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The Axisymmetric Indentation of Semi-Infinite Transversely Isotropic Space by Heated Annular Punch

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1. INTRODUCTION

The problem of determining the distribution of stress in a semi – infinite elastic solid when a rigid body of prescribed shape is pressed against its free surface is associated with the name of Boussinesq, since it was first discussed in classical Treatise [1]. A detailed account of punch problem may be formed in Sneddon [2] and Green and Zerna [3]. Recently, Shibuya et.al. [4] devised a novel technique for determining stress distribution in elastic half space indented by flat annular punch. Shibuya et.al. [5] also extended this technique to determine stress distribution in an elastic slab indented by a pair of flat rigid annular punches.

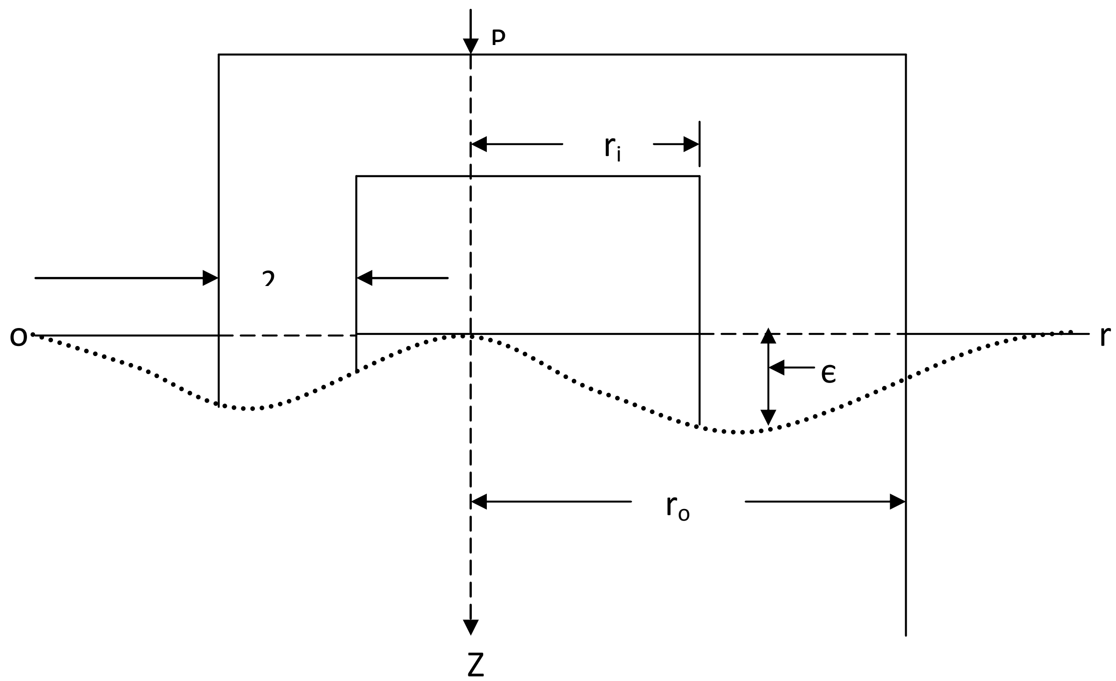
George and Sneddon [6] were first to study the axially symmetric problem of elastic half space indented by heated punch. Keer and Fu [7] also studied the thermo – elastic stress distribution problem due to combined loading of rigid, non– symmetrical, circular punches indenting thick elastic plate. The axisymmetric Boussinesq problem for heated annular punch was discussed by Kumar and Hiremath [8]. The problem of determining axisymmetric distribution in a thick elastic plate indented by a pair of heated annular punches was also studied by Kumar and Hiremath [9].

The present paper extends the method of Kumar and Hiremath [8, 9] to study the problem of determining stress distribution in a transversely isotropic half space indented by a heated annular rigid punch. The mixed boundary value problem is reduced to the solution of triple integral equations, which in turn are reduced to the solution of linear simultaneous algebraic equations. These are solved numerically.

2. FORMULATION OF THE PROBLEM

It is assumed that the axis of the annular punch is normal to the boundary plane of the transversely isotropic elastic solid. If we take the undisturbed boundary to be plane z = 0 and the print, at which the tip of the punch begins to indent the solid to be origin of coordinates, then a typical point of solid in cylindrical coordinates is described as (r, θ, z). (See Fig. 1). Because of axial symmetry, the only non – zero components of displacement vector are u and w and that of stress tensor are σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz} and σ_{rz}. Since the bodies in contact are smooth, we have

$$\sigma_{rz}(r, 0) = 0 \quad ; \quad r \geq 0 \tag{2.1}$$



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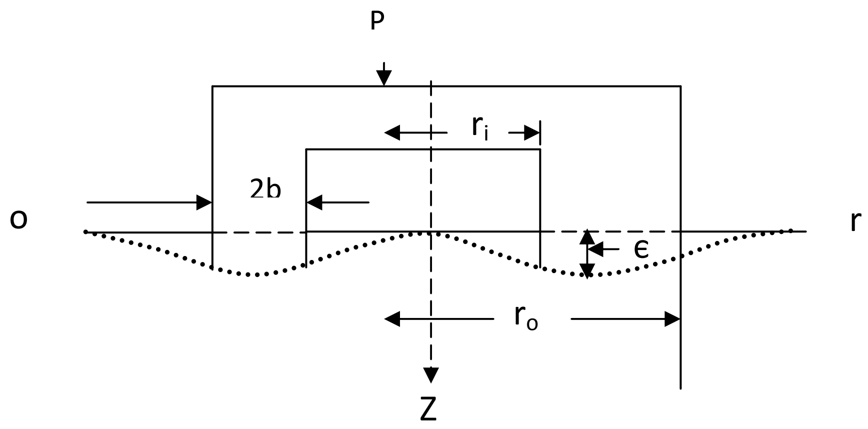


Fig – 1 Geometry of the problem

The mixed conditions on $z = 0$ are

$$w(r, 0) = \in \quad ; \quad r_i \le r \le r_0 \tag{2.2}$$

$$\sigma_{zz}(r, 0) = 0 \quad ; \quad r < r_i \text{ and } r > r_0 \tag{2.3}$$

Where $\in$ is depth of penetration of the punch.

The temperature at the point (r, θ, z) of the elastic solid is taken to be $T(r,z)$, where T is the temperature of the solid in a state of zero stress and strain. We are assuming that the heating of the annular punch is also axisymmetric. Following types of temperature condition are considered.

a) Temperature gradient is prescribed

$$\partial T/\partial z]_{z=0} = -T_0 \;;\; r_i \le r \le r_0$$

$$T(r, 0) = 0 \quad ; \quad r < r_i\,,\; r > r_0 \tag{2.4}$$

b) Temperature field due to surface conditions are

$$T(r, 0) = \begin{cases} T_1 & ; \quad r_i \le r \le r_0 \\ 0 & ; \quad r < r_i,\; r > r_0 \end{cases} \tag{2.5}$$

Fig – 1 Geometry of the problem

For transversely isotropic solid, the equation of thermoelastic equilibrium of u and w are

$$C_{11}\left(\frac{\partial^2 u/}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right) + C_{44}\frac{\partial^2 u}{\partial z^2} + \left(C_{13} + C_{44}\right)\frac{\partial^2 w}{\partial r \partial z} = \alpha_1 \frac{\partial T}{\partial r} \tag{2.6}$$

$$C_{44}\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + C_{33}\frac{\partial^2 w}{\partial z^2} + \left(C_{13} + C_{44}\right)\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = \alpha_2 \frac{\partial T}{\partial z} \tag{2.7}$$

In study state, the temperature of a transversely isotropic solid is governed by

$$\frac{\partial^2 \mathbf{T}}{\partial r^2} + \frac{\mathbf{1}}{r}\frac{\partial \mathbf{T}}{\partial r} + \frac{\mathbf{1}}{s^2}\frac{\partial^2 \mathbf{T}}{\partial^2 T} = 0 \tag{2.8}$$

where s^2 is ratio of conductivity coefficients.
The stress and displacement components appropriate to the problem are

$$u(r, z) = -\lambda_1 \int_0^\infty y^{-1} A(y)\, e^{-ysz} J_1(yr)\, dy - \int_0^\infty y^{-1} A_1(y)\, e^{-yz/\nu_1} J_1(yr)\, dy - \int_0^\infty y^{-2} A_2(y)\, e^{-yz/\nu_2} J_1(yr)\, dy \tag{2.9}$$

$$w(r, z) = -\lambda_2 s \int_0^\infty y^{-1} A(y)\, e^{-ysz} J_0(yr)\, dy - \frac{\mu_1}{\nu_1} \int_0^\infty y^{-1} A_1(y)\, e^{-yz/\nu_1} J_0(yr)\, dy - \frac{\mu_2}{\nu_2} \int_0^\infty y^{-1} A_2(y)\, e^{-yz/\nu_2} J_0(yr)\, dy \tag{2.10}$$

$$\boldsymbol{\sigma_{zz}\ (r, z) = (C_{33}\lambda_2 s^2 - \alpha_2 - C_{13}\lambda_1) \int_0^\infty A(y)\, e^{-ysz} J_0(yr)\, dy + \left\{ \frac{C_{33}\mu_1}{\nu_1^2} - C_{13} \right\} \times}$$
$$\int_0^\infty A_1(y)\, e^{-yz/\nu_1} J_0(yr)dy + \left\{ \frac{C_{33}\mu_2}{\nu_2^2} - C_{13} \right\} \int_0^\infty A_2(y) e^{-yz/\nu_2} J_0(yr)\, dy \tag{2.11}$$

$$\boldsymbol{\sigma_{rz}\ (r, z) = C_{44}\ (\lambda_1 + \lambda 2)s + \int_0^\infty A(y)\, e^{-yzs}\ J_1(yr)\, dy + \frac{C_{44}}{\nu_1} (1 + \mu_1)}$$

$$\int_0^\infty A_1(y)\, e^{-yz/\nu_1} J_1(yr)dy + \frac{C_{44}}{\nu_2}(1 + \mu_2) \int_0^\infty A_2(y) e^{-yz/\nu_2} J_1(yr)\, dy \tag{2.12}$$

$$T(r, z) = \int_0^\infty A(y)\, e^{-ysz}\ J_0(yr)\, dy \tag{2.13}$$

The functions $A(y)$, $A_1(y)$ and $A_2(y)$ are determined from the mixed boundary conditions.

3. STANDARD RESULTS

We shall use following standard results frequently. They may be found in Erdelyi [10] and also Erdelyi [11]

$$\int_0^\infty y\, J_0(yr)\, Z_n(y)\, dy = \begin{cases} \dfrac{\cos n\phi}{\pi\, b\, r_c \sin\phi}, & r_i \le r \le r_0 \\ 0, & r < r_i,\ r > r_0 \end{cases} \tag{3.1}$$

and $0 \le \phi \le \pi$

It is simple to drive the following using (3.1)

$$\int_0^\infty y[Z_{n-1}(y) - Z_{n+1}(y)]\, J_0(yr)\, dy = \begin{cases} \dfrac{2\sin n\phi}{\pi\, b\, r_c}, & r_i \le r \le r_0 \\ 0, & r < r_i,\ r > r_0 \end{cases} \tag{3.2}$$

where

$$Z_n(y) = J_n(yr_c)\ J_n(yb) \tag{3.3}$$

and

$$r^2 = r_c^2 + b^2 - 2r_c b \cos\phi, \quad 2r_c = r_i + r_0 \tag{3.4}$$

We shall also use the result,

$$\int_0^\pi \cos n\phi\ J_0(\xi \sqrt{\{r_c^2 + b^2 - 2r_c\, b\cos\phi\}}\, d\phi = \pi\, Z_n(\xi) \tag{3.5}$$

It is possible to derive that

$$J_0(yr) = Z_0(y) + 2\sum_{m=1}^{\infty} Z_m(y)\, \cos m\phi \tag{3.6}$$

$m = 1, 2, \ldots.., \infty$, and for $r_i \le r \le r_0$

Further, using results of Erdelyi [11, p.:53]

$$I_0^n = \begin{cases} \dfrac{\Gamma(n+1/2)}{\Gamma(n+1)\,\Gamma(1/2)\, r_c}\left(\dfrac{b}{r_c}\right)^n [F\{1/2,\ n+1/2,\ n+1;\ \sin^2\phi\}, \\ \qquad F\{1/2,\ n+1/2,\ 1;\ \sin^2\psi\}];\ r < r_i \\ \dfrac{(-1)^n}{\pi r}\left\{\dfrac{\Gamma(n+1/2)}{\Gamma(n+1)}\right\}^2 \left(\dfrac{br_c}{r^2}\right)^n [F\{n+1/2,\ n+1/2,\ n+1;\ \sin^2\phi\}] \\ \qquad F\{n+1/2,\ n+1/2,\ n+1;\ \sin^2\psi\},\ r > r_0 \end{cases} \tag{3.7}$$

where F (α, β, γ; x) is the Gauss hyper geometric series and

$$\begin{bmatrix} \psi \\ \text{----} \\ \phi \end{bmatrix} = \begin{cases} \dfrac{1}{2}\left[\sin^{-1}\dfrac{r+b}{r_c} \pm \sin^{-1}\dfrac{r-b}{r_c}\right]; & r < r_i \\ \dfrac{1}{2}\left[\sin^{-1}\dfrac{r_0}{r} \pm \sin^{-1}\dfrac{r_i}{r_0}\right]; & r < r_0 \end{cases} \tag{3.8}$$

4. DETERMINATION OF TEMPERATURE FIELD

We shall determine the temperature field T(r, z) using two types of boundary conditions (2.4) and (2.5).
CASE (a): The use of boundary conditions (2.4) with (2.13), gives us

$$\left.\frac{\partial T}{\partial Z}\right|_{z=0} = \int_0^\infty y\ A(y)\ J_0(yr)dy = T_0/s, \quad r_i \le r \le r_0 \tag{4.1}$$

$$T(r, 0) = \int_0^\infty A(y)\ J_0(yr)dy = 0, \quad r < r_i, \quad r > r_0 \tag{4.2}$$

It is possible to express T(r,0) by Fourier sine series with respect to ϕ using (3.4)

$$T(r, 0) = \frac{2}{\pi b r_c} \sum_{n=1}^{\infty} b'_n \sin n\phi, \quad r_i \le r \le r_0 \tag{4.3}$$

In view of (4.3) and (3.2), it is simple to see that (4.2) is satisfied, if A(y) is chosen as

$$A(y) = \sum_{n=1}^{\infty} b'_n\ G_n(y) \tag{4.4}$$

where

$$G_n(y) = y\,[Z_{n-1}(y) - Z_{n+1}(y)] \tag{4.5}$$

This choice of A(y) reduces (4.1) to

$$\sum_{n=1}^{\infty} b_n \int_0^\infty yG_n(y)\ J_0(yr)\ dy = 1\ ; \quad r_i \le r \le r_0 \tag{4.6}$$

where

$$b_n = sb'_n / T_0 \tag{4.7}$$

Substitute $J_0(yr)$ from (3.6) and compare coefficient of cosmϕ to obtain following:

$$\sum_{n=1}^{\infty} b_n \int_0^\infty y\ G_n(y)\ Z_m(y)\ dy = s_0, m; \quad m = 1, 2, \ldots, \infty \tag{4.8}$$

Now, subtract $(m + 1)^{th}$ equation from $(m - 1)^{th}$ equation to get symmetrical form of infinite set of simultaneous equations in unknowns b_n.

$$\sum_{n=1}^{\infty} b_n \int_0^{\infty} G_n(y)\, Z_m(y)\, dy = \delta_1, m; \quad m = 1, 2, \ldots, \infty \tag{4.9}$$

The set of simultaneous equations is solved for b_n by the process described in Section 7. Since coefficients b_n are known, it is now possible to express

$$T(r, 0) = 2T_0/\pi s b r_c \sum^{\infty} b_n \sin n\phi ; \qquad r_i \le r \le r_0 \tag{4.10}$$

CASE (b): The boundary condition (2.5) gives us

$$\int_0^{\infty} A(y)\, J_0(yr)dy = \begin{cases} T_1, & r_i \le r \le r_0 \\ 0, & r < r_i,\ r < r_0 \end{cases} \tag{4.11}$$

It is simple matter due to express T(r, z) in following form

$$T(r, z) = T_1 \int_0^{\infty} [r_0 J_1(\xi r_0) - r_i J_1(\xi r_i)]\, e^{-\xi z} J_0(\xi r)d\xi; \quad r_i \le r \le r_0 \tag{4.12}$$

5. SOLUTION OF THERMOELASTIC PROBLEM

The boundary condition (2.1) is satisfied if

$$C_{44}/\nu_1 (1 + \mu_1)\, A_1(y) + C_{44}/\nu_2 (1 + \mu_2)\, A_2(y) = -C_{44} (\lambda_1 + \lambda_2) s A(y) \tag{5.1}$$

The boundary conditions (2.2) and (2.3) yield following triple integral equations

$$w(r, 0) = \int_0^{\infty} N(y)\, J_0(yr)dy = \frac{p_2}{p_4} \in + p(r), \quad r_i \le r \le r_0 \tag{5.2}$$

$$\sigma_{zz} (r, 0) = \int_0^{\infty} y\, N(y)\, J_0(yr)dy = 0, \quad r < r_i, \quad r > r_0 \tag{5.3}$$

where

$$y\, N(y) = p_1 A(y) + p_2 A_2(y) \tag{5.4}$$

$$p_1 = C_{33}\lambda_2 s^2 - \alpha_2 - C_{13}\lambda_1 - (C_{33}\mu_1 - C_{13}\nu_1^2)(\lambda_1 + \lambda_2)s/(1 + \mu_1)\, \nu_1$$

$$p_2 = C_{33}\, \mu_2 - C_{13}\, \nu_2^2/\nu_2^2 - (C_{33}\, \mu_1 - C_{13}\, \nu_1^2)(1 + \mu_2)/(1 + \mu_1)\, \nu_1\nu_2$$

$$p_3 = -\lambda_2 S + \mu_1 (\lambda_1 + \lambda_2)s/(1 + \mu_1)$$

$$p_4 = \mu_1(1 + \mu_2)/(1 + \mu)\mu_2 - (\mu_2/\nu_2)$$

$$p(r) = \left(p_1 - \frac{p_2\, p_3}{p_4} \right) \int_0^{\infty} y^{-1} A(y)\, J_0(yr)dy \tag{5.5}$$

It is well known that the normal stress σ_{zz} (r, 0) will have singularities of the form $(r^2 - r_i^{2)-1/2}$ at $r = r_i$ and $(r_0^2 - r^2)^{-1/2}$ at $r = r_0$ (see George and Sneddon [6]).

Hence in the region of annular punch, we can express

$$\sigma_{zz}(r, 0) = \in f(r)/\sqrt{\{(r_0^2 - r^2)(r^2 - r_i^2)\}} \qquad (5.6)$$

where f(r) is unknown function in $r_i \le r \le r_0$ and $\in$ is depth to which heated punch penetrates. From definition of variable r_c and b of equation (3.4), we find that the variable r in $r_i \le r \le r_0$ can be replaced by a new variable ϕ with property that $\phi = 0$ and π respectively at $r = r_i$ and $r = r_0$. It is possible to express f(r) in Fourier series with respect to ϕ

$$f(r) = \sum_{n=0}^{\infty} a_n \cos n\phi \quad \text{and} \qquad (5.7)$$

$$\sigma_{zz}(r, 0) = \frac{\in}{2br_c} \sum_{n=0}^{\infty} a_n \frac{\cos n\phi}{\sin\phi}; \qquad r_i \le r \le r_0 \qquad (5.8)$$

where a_n is unknown coefficient. Equations (5.3) and (5.8) and the Hankel inversion transform gives us

$$N(y) = \frac{\in}{2} \sum_{n=0}^{\infty} a_n \int_0^{\pi} \cos n\phi\, J_0(\xi r)d\phi \qquad (5.9)$$

The use of result (3.5) gives us

$$N(y) = \frac{\pi\in}{2} \sum_{n=0}^{\infty} a_n Z_n(y) \qquad (5.10)$$

where $Z_n(y)$ is defined by (3.3). Substitution of this N(y) in (5.2), we get

$$\frac{\pi\in}{2} \sum_{n=0}^{\infty} a_n \int_0^{\infty} Z_n(y)\, J_0(yr)dy = \frac{p_2}{p_4} \in + pr \qquad (5.11)$$

Substituting $J_0(yr)$ from (3.6) in (5.11), we get

$$\sum_{n=0}^{\infty} a_n \int_0^{\infty} Z_m(y)\, Z_n(y)dy = \frac{2}{\pi}\,\frac{p_2}{p_4}\,\delta_{0,m} + \frac{2}{\pi\in} \sum_{n=0}^{\infty} b_n \int_0^{\infty} G_n(y)\, Z_m(y)dy \qquad (5.12)$$

where m = 0, 1, 2, ……, ∞.

This is an infinite set of simultaneous equations in unknown a_n. This can be solved numerically for a_n and the procedure is described in section 7. The N(y) is therefore assumed to be known.

6. QUANTITES OF PHYSICAL INTEREST

The shape of deformed surface in the region $r < r_i$ and $r > r_0$, is given by

$$W(r, 0) = \int_0^{\infty} \left[\frac{p_4}{p_2} yN(y) + \left(p_3 - \frac{p_1 p_4}{p_2} \right) A(y) \right] y^{-1} J_0(yr)dy$$

$$= \frac{\pi}{2}\,\frac{p_4}{p_2} \sum_{n=0}^{\infty} a_n I_0^n + \left(p_3 - \frac{p_1 p_4}{p_2} \right) \frac{T_0}{s} \sum_{n=1}^{\infty} b_n \left[I_0^{n-1} - I_0^{n+1} \right] \qquad (6.1)$$

where I_k^n is given by (3.7).

The normal stress under punch ($r_i \le r \le r_0$) is given by

$$\sigma_{zz}(r, 0) = \frac{\in}{2br_c} \sum_{n=0}^{\infty} a_n \frac{\cos n\phi}{\sin\phi}; \qquad r_i \le r \le r_0 \qquad (6.2)$$

where $\phi = \cos^{-1}(r_c^2 + b^2 - r^2 \quad) / 2br_c$

The total load P that must be applied to maintain the prescribed displacement is

$$P = -2\pi \int_{r_i}^{r_0} r\, \sigma_{zz}(r, 0)\, dr$$

$$= -\pi\ a_0 \in \tag{6.3}$$

7. NUMERICAL CALCULATIONS

We now describe the method of solving infinite set of linear simultaneous equations (5.12) and (4.8). The procedure is stated for (5.12) and (4.8) is also solved by the same method. The general element of set (5.12) is written as

$$A_{nm} = A_{mn} = \int_0^{\lambda} Z_n(y)\, Z_m(y) dy \tag{7.1}$$

Using asymptotic expansions for Bessel function for large value of argument y, we can rewrite (7.1) as

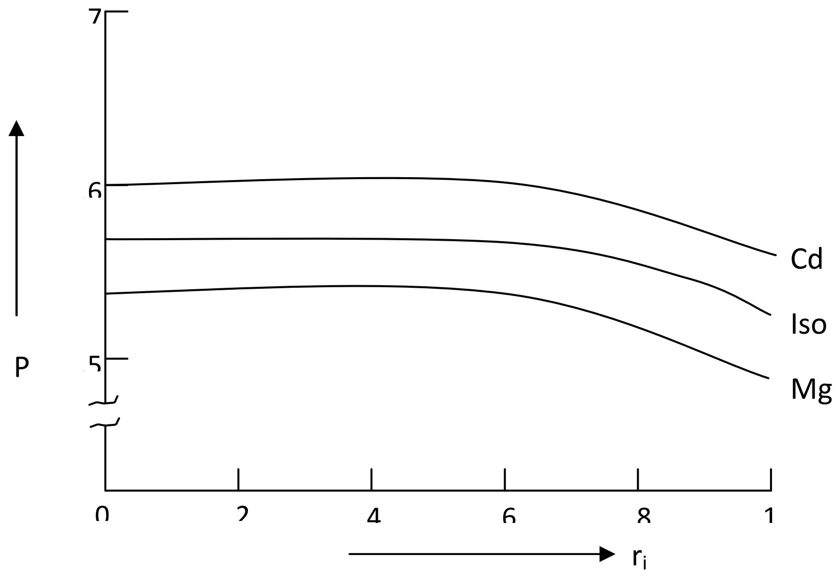
$$A_{nm} = \int_0^{\lambda} Z_n(y)\, Z_m(y) dy + A'_{nm} \tag{7.2}$$

where $A'_{nm} = 1/\pi^2 br_c\, [\lambda^{-1} \cos^2 \lambda r_i + r_i\, Si(2\lambda\, r_i) + \{(-1)^m + (-1)^n\}\ \{\lambda^{-1}\ \{\lambda^{-1} \sin\lambda\, r_0 \cos \lambda r_i + r_0\, Ci(2\lambda\, r_0) - b\, Ci(2\lambda b)\} + (-1)^{m+n+2}\ \{\lambda^{-1} \sin^2 \lambda r_0 - r_0\, Si(2\lambda r_0)\}\}]$ (7.3)

$$Si(x) = \int_{\infty}^{x} \frac{\sin t}{t} dt,$$

$$Ci(x) = \int_{\infty}^{x} \frac{\cos t}{t} dt,$$

The first integral of (7.2) is evaluated using Gauss Legendre formula. The upper limit λ is fixed equal to 20. The second term is also evaluated numerically. Thus, A_{nm} will be known. The outer radius r_0 of annual punch is taken as the unit of length and is fixed equal to 1.0. The inner radius r_i is made to vary from 0.1 to 0.9 in step of 0.2. The calculations are performed for transversely isotropic crystals Mg, and Cd values for C_{44}, ν_1, ν_2 etc are taken from [12]. The variation of total load P is shown in Fig. 2. It is noticed that variation of P with r_i is ordered as Cd > ISO > Mg. The values of total load P for isotropic medium (ISO) are taken from [8] and plotted.



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