Some Coincidence and Fixed Point Results for Hybrid Contraction

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ABSTRACT
Fixed point theory for multi-valued mappings has many useful applications in applied sciences, in particular in Game theory and Mathematical Economics. Thus it is natural to try of extending the known fixed point results for single-valued mappings to the setting of multi-valued mappings. Some theorems of existence of fixed points of single-valued mappings have been extended to the multi-valued case. For example, in 1969 Nadler extended the Banach contraction principle to multi-valued contractive mappings in complete metric space. However, many other questions remain open. Moreover, the study of existence of coincidence and fixed points for hybrid contraction that is a pair of single-valued and multi-valued maps became more interesting due to the recent investigation of Corley in 1986. He gave a good relationship between hybrid fixed points and optimization problems. Here we present the main known results and current research direction in this subject .This talk can be considered as a survey, but some new results are also included.
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1.*THE BACKGROUND OF METRICAL FIXED POINT THEORY :*

Let X be a nonempty set and $T : X \to X$ a self-map. We say that $x \in X$ is a fixed point of T if,

$$T(x) = x$$

and denote the set of all fixed points of T by $F_T = \{x \in X / T(x) = x\}$ or by Fix T.

EXAMPLE 1.1.

1) If $X = R$ and $T(x) = x^2 + 5x + 4$, then $F_T = \{-2\}$;

2) If $X = R$ and $T(x) = x^2 - x$, then $F_T = \{0, 2\}$;

3) If $X = R$ and $T(x) = x + 2$, then $F_T = \phi$;

4) If $X = R$ and $T(x) = x$, then $F_T = R$.

Let X be any set and $T : X \to X$ a self-map. For any given $x \in X$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the n^{th} iterate of x under T. In order to simplify the notations we will often use Tx instead of $T(x)$.

The mapping $T^n\,(n \ge 1)$ is called the n^{th} iterate of T. For any $x \in X$, the sequence $\{x_n\}_{n \ge 0} \subset X$ given by

$$x_n = Tx_{n-1} = T^n x_0, \qquad n = 1, 2..... \qquad (1)$$

is called the sequence of successive approximations with the initial value x_0. It is also known as the Picard iteration.

For a given self-map the following properties obviously hold:

1) $F_T \subset F_{T^n}$, for each $n \in N^*$;

2) $F_{T^n} = \{x\}$, for each $n \in N^* \Rightarrow F_T = \{x\}$;

The reverse of 2) is not true, in general, as shown by the next example.

EXAMPLE 1.2. Let $T:\{1,2,3\}\to\{1,2,3\}, T(1)=3, T(2)=2$ and $T(3)=1$. Then $F_T=\{2\}$ and $F_{T^2}=\{1,2,3\}$.

The fixed point theory is concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator $T:X\to X$, in order to obtain results on:

a) the existence (and uniqueness) of fixed points;
b) the construction of fixed points.

The ambient spaces X involved in fixed point theorems cover a variety of spaces: lattice, metric space, normed space, generalized metric space, uniform space, linear topological space etc., while the conditions imposed on the operator T are generally metrical or compactness type conditions. In order to introduce the most important ones, we need some minimal functional analysis background.

DEFINITION 1.1. Let (X,d) be a metric space. A mapping $T:X\to X$ is called:

(a_1) Lipschitzian (or L-Lipschitzain) if there exist $L>0$ such that

$d(Tx,Ty)\le Ld(x,y)$, for all $x,y\in X$;

(a_2) (strict) contraction (or a-contraction) if there exists a constant $a\in(0,1]$ such that T is a-Lipschitzian;

(a_3) nonexpansive if T is 1-Lipschitzian;

(a_4) contraction if $d(Tx,Ty)<d(x,y)$, for all $x,y\in X, x\neq y$;

(a_5) isometry if $d(Tx,Ty)=d(x,y)$, for all $x,y\in X$;

EXAMPLE 1.3.

1) $T:R\to R\ T(x)=\frac{1}{2}x+3,\quad x\in R$, is a strict contraction and $F_T=\{6\}$;

2) The function $T:\left[\frac{1}{2},2\right]\to\left[\frac{1}{2},2\right], T(x)=\frac{1}{x}$, is 4-Lipschitzian with $F_T=\{1\}$, while the function T in Example 1.1(3-4) are isometrics;

3) $T:[1,+\infty]\to[1,+\infty], T(x)=x+\frac{1}{x}$, is contractive and $F_T=\phi$.

The following theorem is of fundamental importance in the metrical fixed point theory.

THEOREM 1.1 [Banach Contraction Principle]. Let T be a contraction mapping, with Lipschitz constant k of a complete metric space X into itself. Then T has a unique fixed point.

In 1969, Nadler [38] extended the Banach contraction principle for multi-valued maps and introduced the concept of multi-valued contraction mapping. In his result he established that a multi-valued contraction mapping posses a fixed point in complete metric space.

Let (X,d) be a metric space. Then following Nadler [op.cit] define,

$$CL(X)=\{A:A \text{ is a non-empty closed subset of } X\}.$$

$$CB(X)=\{A:A \text{ is a non-empty closed and bounded subset of } X\}.$$

$$C(X)=\{A:A \text{ is a non-empty compact subset of } X\}.$$

For $A,B\in CB(X)$, and $x\in X$ define

$$D(A,B)=\inf\{d(a,b);a\in A,b\in B\}.$$

$$d(x,A)=\inf\{d(x,a);a\in A\}.$$

$$\delta(A,B)=\sup\{d(a,b);a\in A,b\in B\}.$$

$$H(A,B)=\max\left(\sup\{D(a,B);a\in A\},\sup\{D(A,b);b\in B\}\right).$$

H is called generalized Hausdorff distance function for $CL(X)$ induced by d. If $H(A,B)$ is defined for $A,B \in CB(X)$ (resp. $CL(X)$), then the pair $(CB(X),H)$ (resp. $CL(X),H$) is a metric space and H is called the Hausdorff metric.

DEFINITION 1.2 [38]. Let $T: X \to 2^X$ be a multi-valued mapping, then a point $x \in X$ is said to be a fixed point of T if $x \in Tx$.

The following lemma was proved in Nadler [op.cit].

LEMMA 1.1 [38]. Let A,B be in $CB(X)$. Then for all $\varepsilon > 0$ and $a \in A$, there exist $b \in B$ such that

$$d(a,b) \le H(A,B) + \varepsilon.$$

If A,B are in $C(X)$, then one can choose $a \in A$ and $b \in B$ such that

$$d(a,b) \le H(A,B).$$

The following theorem is an extension of Banach contraction principle for multi-valued mapping, which was obtained by Nadler [op.cit].

THEROEM 1.2 [38]. Let (X,d) be a complete metric space, and let $F: X \to CB(X)$ be a multi-valued mapping. Assume that there exist $r \in [0,1)$ such that for all $x, y \in X$

$$H(Fx,Fy) \le rd(x,y).$$

Then there exist $z \in X$, such that $z \in Fz$.

EXAMPLE 1.4. Let $X = [0,1]$ and $f: X \to X$ such that

$$f(x) = \begin{cases} \frac{1}{2}x + \frac{1}{2} & 0 \le x \le \frac{1}{2} \\ -\frac{1}{2}x + 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Define $F: X \to 2^X$ by $F(x) = \{0\} \cup \{f(x)\}$ for each $x \in X$. Then one can easily check that F is a multi-valued contraction mapping and the set of fixed points of F is $\left\{0, \frac{2}{3}\right\}$.

We now discuss contractive multi-functions and state a theorem, proved by Smithson [57], which extends Edelstein's FPT for contractive single-valued mappings to multi-functions

DEFINITION 1.3. An orbit $O(x)$ of a multi-function $T: X \to CB(X)$ at the point x is a sequence $\{x_n : x_n \in T(x_{n-1})\}$ where $x_0 = x$. $O(x)$ is called regular iff

$$d(x_{n+1}, x_{n+2}) \le d(x_n, x_{n+1}) \text{ and } d(x_{n+1}, x_{n+2}) \le H(T(x_n), T(x_{n+1})).$$

DEFINITION 1.4. A multi-function T is said to be contractive iff for each $x_1, x_2 \in X$ with $x_1 \ne x_2$, $H(T(x_1), T(x_2)) < d(x_1, x_2)$.

An immediate consequence of the definition is the following: If $y_1 \in T(x_1)$, then there is an element $y_2 \in T(x_2)$ such that $d(y_1, y_2) < d(x_1, x_2)$.

REMARK 1.1. Let T be a point compact, contractive multi-function. Define an orbit $O(x)$ by choosing $x_n \in T(x_{n-1})$ such that

$$d(x_{n-1}, x_n) = D(x_{n-1}, T(x_{n-1})) = \inf\{d(x_{n-1}, y) : y \in T(x_{n-1})\}.$$

Since T is contractive, the orbit $O(x)$ is regular.

THEOREM 1.3 [57]. Let T be a point closed, contractive multi-function. If there is a regular orbit $O(x)$ for T which contains a subsequence $\{x_{n_i}\}$ converging to y_0 such that $x_{n_i+1} \to y_1$, then $y_1 = y_0$, that is, T has a fixed point.

There have been several extensions of known fixed point theorems for multi-valued mappings which take each point of a metric space (X,d) into a closed subset K of X. However, in many applications, the mappings involved is not a self-mapping of K. Assad and Krik [3] gave sufficient conditions for such mappings to have a fixed point by proving a fixed point theorem for multi-valued contraction mappings on a complete metrically convex metric space and by putting certain boundary conditions on the mappings. Similar results for multi-valued contractive mappings were obtained by Assad [2]. Itoh [23] extended the results given in Assad and Krik [op.cit] and Assad [2] for more general types of contraction and contractive mappings. Khan [31] extended the result of Itoh [op.cit] for a pair of generalized contraction and contractive mappings. He also partially generalized the fixed point theorem of Iseki [22] and Rus [48]. Later on Ciric [9] extend the result of Nadler [op.cit] by introducing the generalized multi-valued contraction. After this many Mathematician investigated and generalized various result on existence of fixed point for multi-valued mappings (see for instance [1], [7], [12], [13], [14], [19], [28], [29], [31], [32], [34], [35] [42], [44], [45], [56], [58] and references therein).

2. HYBRID CONTRACTION:

Let (X,d) be a metric space, $P: X \to X$ and $T: X \to CL(X)$, the set of (nonempty) closed subset of X. Consider the following conditions on T for x, y in X and some positive number $k < 1$,

(1) $$H(Tx,Ty) \le kd(Px,Py)$$

and

(2) $$H(Tx,Ty) \le k \max\left\{d(Px,Py), D(Py,Ty), \frac{1}{2}\left[D(Px,Ty)+D(Py,Tx)\right]\right\},$$

where H is a generalized Hausdorff metric induced by d.

The above conditions are generally termed as **hybrid contractions** (see for instance, [4], [39]). Note that (1) implies (2). We say that a point z in X is:

(i) A coincidence point of T and P iff $Pz \in Tz$;

(ii) A fixed point of T and P iff $z = Pz \in Tz$

and

(iii) A hybrid fixed point iff $Pz \in TPz$.

We have to emphasize that T and P satisfying (1) with $T(X) \subset P(X)$ need not have a common fixed point in complete X even if P and T are continuous and commuting, that is, $PTx \subset TPx, x \in X$ (cf. Itoh-Takahashi [24]). We refer [30], [39], [49] and [52] for counterexamples and a good discussion on this aspect. Further, the condition (1) with $Px = x, (x \in X)$ contains the Nadler's (new classic) multi-valued contraction [38]. Interesting generalizations of Nadler's contraction [op. cit.] due to Ciric [op.cit], Iseki [op.cit], Ray [42] and Reich [44], are the special cases of (2). For example, the condition (2) with $Px = x, (x \in X)$ was first studied by Ciric [op. cit]. The condition (2) with $T: X \to X$ and $Px = x, (x \in X)$ is the condition $(21')$ of Rhoades [46, P. 267] (see also [20], [48],)

Hybrid fixed point theory is a recent development is the ambit of fixed point theorems for contracting single-valued and multi-valued maps in metric spaces. Indeed the study of such maps was initiated during 1980-1983 by Bhaskaran and Subramanyam [6], Hadzic [18], Singh- Kulshresta [54], Kaneko [29], Kaneko-Sessa [30], Naimpally et al. [39], Rhoades et al. [47] etc. For a history of fundamental work on this line refer to Singh and Mishra [55] and for more recent work on this line Beg and Azam [5], Jungck and Rhoades [27], Kaneko [28], Mishra- Singh and Talwar [37], Pathak et al. [40] etc.

Hybrid fixed point theory has potential application in Functional Inclusions, Optimization Theory, Fractal Graphics and Discrete dynamics for set-valued Operator (see, [10],[60]) . Recent investigations of Corley [10], gave a good relationship between hybrid fixed points and optimization problems.

In particular, he has shown that a Pareto type of maximization problem is equivalent to a hybrid fixed point problem. So, this is an additional motivation which attracts the researchers to work in this direction.

The following fundamental coincidence theorem for a pair of multi-valued and single-valued maps is essentially due to Singh- Kulshrestha [54] (see also Kulshrestha [33] and Singh-Mishra [55]).

THEOREM 2.1 [54]. Let (X,d) be a metric space and $(CL(X),H)$ is the Hausdorff metric space induced by d, where $CL(X)$ is the collection of all non-empty closed subset of X. Let $P:X\to CL(X)$ and $f:X\to X$ be such that $P(X)\subseteq f(X)$ and

$$H(Px,Py)\le q\max\left\{d(fx,fy),d(fx,Px),d(fy,Py),\frac{1}{2}\left[d(fx,Py)+d(fy,Px)\right]\right\},$$

for all $x,y\in X$, where $0\le q<1$. If $f(X)$ [or $P(X)$] is a complete subspace of X, then P and f have coincidence that is , there exist a point $z\in X$ such that $fz\in Pz$.

Later on Singh et al. [52] proved a fixed point theorem for hybrid maps by using the concept of asymptotic regularity condition and investigate different sets of conditions under which the fixed point equation $x=fx\in Sx\bigcap Tx$ for $x\in X$ possesses a solution where f is a single valued map from X to X and S and T are multivalued maps from X to CL(X).

2.1 MATKOWSKI'S CONTRACTION

Recently Baillon-Singh [4] motivated by the work of Corley [10], Gairola et al. [17], Czerwik [11], Reddy-Subrahmanyam [43], Singh-Gairola [51], Singh et al. [52], Matkowski [36], have introduced coordinatewise weakly commuting system of single-valued and multi-valued maps on the product of metric spaces and proved a coincidence theorem .

In 1997, Gairola et al. [16] introduced the concept of coordinatewise asymptotically commuting systems of single-valued and multi-valued maps on the product of metric spaces and give a coincidence theorem for such a system of multi-valued and two system of single-valued maps on the product of n metric space. We use the following notations.

Let (a_{ik}) be an $n\times n$ square matrix with non-negative entries defined in Czerwik [11] and Matkowski [op.cit].

$$c_{ik}^{1}=\begin{cases}a_{ik} & i\ne k\\ 1-a_{ik} & i=k\end{cases}\qquad i,k=1,........n \tag{A}$$

$$c_{ik}^{t+1}=\begin{cases}C_{11}^{t}C_{i+1,k+1}^{t}+C_{i+1,1}^{t}C_{1,k+1}, & i\ne k\\ C_{11}^{t}C_{i+1,k+1}^{t}-C_{i+1,1}^{t}C_{1,k+1}, & i=k\end{cases} \tag{B}$$

$$t=1,.............,n-1,\quad i,k=1,.............,n-t$$

Let $$c_{i,i}^{t}>0,\quad t=1,............n,\quad i=1,.............n-t+1. \tag{C}$$

Throughout this lecture we shall assume that (X_i,d_i) are metric spaces $(CL(X_i),H_i)$ the generalized Hausedorff metric spaces induced by d_i. Further, let P_i and Q_i stand for multi-valued maps from $X=X_1\times......\times X_n=(X_1,.....,X_n)$ to $CL(X_i)$, and $T_i:X\to X_i, i=1,......,n.$ For $X\supset A=(A_1,......,A_n)$, we (as in [4] use the notation $T(A)=(T_1A_1,.......,T_nA_n)$.

DEFINITION 2.1.1 [4]. Two systems of maps $\{T_1,.......,T_n\}$ and $\{P_1,.....,P_n\}$ are co-ordinatewise commuting (or simply commuting) at a point $x\in X$ if and only if

$$T_i(P_1x,......,P_nx)\subseteq P_i(T_1x,.....,T_nx),\quad i=1,2,.....,n.$$

For $n=1$, this definition is that of Itoh-Takahashi [24]. For $n=1$, the following definition is investigated in [29] and [52].

DEFINITION 2.1.2 [4]. Two systems of maps $\{T_1,.......,T_2\}$ and $\{P_1,......,P_n\}$ are co-ordinatewise weakly commuting (or simply weakly commuting) at a point $x \in X$ if and only if

$$H_i\left(T_i\left(P_1x,....,P_nx\right),P_i\left(T_1x,.......,T_nx\right)\right)\le D_i\left(P_ix,T_ix\right),\ i=1,........,n.$$

Two system are co-ordinatewise weakly commuting on X if and only if they are co-ordinatewise weakly commuting at every point of X.

An equivalent formulation of Definition 2.1.2 for two system of single-valued maps on X appears in [17].

We should remark that, in general, co-ordinatewise weakly commuting systems of maps need not to be co-ordinatewise commuting. However, the commuting systems are necessarily weakly commuting (see [4], [17], [51]).

DEFINITION 2.1.3 [16]. Two system of maps $\{T_1,........,T_n\}$ and $\{P_1,........,P_n\}$ are coordinate wise asymptotically commuting (or simply asymptotically commuting) if and only if

$$H_i\left(P_i\left(T_1x^m,.......,T_nx^m\right),T_i\left(P_1x^m,.......,P_nx^m\right)\right)\to 0 \quad \left(as \quad m\to\infty\right),$$

whenever $\{x^m\}$ is a sequence in X such that

$$P_ix^m \to M_i \in CL\left(X_i\right) \text{ and } T_ix^m \to x_i \in M_i.$$

DEFINITION 2.1.4 [16]. The mappings $T_1 : X_1 \to X_1$ and $P_1 : X_1 \to CL\left(X_1\right)$ are asymptotically commuting (called compatible in [5] and [51] for $T_1 : X_1 \to X_1$ and $P_1 : X_1 \to CB\left(X_1\right)$ if and only if $H_1\left(P_1T_1x^m,T_1P_1x^m\right)\to 0$ (as $m\to\infty$) whenever $\{x^m\}$ is a sequence in X_1 such that $P_1x^m \to M_1 \in CL\left(X_1\right)$ and $T_1x^m \to u_1 \in M_1$.

If the map P_1 in this definition is single-valued then M_1 has just a single element u_1, and we get the definition of asymptotically commuting (or compatible) single-valued maps independently introduced by Tivari-Singh [59] and Jungck [26]. Since a sequence in the limiting tone is the main aspect in Definition 2.1.3-2.1.4, the name "asymptotically commuting maps" seems to slightly better fit to the situation that "compatible maps". So, following [59], we shall henceforth prefer the name "asymptotically commuting".

REMARK 2.1.1. The class of asymptotically commuting maps includes commuting and weakly commuting maps. Commuting maps are necessarily weakly and asymptotically commuting both (see, for instance, [4], [5], [25], [26], [29], [50], [51], [52] and the following example).

EXAMPLE 2.1.1. Let $X_1=[1,\infty)$ and $X_2=[0,\infty)$ be metric spaces with the absolute value metric. Let $x=\left(x_1,x_2\right),P_1x=\left[1,x^2\right],P_2x=\left[\frac{x^2}{4},\frac{x^2}{2}\right],T_1x=2x^3-1$ and $T_2x=\frac{x^2}{5}$. It can easily be verified that the systems of maps $\{P_1,P_2\}$ and $\{T_1,T_2\}$ are not co-ordinatewise weakly commuting but co-ordinatewise asymptotically commuting on $x=X_1\times X_2$. Note that the above two systems are co-ordinatewise commuting at $x=(1,0)$.

REMARK 2.1.2. At any point of coincidence of two (or two systems of) maps, their commutativity, weak commutativity and asymptotic commutativity are equivalent at that point (see [4], [25], and [29]).

Now we state the result proved in [17]

THEOREM 2.1.1 [17]. Let $\left(X_i,d_i\right),i=1,..........n,$ be a complete metric space and assume that $P_i : X \to CL\left(X_i\right),S_iT_i : X\to X_i,i=1,........n$ are continuous maps such that

$$P_i\left(X\right)\subset S_i\left(X\right)\cap T_i\left(X\right),\quad i=1,...........n. \tag{2.1.1}$$

The system $\{P_1..........P_n\}$ is asymptotically commuting with both the systems $\{S_1.........S_n\}$ and $\{T_1.........T_n\}$. $(2.1.2)$

If there exist non-negative numbers $b<1$ and a_{ik} defined in (A) and (B) such that (C) and the following hold

$$H_i\left(P_ix,P_iy\right)\le \max\left(\sum_{k=1}^{n} a_{ik}d_k\left(S_kx,T_ky\right), b\max\begin{Bmatrix} D_i\left(S_ix,P_ix\right), D_i\left(T_iy,P_iy\right) \\ \frac{1}{2}\left[D_i\left(S_ix,P_iy\right)+D_i\left(T_iy,Px_i\right)\right]\end{Bmatrix}\right) \quad (2.1.3)$$

for all $x=\left(x_1,.........x_n\right), y=\left(y_1,.........y_n\right)\in X$, then there exist a point $u\in X$ such that $S_iu\in P_1u$ and $T_iu\in P_iu, \quad i=1,........,n.$

DEFINITION 2.1.5 [15]. Two systems of maps $\{T_1,.....,T_n\}$ and $\{P_1,.......,P_n\}$ are co-ordinatewise R-weakly commuting at a point $x\in X$ if and only if

$$H_i\left(P_i\left(T_1x,....,T_nx\right),T_i\left(P_1x,.......,P_nx\right)\right)\le RD_i\left(T_ix,P_ix\right),\ i=1,........,n,\ R\ge 0.$$ two systems are co-ordinatewise R-weakly commuting on X if and only if they are co-ordinatewise R-weakly commuting at every point of X. As a special case of the above definition $(n=1)$ we have the following.

DEFINITION 2.1.6 [15]. The mappings $T_1:X_1\to X_1$ and $P_1:X_1\to CL\left(X_1\right)$ are R-weakly commuting if and only if

$$H_1\left(P_1T_1x,T_1P_1x\right)\le RD_1\left(P_1x,T_1x\right),\quad R\ge 0.$$

This example illustrate non co-ordinatewise weakly commuting maps are co-ordinatewise R-weakly commuting.

EXAMPLE 2.1.2. Let $X_1=[1,\infty)$ and $X_2=[0,1]$ be metric spaces with the absolute value metric. Let $x=\left(x_1,x_2\right)$,

$$P_1x=\left\{x_1^2\right\},\quad P_2x=\{0,1\},\quad T_1x=2x_1-1,\quad T_2x=1-x_2;$$

then

$$H_1\left(T_1\left(P_1x,P_2x\right),P_1\left(T_1x,T_2x\right)\right)=H_1\left(\left\{2x_1^2-1\right\},\left\{\left(2x_1-1\right)^2\right\}\right)$$
$$=2\left(x_1-1\right)^2=2D_1\left(2x_1-1,\left\{x_1^2\right\}\right)=2D_1\left(T_1x,P_1x\right),$$

and

$$H_2\left(T_2\left(P_1x,P_2x\right),P_2\left(T_1x,T_2x\right)\right)=H_2\left(\{0,1\},\{(0,1)\}\right)=0\le D_2\left(T_2x,P_2x\right).$$

The systems of maps $\{T_1,T_2\}$ and $\{P_1,P_2\}$ are thus co-orddinatewise R-weakly commuting with $R=2$ but not co-ordinatewise weakly commuting.

REMARK 2.1.3. Co-ordinatewise weakly commuting maps are co-ordinatewise R-weakly commuting, however, R-weakly commutativity implies co-ordinatewise weak commutativity only when $R\le 1$.

This example illustrate non co-ordinatewise asymptotically commuting maps are co-ordinatewise R-weakly commuting.

EXAMPLE 2.1.4. Let $X_1=X_2=[2,6]$ be a metric spaces with usual metric such that

$$P_1(x_1,x_2)=\begin{cases}\{2\} & \textit{if } x_1<3\\ \{4,5\} & \textit{if } x_1=3\\ \left\{\dfrac{x_1+21}{12}\right\} & \textit{if } 3<x_1\le 6\end{cases};$$

$$P_2(x_1,x_2)=\begin{cases}\{2\} & \textit{if } x_2<3\\ \{4,5\} & \textit{if } x_2=3\\ \dfrac{x_2+21}{12} & \textit{if } 3<x_2\le 6\end{cases};$$

$$T_1(x_1,x_2)=\begin{cases}2 & \textit{if } x_1=2\\ 2x_1 & \textit{if } 2<x_1\le 3\\ \dfrac{2x_1}{3} & \textit{if } 3<x_1\le 6\end{cases};$$

$$T_2(x_1,x_2)=\begin{cases}2 & \textit{if } x_2=2\\ 2x_2 & \textit{if } 2<x_2\le 3\\ \dfrac{2x_2}{3} & \textit{if } 3<x_2\le 6\end{cases}.$$

To see that the systems of maps $\{T_1,T_2\}$ and $\{P_1,P_2\}$ are co-ordinatewise non asymptotically commuting, consider a decreasing sequence $\{(x_1^m,x_2^m)\}$ in $X_1\times X_2$ such that $3<x_i<4,\ i=1,2$ and $\lim(x_1^m,x_2^m)\to(3,3)$. Then

$$T_1(x_1^m,x_2^m)\to 2,\qquad T_2(x_1^m,x_2^m)\to 2,$$
$$P_1(x_1^m,x_2^m)\to\{2\},\quad P_2(x_1^m,x_2^m)\to\{2\},$$
$$T_1\left(P_1(x_1^m,x_2^m),P_2(x_1^m,x_2^m)\right)\to 4,$$
$$T_2\left(P_1(x_1^m,x_2^m),P_2(x_1^m,x_2^m)\right)\to 4,$$
$$P_1\left(T_1(x_1^m,x_2^m),T_2(x_1^m,x_2^m)\right)\to\{2\},$$

and

$$P_2\left(T_1(x_1^m,x_2^m),T_2(x_1^m,x_2^m)\right)\to\{2\}.$$

Moreover, the systems of maps are co-ordinatewise R-weakly commuting.

We, motivated by the work of Baillon-Singh [4] and Gairola-Mishra-Singh [16], prove coincidence theorems for Ciric-Matkowski type hybrid contractive condition for systems of multi-valued and single-valued maps.

THEOREM 2.1.2 [15]. Let $(X_i,d_i),\ i=1,....,n$ be complete metric space and $P_i,Q_i:X\to CL(X_i),T_i:X\to X_i,\ i=1,.....,n$ are continuous maps such that

(i) $P_i(X)\cup Q_i(X)\subset T_i(X),\ i=1,.......,n;$

(ii) The system $\{T_1,........,T_n\}$ is R-weakly commuting with the systems $\{P_1,.........,P_n\}$. and $\{Q_1,......,Q_n\}$.

If there exist non negative numbers $b<1$ and a_{ik} defined in $(A),(B)$ and (C) and the following hold:

(iii) $$H_i\left(P_i x, Q_i y\right) \le \max\left\{\sum_{k=1}^{n} a_{ik} d_k\left(T_k x, T_k y\right), b\max\left\{\begin{matrix} D_i\left(T_i x, P_i x\right), D_i\left(T_i y, Q_i y\right), \\ \frac{1}{2}\left[D_i\left(T_i x, Q_i y\right) + D_i\left(T_i y, P_i x\right)\right]\end{matrix}\right\}\right\},$$

for all $x=\left(x_1,......,x_n\right)=x\left(1,n\right)\in X, y=\left(y_1,.......,y_n\right)=y\left(1,n\right)\in X$, then there exist a point $u=u\left(1,n\right)\in X$ such that

(iv) $T_i u \in P_i u \bigcap Q_i u, \quad i=1,.......,n.$

COROLLARY 2.1.1 [15]. Let $\left(X_i,d_i\right), i=1,....,n$ be complete metric space and $P_i,Q_i : X \to CL\left(X_i\right), i=1,.....,n$ be multivalued maps such that

$$H_i\left(P_i x, Q_i y\right) \le \max\left\{\sum_{k=1}^{n} a_{ik} d_k\left(x_k, y_k\right), b\max\left\{\begin{matrix} D_i\left(x_i, P_i x\right), D_i\left(y_i, Q_i y\right), \\ \frac{1}{2}\left[D_i\left(x_i, Q_i y\right) + D_i\left(y_i, P_i x\right)\right]\end{matrix}\right\}\right\},$$

for all $x,y\in X$, where $b\in[0,1)$ and a_{ik} defined in $(A),(B)$ and (C) then the system of multivalued maps $\left(P_1,......,P_n\right)$ and $\left(Q_1,......,Q_n\right)$ has a common fixed point.

REMARK 2.1.4 The result of corollary includes a multitude of contractive condition for single and multivalued maps (see, for instance [9], [11], [28], [38], [47], [48], [51], [52] and [54]). The following result in corollary with $\left(Y,d\right)=(X_i,d_i), \quad P=P_i, \quad Q=Q_i, \; i=1,...,n$ and $n=1, \; \max\{a_{11},b\}=k(say)$.

COROLLARY 2.1.2. Let $\left(Y,d\right)$ be a complete metric space and $P,Q:Y\to CL(Y)$. oif there exists $k, \; 0<k<1$ such that for all $x,y\in Y$.

$$H\left(Px,Qy\right)\le k\max\left\{d(x,y), D(x,Px), \frac{1}{2}[D(x,Qy)+D(y,Px)]\right\},$$

then P and Q have a cmmon fixed point.

2.2 BRANCIARI CONTRACTION:

We need the following definitions before stating the reults

DEFINITION 2.2.1 [53]. Maps $S:X\to X$ and $T:X\to CL(X)$ are said to be (IT)-commuting at a point $v\in X$ if $STv\subset TSv$. Further S and T are (IT)-commuting on X if they are (IT)-commuting at each point $v\in X$.

We remark that the (IT)-commutativity of a hybrid pair (S,T) at a point v is more general than its compatibility (cf. [53]) and weak compatibility at a point v. Maps S and T are commuting at $v\in X$ when $STv=TSv$. Clearly a commuting hybrid pair of maps is (IT)-commuting and the reverse implication is not true (cf. [24], see also [53] and [55]).

DEFINITION 2.2.2 [53]. Let $A:X\to CL(X)$ and $S:X\to X$. Then A and S will satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = M\in CL(X) \text{ and } \lim_{n\to\infty} Sx_n = t\in M.$$

Motivated by Singh-Hashim [53] and Branciari [8]. Rawat [41] proved the following results for the hybrid contraction.

Define $\phi = \{ f : f : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative} and f satisfies the following inequality

(i) $\int_0^s f(t)dt > 0$ for each $s > 0$.

First we present the basic result for hybrid pair of maps proved in [41].

THEOREM 2.2.1. Let (X,d) be a metric space and $A : X \to CL(X)$ and $S : X \to X$ be such that

(2.2.1) $AX \subset SX$;

(2.2.2) the pair (S, A) satisfies the property (E.A)

(2.2.3) $\int_0^{H(Ax,Ay)} \varphi(s)ds < m(x,y)$ when $m(x,y) > 0$, where

$$m(x,y) = \max\left\{ \int_0^{d(Sx,Sy)} \varphi(s)ds, \alpha\int_0^{[d(Sx,Ax)+dSy,Ay]} \varphi(s)ds, \alpha\int_0^{[d(Sx,Ay)+d(Sy,.Ax)]} \varphi(s)ds \right\},$$

$0 \le \alpha < 1$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and satisfy the condition

(2.2.4) $\int_0^{\varepsilon} \varphi(s)ds > 0$ for each $\varepsilon > 0$.

If $A(X)$ or $S(X)$ is a complete subspace of X then $C(S, A)$ is nonempty. Further A and S have a common fixed point provided that $SSz = Sz$ and A and S are (IT)- commuting at $z \in C(S, A)$.

Now we state the result for a hybrid quadruple of maps on an arbitrary nonempty set

THEOREM 2.2.2. Let (X,d) be a metric space and $A, B : X \to CL(X)$ and $S, T : X \to X$ such that

(2.2.5) $AX \subset TX$ and $BX \subset SX$,

(2.2.6) One of the pair (S, A) and (T, B) satisfies the property (E.A)

(2.2.7) $\int_0^{H(Ax,Ay)} \varphi(s)ds < m(x,y)$ when $m(x,y) > 0$, where

$$m(x,y) = \max\left\{ \int_0^{d(Sx,Ty)} \varphi(s)ds, \alpha\int_0^{[d(Sx,Ax)+dTy,By]} \varphi(s)ds, \alpha\int_0^{[d(Ty,Ax)+d(Sx,By)]} \varphi(s)ds \right\},$$

$0 \le \alpha < 1$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and satisfy the condition

(2.2.8) $\int_0^{\varepsilon} \varphi(s)ds > 0$ for each $\varepsilon > 0$.

If $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is a complete subspace of X then $C(S, A)$ and $C(B,T)$ are nonempty. Further,

(I) A and S have a common fixed point Su provided that $SSu = Su$ and A, S are (IT)-commuting at $u \in C(S, A)$.

(II) B and T have a common fixed point Tv provided that $TTv = Tv$ and B, T are (IT)-commuting at $v \in C(T, B)$.

(III) A, B, S and T have a common fixed point provided that (I) and (II) are true.

Now we are stating the another version of the above Theorem.

THEOREM 2.2.3. Let (X,d) be a metric space and $A, B : X \to CL(X)$ and $S, T : X \to X$ such that

(2.2.9) $AX \subset TX$ and $BX \subset SX$,

(2.2.10) One of the pair (S, A) and (T, B) satisfies the property (E.A)

(2.2.11) $\int_0^{H(Ax,Ay)} \varphi(s)ds < m(x,y)$ when $m(x,y) > 0$, where

$$m(x,y) = \max\left\{\int_0^{d(Sx,Ty)} \varphi(s)ds, \tfrac{\alpha}{2}\int_0^{[d(Sx,Ax)+d(Ty,By)]} \varphi(s)ds, \tfrac{1}{2}\int_0^{[d(Ty,Ax)+d(Sx,By)]} \varphi(s)ds\right\},$$

$0 \le \alpha < 2$ and $\varphi : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative and satisfy the condition

(2.2.12) $\int_0^{\varepsilon} \varphi(s)ds > 0$ for each $\varepsilon > 0$.

If $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is a complete subspace of X then $C(S,A)$ and $C(B,T)$ are nonempty. Further,

(Ia) A and S have a common fixed point Su provided that $SSu = Su$ and A, S are (IT)-commuting at $u \in C(S,A)$.

(IIa) B and T have a common fixed point Tv provided that $TTv = Tv$ and B, T are (IT)-commuting at $v \in C(T,B)$.

(IIIa) A, B, S and T have a common fixed point provided that (I) and (II) are true.

REMARK 2.2.1. Every contractive condition of integral type automatically include a corresponding contractive condition not involving integrals by setting $\varphi(s) = R^+$ over R^+.

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