTRACT
In this research paper to obtained a common unique fixed point theorem for two continuous surjective random operators defined on a non empty closed subset of separable Hilbert space. The corresponding result for non-random case is also obtained.
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Keyboards : Separable Hilbert space, random operators, common random fixed point.

1. Introduction

The study of random fixed points has been an active area of contemporary research in mathematics. Some of the recent works in this field are noted in [2, 5, 6, 1].

In this paper, we construct a sequence of measureable functions and consider its convergence to the common unique random fixed point of two continuous surjective random operators, defined on a non-empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of two continuous, surjective random operators, we have used a rational inequality.

Throughout this paper (Ω, ε) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subset of Ω, H stands for a separable Hilbert space and C is a non-empty closed subset of H.

2. Preliminaries :

Definition 2.1 : A function $f : \Omega \to C$ is said to be measurable if $f^{1}(B \cap C) \varepsilon \Sigma$ for every Borel subset B of H.

Definition 2.2 : A function $F : \Omega \times C \to C$ is said to be a random operator if $F(., x) : \Omega \to C$ is measurable for every $x \in C$.

Definition 2.3 : A measurable function $g : \Omega \to C$ is said to be a random fixed point of the random operator $F : \Omega \times C \to C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

Definition 2.4 : A random operator $F : \Omega \times C \to C$ is said to be contin continuous if for fixed $t \in \Omega$, $F(t, .) : C \to C$ is continuous.

Condition A : Two mappings $p, T : C \to C$, where C is a non-empty subset of a Hilbert space H, is said to satisfy condition (A) it.

$$\| px - Ty \|^2 \geq a_1 \frac{\| x - px \|^2 \| y - Ty \|^2}{\| x - y \|^2} + a_2 \frac{\| x - Ty \|^2 \| y - Py \|^2}{\| x - y \|^2} +$$

$$a_3 \| x-px \|^2 + a_4 \| y-Ty \|^2 + a_5 \| x-y \|^2 + a_6 \frac{\| y - px \|^2 \| x - Ty \|^2 + \| x - px \|^2}{1 + \| y - Px \| \, \| x - Ty \| \, \| x - Px \|}$$

where $a_1+a_3+a_4+a_5+a_6 > 1$, $a_2+a_6 > 1$ and $a_1, a_2, a_3, a_4, a_5, a_6 > 0$.....(2.2)

We construct a sequence of functions $\{g_n\}$ as

$$g_0 : \Omega \to C \qquad .... (2.3)$$

is arbitrary measurable function for $t \in \Omega$ and n = 0, 1, 2 ....

$$g_{2n}(t) = P(t, g_{2n+1}(t)), \; g_{2n+1}(t) = T(t, g_{2n+2}(t)) \qquad ..... (2.4)$$

3. Main Results :

Theorem 3.1 : Let C be a non-empty closed subset of a separable Hilbert space H. let P and T be two continuous, surjective random operators defined on C such that for $t \in \Omega$, $P(t, .)$ $T(t, .) : C \to C$ satisfy conditino (A). Then the sequence $\{g_n\}$ obtained in (2.3) and (2.4) converges to the unique common random fixed point of P and T.

Proof : for fixed $t \in \Omega$, n = 1, 2, 3, ...

$$\|g_{2n-1}(t) - g_{2n}(t)\|^2 \approx \|T(t, g_{2n}(t) - P(t, g_{2n+1}(t)\|^2 =$$
$$\|P(t, g_{2n+1}(t) - T(t, g_{2n}(t)\|^2$$

$$\geq a_1 \frac{\|g_{2n+1}(t) - P(t, g_{2n+1}(t)\|^2 \|g_{2n}(t) - T(t, g_{2n}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} +$$

$$a_2 \frac{\|g_{2n+1}(t) - T(t, g_{2n}(t)\|^2 \|g_{2n}(t) - P(t, g_{2n+1}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} +$$

$$a_3\| g_{2n+1}(t) - P(t, g_{2n+1}(t)\|^2 + a_4 \|g_{2n}(t) - T(t, g_{2n}(t))\|^2 + a_5 \| g_{2n+1}(t) - g_{2n}(t)\|^2$$

$$+ a_6 \frac{\|g_{2n}(t) - P(t, g_{2n+1}(t)\|^2 \|g_{2n+1}(t) - T(t, g_{2n}(t)\|^2 + \|g_{2n+1}(t) - P(t, g_{2n+1}(t)\|^2}{1 + \|g_{2n}(t) - P(t, g_{2n+1}(t)\| \, \|g_{2n+1}(t) - T(t, g_{2n}(t)\| \, \|g_{2n+1}(t) - P(t, g_{2n+1}(t)\|}$$

$$= a_1 \frac{\|g_{2n} + (t) - g_{2n}(t)\|^2 \|g_{2n}(t) - g_{2n} - (t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} + a_2 \frac{\|g_{2n+1}(t) - g_{2n-1}(t)\|^2 \|g_{2n}(t) - g_{2n}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2}$$

$$+ a_3\|g_{2n+1}(t) - g_{2n}(t)\|^2 + a_4\|g_{2n}(t) - g_{2n-1}(t)\|^2 + a_5\|g_{2n+1}(t) - g_{2n}(t)\|^2$$

$$+ a_6 \left[\frac{\|g_{2n}(t) - g_{2n}(t)\| 2 \|g_{2n+1}(t) - g_{2n-1}(t)\|^2 + \|g_{2n+1}(t) - g_{2n}(t)\|_2}{1 + \|g_{2n}(t) - g_{2n}(t)\| \, \|g_{2n+1}(t) - g_{2n-1}(t)\| \|g_{2n+1}(t) - g_{2n}(t)\|} \right]$$

$$= a_1 \frac{\|g_{2n+1}(t) - g_{2n}(t)\|^2 \|g_{2n}(t) - g_{2n-1}(t)\|^2}{\|g_{2n+1}(t) - g_{2n}(t)\|^2} + a_3\|g_{2n+1}(t) - g_{2n}(t)\|^2$$

$$+ a_4 \| g_{2n}(t) - g_{2n-1}(t)\|^2 + a_5\|g_{2n+1}(t) - g_{2n}(t)\|^2 + a_6\|g_{2n+1}(t) - g_2n(t)\|^2$$
$$(a_1+a_4)\|g_{2n}(t) - g_{2n-1}(t)\|^2 + (a_3+a_5+a_6)\|g_{2n+1}(t) - g_{2n}(t)\|^2$$

$$\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\|^2 \leq \left[\frac{1 - (a_1 + a_4)}{a_6 + a_5 + a_6} \right] \| g_{2n-1}(t) - g_{2n}(t)\|^2$$

$$\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\| \leq \left[\frac{1 - (a_1 + a_4)}{a_3 + a_5 + a_6} \right]^{½} \| g_{2n-1}(t) - g_{2n}(t)\|$$

$$\Rightarrow \|g_{2n}(t) - g_{2n+1}(t)\| \leq K_1 \ \|g_{2n-1}(t) - g_{2n}(t)\| \quad .... (3.1)$$

Where $K_1 = \left[\frac{1 - (a_1 + a_4)}{a_3 + a_5 + a_6} \right] < 1$ (as $a_1 + a_3 + a_4 + a_5 + a_6 > 1$)

For fixed $t \in \Omega$, n = 1, 2, 3 ......

$$\| g_{2n-2}(t) - g_{2n-1}(t)\|^2 = P(t, g_{2n-1}(t) - T(t, g_{2n}(t))\|^2$$

$$\geq a_1 \frac{\|g_{2n-1}(t) - P(t, g_{2n-1}(t))\|^2 \|g_{2n}(t) - T(t, _{g2n}(-)\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2}$$

$$+ a_2 \frac{\|g_{2n-1}(t) - T(t, g_{2n-1}(t))\|^2 \|g_{2n}(t) - P(t, g_{2n-1}(t)\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2}$$

$$a_3\| g_{2n-1}(t) - P(t, g_{2n-1}(t)\|^2 + a_4 \| g_{2n}(t) - T(t, g_{2n}(t)\|^2 + a_5 \| g_{2n-1}(t) g_{2n}(t)\|^2$$

$$+ a_6 \left[\frac{\|g_{2n}(t) - P|t, g_{2n-1}(t)\|^2 \|g_{n-1}(t) - T(t, g_{2n}(t)\|^2 + \|g_{2n-1}(t) - P(t, g_{2n-1}(t)\|^2}{1 + \|g_{2n}(t) - P(t, g_{2n-1}(t)\| \, \|g_{2n-1}(t) - T(t, g_{2n}(t))\| \, \|g_{2n-1}(t) - P(t, g_{2n-1}(t)\|} \right]$$

$$= a_1 \frac{\|g_{2n-1}(t) - g_{2n-2}(t)\|^2 \ \|g_{2n}(t) - g_{2n-1}(t)\|^2}{\| g2n - 1(t) - g2n(t)\| 2}$$

$$+ a_2 \frac{\|g_{2n-1}(t) - g_{2n-1}(t)\|^2}{} \frac{\|g_{2n}(t) - g_{2n-2}(t)\|^2}{\|g_{2n-1}(t) - g_{2n}(t)\|^2} |$$

$$+ a_3\|g_{2n-1}(t) - g_{2n-2}(t)| + a_4\|g_{2n}(t) - g_{2n-1}(t)\|^2 + a_5 \| g_{2n-1}(t) - g_{2n}(t)\|^2$$

$$+a_6\left[\frac{\|g_{2n}(t)-g_{2n-2}(t)\|^2\,\|g_{2n-1}(t)-g_{2n-1}(t)\|^2+\|g_{2n-1}(t)-g_{2n-1}(t)\|^2}{1+\|g_{2n}(t)-g_{2n-2}(t)\|\,\|g_{2n-1}(t)-g_{2n-1}(t)\|\,\|g_{2n-1}(t)-g_{2n-2}(t)\|}\right]$$

$= a_1 \| g_{2n-1}(t) - g_{2n-2}(t)\|^2 + a_3 \|g_{2n-1}(t) - g_{2n-2}(t)\|^2 +$
$a_4 \| g_{2n}(t) - g_{2n-1}(t)\|^2 + a_5 \| g_{2n-1}(t) - g_{2n}(t)\|^2 + a_6 \| g_{2n-1}(t) - g_{2n-2}\|^2$
$= (a_1+a_3+a_6) \|g_{2n-1}(t) - g_{2n-2}(t)\|^2 + (a_4+a_5) \| g_{2n-1}(t) - g_{2n}(t)\|^2$

$$\Rightarrow \|g_{2n-1}(t) - g_{2n}(t)\|^2 \le \left[\frac{1-(a_1+a_3+a_6)}{a_4+a_5}\right] \| g_{2n-2}(t) - g_{2n-1}(t)\|^2$$

$$\Rightarrow \|g_{2n-1}(t) - g_{2n}(t)\| \le \left[\frac{1-(a_1+a_3+a_6)}{a_4+a_5}\right]^{1/2} \| g_{2n-2}(t) - g_{2n-1}(t)\|$$

$\Rightarrow g_{2n-1}(t) - g_{2n}(t)\| \le K_2 \| g_{2n-2}(t) - g_{2n-1}(t)\|$ ...... (3.2)

The inequalities (3.1) and (3.2) are jointly imply that for all
$t \in \Omega$, $n = 1, 2, 3$ ....
$\| g_n(t) - g_{n+1}(t)\| \le K \| g_{n-1}(t) - g_n(t) - g_n(t)\|$

Where $K = \max\{k_1, k_2\} < 1$

$\Rightarrow \quad \| g_n(t) - g_{n+1}(t)\| \le K^n \| g_0(t) - g_1(t)\|$ for $t \in \Omega$ ..... (3.3)

Now we shall prove that for $t \in \Omega$ $\{g_n(t)\}$ is a cauchy sequence. For this for every positive integer r, we have $\| g_n(t)-g_{n+r}(t)\| = \| g_n(t) - g_{n+1}(t) + g_{n+1}(t)$ .... $+ g_{n+r-1}(t) - g_{n+r}(t)\|$

$\le \| g_n(t) - g_{n+1}(t)\| + \| g_{n+1}(t) - g_{n+2}(t)\|$
$+ \ldots\ldots + \| g_{n+r-1}(t) - g_{n+r}(t)\|$
$\le [k^n + k^{n+1} + k^{n+2} + \ldots\ldots + K^{n+r-1}] \| g_o(t) - g_1(t)\|$ (by (3.3))
$= K^n [1+k+k^2 + \ldots + k^{r-1}] \| g_o(t) - g(t)\|$

$$< \frac{k^n}{(1-k)} \| g_o(t) - g_1(t)\| \text{ for } t \in \Omega$$

As $n \to \infty$, $\| g_n - g_{n+r}(t)\| \to 0$, it follows that for $t \in \Omega$, $\{g_n(t)\}$ is a cauchy sequence and hence is convergent in Hilbert space H.

For $t \in \Omega$, let

$\{g_n(t) \to g(t)$ as $n \to \infty$ .... (3.4)

Since C is closed, g is a fnction from C to C.

Since P and T are surjective maps. So there exists two functions
$g : \Omega \to C$ and $g'' : \Omega \to C$ such that
$g(t) = p(t, g'(t))$ and $g(t) = T(t, g''(t))$ .... (3.5)

For $t \in \Omega$
$\| g_{2n}(t) - g(t)\|^2 = \|p(t, g_{2n+1}(t)) - T(t, g''(t)\|^2$

$$\ge a_1 \frac{\|g_{2n+1}(t)-P(t,g_{2n+1}(t)\|^2\,\|g''(t)-T(t,g''(t)\|^2}{\|g_{2n+1}(t)-g''(t)\|^2}$$

$$+a_2 \frac{\|g_{2n+1}(t)-T(t,g''(t)\|^2\,\|g''(t)-P(t,g_{2n+1}(t)\|^2}{\|g_{2n+1}(t)-g''(t)\|^2}$$

$+a_3\| g_{2n+1}(t) - P(t, g_{2n+1}(t))\|^2 + a_4 \| g''(t) - T(t, g''(t)\|^2 + a_5\| g_{2n+1}(t) - g''(t)\|^2$

$$+a_6\left[\frac{\|g''(t)-P(t,g_{2n+1}(t)\|^2\,\|g_{2n+1}(t)-T(t,g''(t)\|+\|g_{2n+1}(t)-P(t,g_{2n+1}(t)\|^2}{1+\|g''(t)-P(t,g_{2n+1}(t)\|\,\|g_{2n+1}(t)-T(t,g''(t)\|\,\|g_{2n+1}(t)|-P(t,g_{2n+1}(t)\|}\right]$$

$$=a_1 \frac{\|g_{2n+1}(t)-g_{2n}(t)\|^2 g''(t)-T(t,g''(t)\|^2}{\|g_{2n+1}(t)-g''(t)\|^2}+$$

$$\frac{a_2\,\|g_{2n+1}(t)-T(t,g''(t)\|^2\,\|g''(t)-g_{2n}(t)\|^2}{\|g_{2n+1}(t)-g''(t)\|^2}$$

$+\,a_3 \|g_{2n+1}(t)-g_{2n}(t)\|^2 + a_4 \|g''(t)-T(t,g''(-)\|^2 + a_5 \| g_{2n+1}(t) - g''(t)\|^2$

$$+a_6\left[\frac{\|g''(t)-g_{2n}(t)\|^2\,\|g_{2n+1}(t)-T(t,g''(t)\|^2+\|g_{2n+1}(t)-g_{2n}(t)\|^2}{1+\|g''(t)-g_{2n}(t)\|^2\,\|g_{2n+1}(t)-T(t,g''(t)\|\,\|g_{2n+1}(t)|-g_{2n}(t)\|}\right]$$

Making $n \to \infty$ in the above inequality we have by virtue of (3.4) for all $t \in \Omega$,

$0 \geq (a_4 + a_5) \| g(t) - g''(t) \|^2$

$\Rightarrow g(t) - g''(t) \|^2 = 0$ [as $a_4 + a_5 > 0$]

$\Rightarrow g(t) = g''(t)$ for $t \in \Omega$ .... (3.6)

In an exactly similar way by using $(a_3 + a_5) > 0$ we can prove that

$g(t) = g'(t)$ for $t \in \Omega$ .... (3.7)

Thus by (3.5), (3.6) and (3.7) we have, for $t \in \Omega$

$P(t, g(t)) = g(t)$ .... (3.8)

and $T(t, g(t) = g(t)$ .... (3.9)

Again, if $A: \Omega \times C \to$ is a continuous random operator on a non-empty subset C of a separable Hilbert space H, then for any measurable function $f: \Omega \to C$, the function $h(t) = A(t, f(t))$ is also measurable [3]

It follows from the construction of $\{g_n\}$ (2.3) and (2.4) and the above consideration that $\{g_n\}$ is a sequence of measurable functions from (3.4) it follows that g is also a measurable function. This fact along with (3.8) and (3.9) show that $g: \Omega\ C \to$ is a common random fixed point of P and T.

Next we prove that uniqueness let $h: \Omega \to C$ be another random fixed point common to P and T, that is for $t \in \Omega$.

$P(t, h(t)) = h(t)$ and $T(t, h(t) = h(t)$ .... (3.10)

Then for $t \in \Omega$

|| Then for $t \in \Omega$

$\| g(t) - h(t) \|^2 = \| P(t, g(t)) - T(t, h(t)) \|^2$

$\geq a_1 \dfrac{\| g(t)\text{-}P(t, g(t) \|^2\ \|h(t)\text{-}T(t, h(t) \|^2}{\| g(t) - h(t) \|^2}$

$+ \dfrac{a_2 \|g(t) - T(t, h(t)\|^2 \| h(t) - P(t, g(t) \|^2}{\| g(t) - h(t)\|^2} + a_3 \|g(t) - P(t, g(t) \|^2$

$+ a_4 \| h(t) - T(t, h(t) \|^2 + a_5 \|g(t) - h(t)\|^2$

$+ a_6 \dfrac{[\| h(t) - P(t, g(t))\|^2 \|g(t) - T(t, h(t)\|^2 + \|g(t) - P(t, g(t)) \|^2]}{1+\|h(t) - p(t, g(t))\|\ \|g(t)\text{-}T(t, h(t)) \|\ \| g(t) - P(t, g(t)\|}$

$\Rightarrow \| g(t) - h(t) \|^2 \geq (a_2 + a_5) \| g(t) - h(t) \|^2$ (by (3.10))

$\Rightarrow \|g(t) - h(t) \|^2 = 0$ [as $(a_2 + a_5) > 1$]

$\Rightarrow g(t) = h(t)$ for all $t \in \Omega$

This complete the proof of the theorem 3.1

References :

1. B.E. Roades, “Interation to obtain random solutions and fixed points of operators in uniformly convex in Banach spaces”. Soochow Journal of Mathematics, 27 (4) (2001), 401-404.
2. B.S. Choudhary, “A common unique fixed point theorem for two random operators in Hilbert space”. IJ MMS, 32 (3) (2002), 177-182
3. C.J. Himmelberg, “Measurable relations”, fund math, 87 (1975), 53-72.
4. Dhagat, V.B., Sharma, A., Bhardwaj, R.K. "Fixed point theorems For random operators in Hilbert spaces," International Journal of Math. Anal. 2 (2008). No. 12, 557-561.
5. S.S. Pagey ,Neeraj Malviya, “A common random fixed point theorem for rational inequality in Hilbert space”. International Journal of Mathematical Analysis, Vol. 4, 133-141, 2010.
6. S.S. Pagey, Neeraj Malviya, “A common unique random fixed point theorem in Hilbert space”. International journal of mathematical analysis, vol. 4, 143-148.