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Fixed Point Theorems for Four Mappings in Fuzzy Metric Space using Implicit Relation

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Abstract
The present paper deals with proving some common fixed point results in a fuzzy metric space using Implicit Relation.
Index Terms: Fixed point, fuzzy metric space, occasionally weakly compatible mappings, t-norm, Implicit Relation.

I. Introduction

It proved a turning point in the development of fuzzy mathematics when the notion of fuzzy set was introduced by Zadeh [12] in 1965. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [15] and George and Veeramani [2] modified the notion of fuzzy metric spaces with the help of continuous t-norm, which shows a new way for further development of analysis in such spaces. Consequently in due course of time some metric fixed points results were generalized to fuzzy metric spaces by various authors. Sessa [20] improved commutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space. Vasuki [20] proved fixed point theorems for R-weakly commuting mapping. Pant [18, 19] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [15] and weakly compatible maps by [9] in fuzzy metric space is generalized by A.Al Thagafi and Naseer Shahzad [1] by introducing the concept of occasionally weakly compatible mappings.

In this paper we give a fixed point theorem on fuzzy metric space with an implicit relation defined in [14]. Our results extend and generalize the result of Wadhwa and Dubey [11] using Chouhan, Khanduja and Singh [14].

We start with some preliminaries:

II. Preliminaries

Definition 2.1 [4]: A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is continuous t-norm if satisfies the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a*1 = a$ for all $a \in [0, 1]$,
(4) $a*b \le c*d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Examples of t- norm are $a*b = \min\{a, b\}$ and $a*b = ab$.

Definition 2.2 [2]: A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0, \infty)$ satisfying the following conditions: (The functions $M(x, y, t)$ denote the degree of nearness between x and y with respect to t, respectively.)

1) $M(x, y, 0) = 0$ for all $x, y \in X$
2) $M(x, y, t) = 1$ for all $x, y \in X$ and $t>0$ if and only if $x = y$
3) $M(x, y, t) = M(y, x, t)$ for all $x, y \in X$ and $t>0$
4) $M(x, y, t)*M(y, z, s) \le M(x, z, t+s)$ for all $x, y, z \in X$ and $t, s > 0$
5) For all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Remark 2.1: In a FM $M(X, M, *)$, $M(x, y, .)$ is non- decreasing for all x.

Example 2.3 (Induced fuzzy metric [2]): – Let (X, d) be a metric space. Denote $a*b = ab$ for all $a, b \in [0, 1]$ and let

M_d be fuzzy sets on $X^2 \times [0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t+d(x,y)}$$

Then $(X, M_d, *)$ is a fuzzy metric space. We call this fuzzy metric induced by a metric d as the standard intuitionistic fuzzy metric.

Definition 2.4: Let $M(X, M, *)$ be a FM - space. Then

(i) A sequence $\{x_n\}$ in X is said to be Cauchy Sequence if for all $t>0$ and $p>0$
$\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$

(ii) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if for all $t>0$,
$\lim_{n\to\infty} M(x_n, x, t) = 1$

Since * is continuous, the limit is uniquely determined from (5) and (11) respectively.

Definition 2.5: A FM-Space M(X, M, *) is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.6: Let A and S be maps from a fuzzy metric M(X, M, *) into itself. The maps A and S are said to be weakly commuting if $M(ASz, SAz, t) \geq M(Az, Sz, t)$ for all $z \in X$ and $t>0$

Definition 2.7 [10]: Let f and g be maps from an FM-space M(X, M, *) into itself .The maps f and g are said to be compatible if for all t>0, $\lim_{n\to\infty} M(fgx_n, gfx_n, t) = 1$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.

Definition 2.8: Two mappings A and S of a fuzzy metric space M(X, M, *) will be called reciprocally continuous if $ASu_n \to Az$ and $SAu_n \to Sz$, whenever $\{u_n\}$ is a sequence such that for some Au_n, $Su_n \to z$ for some $z \in X$.

Definition 2.9: Let M(X, M, *) be a fuzzy metric space. f and g be self maps on X. A point x in X is called a coincidence point of f and g iff fx =gx. In this case w = fx =g x is called a point of coincidence of f and g.

Definition 2.10[8]: A pair of maps S and T is called weakly compatible pair if they commute at the coincident points i.e., if Su = Tu for some u in X then STu = TSu.

Definition 2.11[1]: Two self maps f and g of a set X are called occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

Example 2.11.1[1]: Let R be the usual metric space. Define S, T: R→R by Sx=2x and $Tx=x^2$ for all $x\in R$. Then Sx=Tx for x=0, 2, but ST0=TS0, and ST2≠TS2. S and T are occasionally weakly compatible self maps but not weakly compatible.

Definition 2.12 (Implicit Relation)[14]: Let ϕ_5 be the set of all real and continuous function $\phi : (R^+)^5 \to R$ and such that

2.12 (i) ϕ non increasing in 2nd and 4th argument and

2.12 (ii) for $u, v \geq 0$, $\phi(u,v,v,v,v)\geq 0 \Rightarrow u \geq v$

Example: $\phi(t_1, t_2, t_3, t_4, t_5) = t_1 - \max\{t_1, t_2, t_3, t_4\}$.

Lemma 2.1: Let $\{u_n\}$ be a sequence in a fuzzy metric space M(X, M, *). If there exist a constant $k \in (0, 1)$ such that

$M(u_n, u_{n+1}, kt) \geq M(u_{n-1}, u_n, t)$ for all t>0 and n = 1 , 2 , 3….. Then $\{u_n\}$ is a Cauchy sequence in X.

Lemma 2.2: Let M(X, M, *) be a FM space and for all $x, y \in X$, t>0 and if for a number $q \in (0, 1)$,

$M(x, y, qt) \geq M(x, y, t)$ then x = y.

Lemma 2.3[9]: Let X be a set, f and g be owc self maps of X. If f and g have a unique point of coincidence,

w = fx = gx, then w is the unique common fixed point of f and g.

3. Main Result

Theorem 3.1: Let M(X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs {A, S} and {B, T} be owc. If for $\phi\in\phi_5$ there exist $q \in (0, 1)$ such that

$$\phi\left(\begin{array}{c} M(Ax,By,qt),\ M(Sx,Ax,t),\ \frac{aM(Ax,Ty,t)+bM(By,Sx,t)}{a+b},\ \frac{cM(Sx,Ax,t)+dM(By,Ty,t)}{c+d}, \\ \left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right).\left(\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right) \end{array}\right) \geq 0 \qquad \text{………………... (1)}$$

for all $x, y \in X$ and t>0, and a,b,c,d≥0 with a&b and c&d cannot be simultaneously 0 , there exist a unique point $w \in X$ such that Aw = Sw = w and a unique point $z \in X$ such that Bz = Tz = z. Morever z = w, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs {A, S} and {B, T} be owc, so there are points $x, y \in X$ such that Ax = Sx and By = Ty. We claim that Ax = By. If not by inequality (1)

$$\phi\left(\begin{array}{c} M(Ax,By,qt),\ M(Sx,Ax,t),\ \frac{aM(Ax,Ty,t)+bM(By,Sx,t)}{a+b},\ \frac{cM(Sx,Ax,t)+dM(By,Ty,t)}{c+d}, \\ \left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right).\left(\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right) \end{array}\right) \geq 0$$

$$\phi\left(\begin{array}{c} M(Ax,By,qt),\ M(Ax,Ax,t),\ \frac{aM(Ax,By,t)+bM(By,Ax,t)}{a+b},\ \frac{cM(Ax,Ax,t)+dM(By,By,t)}{c+d}, \\ \left(\frac{1+M(Ax,Ax,t)}{1+M(By,By,t)}\right).\left(\frac{M(Ax,By,t)+M(By,Ax,t)}{2}\right) \end{array}\right) \geq 0$$

$$\phi\big(M(Ax,By,qt),\ 1,\ M(Ax,By,t),\ 1,\ M(Ax,By,t)\big) \geq 0$$

$$\phi\big(M(Ax,By,qt),\ M(Ax,By,t),\ M(Ax,By,t),\ M(Ax,By,t),\ M(Ax,By,t)\big) \geq 0$$

Since, ϕ is non-increasing in 2nd and 4th argument therefore by 2.12 (i) and 2.12 (ii)

$M(Ax,By,qt) \geq M(Ax,By,t)$

Therefore Ax = By i.e. Ax = Sx = By =Ty.

Suppose that there is a unique point z such that Az = Sz then by (1) we have

$$\phi\begin{pmatrix} M(Az,By,qt),\ M(Sz,Az,t),\ \frac{aM(Az,Ty,t)+bM(By,Sz,t)}{a+b},\ \frac{cM(Sz,Az,t)+dM(By,Ty,t)}{c+d}, \\ \left(\frac{1+M(Az,Sz,t)}{1+M(By,Ty,t)}\right).\left(\frac{M(Az,Ty,t)+M(By,Sz,t)}{2}\right) \end{pmatrix} \geq 0$$

$\phi\big(M(Az,By,qt),\ 1,\ M(Az,By,t),\ 1,\ M(Az,By,t)\big) \geq 0$

$\phi\big(M(Az,By,qt),\ M(Az,By,t),\ M(Az,By,t),\ M(Az,By,t),\ M(Az,By,t)\big) \geq 0$

Since, ϕ is non-increasing in 2^{nd} and 4^{th} argument therefore 2.12 (i) and 2.12 (ii)

$M(Az,By,qt) \geq M(Az,By,t)$

Az = By = Sz =Ty, So Ax = Az and w = Ax = Sx the unique point of coincidence of A and S. By lemma (2.3) w is the only common fixed point of A and S. Similarly there is a unique point z ∈ X such that z = Bz =Tz.

Assume that w ≠ z we have

$$\phi\begin{pmatrix} M(Aw,Bz,qt),\ M(Sw,Aw,t),\ \frac{aM(Aw,Tz,t)+bM(Bz,Sw,t)}{a+b},\ \frac{cM(Sw,Aw,t)+dM(Bz,Tz,t)}{c+d}, \\ \left(\frac{1+M(Aw,Sw,t)}{1+M(Bz,Tz,t)}\right).\left(\frac{M(Aw,Tz,t)+M(Bz,Sw,t)}{2}\right) \end{pmatrix} \geq 0$$

$$\phi\begin{pmatrix} M(Aw,Bz,qt),\ M(w,w,t),\ \frac{aM(w,z,t)+bM(z,w,t)}{a+b},\ \frac{cM(w,w,t)+dM(z,z,t)}{c+d}, \\ \left(\frac{1+M(w,w,t)}{1+M(z,z,t)}\right).\left(\frac{M(w,z,t)+M(z,w,t)}{2}\right) \end{pmatrix} \geq 0$$

$\phi\big(M(Aw,Bz,qt),\ M(w,w,t),\ M(w,z,t),\ 1,\ M(w,z,t)\big) \geq 0$

$\phi\big(M(Aw,Bz,qt),\ 1,\ M(w,z,t),\ 1,\ M(w,z,t)\big) \geq 0$

$\phi\big(M(Aw,Bz,qt),\ M(w,z,t),\ M(w,z,t),\ M(w,z,t),\ M(w,z,t)\big) \geq 0$

Since, ϕ is non-increasing in 2^{nd} and 4^{th} argument

$M(Aw,Bz,qt) \geq M(w,z,t)$

Since, w is the only common fixed point of A and we have z = Bz.

$M(Aw,Bz,qt) = M(w,z,qt) \geq M(w,z,t)$

We have $M(w,z,qt) \geq M(w,z,t)$

Hence z = w by Lemma (2.2) and z is a common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (1).

Definition 3.11 (Implicit Relation) [14]: Let ϕ_6 be the set of all real and continuous function $\phi : (R^+)^6 \to R$ and such that

3.11 (i) ϕ non increasing in 2^{nd} and 4^{th} argument and

3.11 (ii) for u, v ≥0, $\phi(u,v,v,v,v,v) \geq 0 \Rightarrow u \geq v$

Theorem 3.2: Let M(X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs {A, S} and {B, T} be owc. If for $\phi \in \phi_6$ there exist q ∈ (0, 1) such that

$$\phi\begin{pmatrix} M(Ax,By,qt),\ M(Sx,Ax,t),\ \frac{aM(Ax,Ty,t)+bM(By,Sx,t)}{a+b},\ \frac{cM(Sx,Ax,t)+dM(By,Ty,t)}{c+d}, \\ \left(\frac{1+M(Ax,Sx,t)}{1+M(By,Ty,t)}\right).\left(\frac{M(Ax,Ty,t)+M(By,Sx,t)}{2}\right),\ M(By,Sx,t) \end{pmatrix} \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

for all x, y ∈ X and t>0, and a,b,c,d≥0 with a&b and c&d cannot be simultaneously 0 , there exist a unique point w ∈ X such that Aw = Sw = w and a unique point z ∈ X such that Bz = Tz = z. Morever z = w, so that there is a unique common fixed point of A, B, S and T.

Proof: Let the pairs {A, S} and {B, T} be owc, so there are points x, y ∈ X such that Ax = Sx and By = Ty. We claim that Ax = By. If not by inequality (2)

$$\phi\begin{pmatrix} M(Ax,By,qt),\ M(Ax,Ax,t),\ \frac{aM(Ax,By,t)+bM(By,Ax,t)}{a+b},\ \frac{cM(Ax,Ax,t)+dM(By,By,t)}{c+d}, \\ \left(\frac{1+M(Ax,Ax,t)}{1+M(By,By,t)}\right).\left(\frac{M(Ax,By,t)+M(By,Ax,t)}{2}\right),\ M(By,Ax,t) \end{pmatrix} \geq 0$$

$\phi(M(Ax,By,qt),\ 1,\ M(Ax,By,t),\ 1,\ M(Ax,By,t),\ M(By,Ax,t)) \geq 0$

$\phi(M(Ax,By,qt),\ M(Ax,By,t),\ M(Ax,By,t),\ M(Ax,By,t),\ M(Ax,By,t),\ M(By,Ax,t)) \geq 0$

Since, ϕ is non-increasing in 2^{nd} and 4^{th} argument therefore 3.11 (i) and 3.11 (ii)

$M(Ax,By,qt) \geq M(Ax,By,t)$

Therefore Ax = By i.e. Ax = Sx = By = Ty. Suppose that there is another point z such that A z =S z then by (2) we have A z = S z = Ty, So Ax = A z and w =Ax = Tx is the unique point of coincidence of A and T. By lemma (2.2) w is a unique point z ∈ X such that w = Bz = Tz. Thus z is a common fixed point of A, B, S and T. The uniqueness of fixed point holds by (2).

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