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Coupled Fixed Point Theorem in Partially Ordered Metric Spaces

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Abstract
The present paper deals with some Coupled fixed point theorem for mapping having mixed monotone property in Partially Ordered Metric space.
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1 Introduction:

The fixed point of mapping in ordered metric spaces are of great use in many mathematical problems in applied and pure mathematics. The first result in this direction was obtained by Ran and Reuring's [2]; in this study the authors present some applications of their obtained results of matrix equations. In [3], Nieto and Lopez extended the result of Ran and Reuring's [5],for non-decreasing mappings and applied their result to get a unique solution for a first order differential equation. While Agrawal et al.[6] and O'Regan and Petrutel [7] studied some results for generalized contractions in ordered metric spaces.

The notion of coupled fixed point was introduced by Chang and Ma [1]. Since then the concept has been of interest to many researchers in metrical fixed point theory. Bhaskar and Lakshmikantham [4] established coupled fixed point theorem in metric space endowed with partial order by employing the following Contractivity condition:

For a mapping $F : X \times X \to X$ there exist $k \in (0,1)$ such that

$$d\big(F(x,y),F(u,u)\big) \le \frac{k}{2}\big[d(x,u)+d(y,v\big], \text{ for all x,y, u, v in X, } x \ge u \,\&\, y \le v$$

Harjani et.al [9]established some fixed point theorem in partially ordered metric spaces by using a contractive condition for a rational type i.e. $F : X \to X$, there exist some $\alpha, \beta \in [0,1]$ with $\alpha + \beta < 1$ such that

$$d\big(F(x,y)\big) \le \alpha \left\{ \frac{d(x,Fx)d(y,Fy))}{d(x,y)} \right\} + \beta d(x,y) \text{For all x, y in X and x} \ne \text{y}$$

L. Ciricet. al.[13] proved coupled fixed point theorem in partially ordered metric spaces by employing some notions of Bhaskar and Lakshmikantham [4] as well as rational type contractive condition. Later Shatanawi, w [11], Abbas M Khan, AR Nazir T [10] proved coupled fixed point theorem in generalised metric space. Jay C. Mehta M. L. Tosh [12] ,RamakantBhardwaj [14]proved coupled fixed point theorem in partially ordered metric space.

In this paper,we derive new coupled fixed point theorem for mapping having mixed monotone property in partially ordered metric space.

2 Preliminaries:

We recall the definitions and results that will be needed in the sequel.

Definition 2.1: A partially ordered set is a set P and a binary relation ≤, denoted by $(X, \le)$ such that for all $a,b,c \in P$

i. $a \le a$, (reflexivity)
ii. $a \le b$ and $b \le c \Rightarrow a \le c$ (transitivity)
iii. $a \le b$ and $b \le a \Rightarrow a = b$ (anti-symmetry)

Definition 2.2: A sequence $\{x_n\}$ in a metric space (X, d) is said to be Cauchy sequence if $\lim_{t\to\infty} d(x_n, x_m) = 0$ *for all n, m>t*

Definition 2.3: A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.4: Let $(X,\leq)$ be a partially ordered set and $F: X\times X \to X$. The mapping F is said to has the mixed monotone property if F(x, y) is monotone non- decreasing in x and is monotone non increase in y, that is for $x, y \in X$

$$\forall x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow F(x_1, y) \leq F(x_2, y) \quad and$$
$$\forall y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow F(x, y_1) \geq F(x, y_2)$$

Definition 2.5: (X,d) be a metric space. An element $(x,y)\in X\times X$ is said to be a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$

3. Main Result

Let $(X,\leq)$ be a partially ordered set and d be metric on X such that (X, d) is a complete metric space. We also endow the product space $X\times X$ with the following partial order.

For all $(x,y),(u,v)\in X\times X$, $(\mathrm{u},\mathrm{v})\leq (x,y) \Leftrightarrow x\geq u, y\leq v$

Theorem 3.1: Let $(X,\leq)$ be a partially ordered metric set and suppose that there exist a metric d on X such that (X; d) is a complete metric space. Let $F: X\times X\to X$ be a continuous mapping having the mixed monotone property such that for some $\alpha,\beta,\gamma\in[0,1)$ for all x, y, u, v in X, $x\neq u$ we have

$$\begin{aligned} d(F(x,y),F(u,v)) \leq \alpha&\left\{\frac{d(x,F(x,y))d(u,F(u,v))}{d(x,u)}+\frac{d(u,F(x,y))d(x,F(u,v))}{d(x,u)}\right\}\\ &+\beta\{d(x,u)\}\\ &+\gamma\{d(x,F(x,y))+d(x,F(u,v))+d(u,F(x,y))+d(u,F(u,v))\} \end{aligned} \quad \text{… … (3.1.1)}$$

Where $\alpha+\beta+4\gamma<1$, then F has a coupled fixed point in X.

Proof: choose $x_0, y_0 \in X\times X$ and set

$$x_1 = F(x_0,y_0)\, and\ \ y_1 = F(y_0,x_0) \text{ and in general}$$
$$x_{n+1} = F(x_n,y_n)\, and\ \ y_{n+1} = F(y_n,x_n) \qquad (3.1.2)$$

With $x_0 \leq F(x_0,y_0) = \mathrm{x}_1$ (say) and $y_0 \geq F(y_0,x_0) = y_1$ (say) (3.1.3)

By iterative process above

$$x_2 = F(x_1,y_1)\, and\ \ y_2 = F(y_1,x_1)$$

Therefore $F^2(x_0,y_0) = F(F(x_0,y_0),F(y_0,x_0)) = F(x_1,y_1) = x_2$ and

$$F^2(y_0,x_0) = F(F(y_0,x_0),F(x_0,y_0)) = F(y_1,x_1) = y_2$$

Due to the mixed monotone property of F; we obtain

$$x_2 = F^2(x_0,y_0) = F(x_1,y_1) \geq F(x_0,y_0) = x_1$$
$$y_2 = F^2(y_0,x_0) = F(y_1,x_1) \leq F(y_0,x_0) = y_1$$

In general, we have for n ϵ N

$$x_{n+1} = F^{n+1}(x_0,y_0) = F(F^n(x_0,y_0),F^n(y_0,x_0))$$
$$y_{n+1} = F^{n+1}(y_0,x_0) = F(F^n(y_0,x_0),F^n(x_0,y_0))$$

It is obvious that

$$x_0 \leq F(x_0,y_0) = \mathrm{x}_1 \leq F^2(x_0,y_0) = x_2 \leq \dots\dots\dots \leq F^n(x_0,y_0) = x_n \leq \dots\dots\dots \quad \text{and}$$
$$y_0 \geq F(y_0,x_0) = y_1 \geq F^2(y_0,x_0) = y_2 \geq \dots\dots\dots \geq F^n(y_0,x_0) = y_n \geq \dots\dots\dots$$

Thus by mathematical induction principle, we have for nϵ N

$$\left.\begin{array}{l} x_0 \le x_1 \le x_2 \le \dots\dots\dots \le x_n \le x_{n+1} \le \dots\dots\dots \\ y_0 \ge y_1 \ge y_2 \ge \dots\dots\dots \ge y_n \ge y_{n+1} \ge \dots\dots \end{array}\right\} \quad \dots\dots\dots\dots \tag{3.1.4}$$

Therefore we have by condition (3.1.1) that

$$d(x_{n+1},x_n) = d(F(x_n,y_n),F(x_{n-1},y_{n-1}))$$

$$\le \alpha\left\{\frac{d(x_n,F(x_n,y_n))d(x_{n-1},F(x_{n-1},y_{n-1}))}{d(x_n,x_{n-1})} + \frac{d(x_{n-1},F(x_n,y_n))d(x_n,F(x_{n-1},y_{n-1}))}{d(x_n,x_{n-1})}\right\}$$

$$+\beta\{d(x_n,x_{n-1})\} + \gamma\{d(x_n,F(x_n,y_n)) + d(x_n,F(x_{n-1},y_{n-1})) + d(x_{n-1},F(x_n,y_n)) + d(x_{n-1},F(x_{n-1},y_{n-1}))\}$$

$$\le \alpha\left\{\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n-1})} + \frac{d(x_{n-1},x_{n+1})d(x_n,x_n)}{d(x_n,x_{n-1})}\right\}$$

$$+\beta\{d(x_n,x_{n-1})\} + \gamma\{d(x_n,x_{n+1}) + d(x_n,x_n) + d(x_{n-1},x_{n+1}) + d(x_{n-1},x_n)\}$$

$$\le \alpha d(x_n,x_{n+1}) + \beta\{d(x_n,x_{n-1})\} + \gamma\{d(x_n,x_{n+1}) + d(x_{n-1},x_n) + d(x_n,x_{n+1}) + d(x_{n-1},x_n)\}$$

$$\le (\alpha+2\gamma)d(x_n,x_{n+1}) + (\beta+2\gamma)d(x_n,x_{n-1})$$

$$\Rightarrow \quad d(x_n,x_{n+1}) \le \left(\frac{\beta+2\gamma}{1-\alpha-2\gamma}\right)d(x_n,x_{n-1}) \tag{3.1.5}$$

Similarly since $y_{n-1} \ge y_n$ and $x_{n-1} \le x_n$, from (3.1.1) we have

$$d(y_n,y_{n+1}) = d(F(y_{n-1},x_{n-1}),F(y_n,x_n))$$

$$\le \alpha\left\{\frac{d(y_{n-1},F(y_{n-1},x_{n-1}))d(y_n,F(y_n,x_n))}{d(y_{n-1},y_n)} + \frac{d(y_n,F(y_{n-1},x_{n-1}))d(y_{n-1},F(y_n,x_n))}{d(y_{n-1},y_n)}\right\}$$

$$+\beta\{d(y_{n-1},y_n)\}$$

$$+\gamma\{d(y_{n-1},F(y_{n-1},x_{n-1})) + d(y_{n-1},F(y_n,x_n)) + d(y_n,F(y_{n-1},x_{n-1})) + d(y_n,F(y_n,x_n))\}$$

$$\le \alpha\left\{\frac{d(y_{n-1},y_n)d(y_n,y_{n+1})}{d(y_{n-1},y_n)} + \frac{d(y_n,y_n)d(y_{n-1},y_{n+1})}{d(y_{n-1},y_n)}\right\}$$

$$+\beta\{d(y_{n-1},y_n)\} + \gamma\{d(y_{n-1},y_n) + d(y_{n-1},y_{n+1}) + d(y_n,y_n) + d(y_n,y_{n+1})\}$$

$$\le \alpha\{d(y_n,y_{n+1})\} + +\beta\{d(y_{n-1},y_n)\} + \gamma\{d(y_{n-1},y_n) + d(y_{n-1},y_n) + d(y_n,y_{n+1}) + d(y_n,y_{n+1})\}$$

$$\Rightarrow \quad d(y_n,y_{n+1}) \le \left(\frac{\beta+2\gamma}{1-\alpha-2\gamma}\right)d(y_n,y_{n-1}) \tag{3.1.6}$$

Adding (3.1.5) and (3.1.6) we get

$$d(x_n,x_{n+1}) + d(y_n,y_{n+1}) \le \left(\frac{\beta+2\gamma}{1-\alpha-2\gamma}\right)d(x_n,x_{n-1}) + \left(\frac{\beta+2\gamma}{1-\alpha-2\gamma}\right)d(y_n,y_{n-1})$$

$$= \left(\frac{\beta+2\gamma}{1-\alpha-2\gamma}\right)[d(x_n,x_{n-1}) + d(y_n,y_{n-1})]$$

Let us denote $h = \dfrac{\beta+2\gamma}{1-\alpha-2\gamma}$ and $d(x_n,x_{n+1}) + d(y_n,y_{n+1})$ by d_n then $d_n \le h.d_{n-1}$

Similarly it can be proved that $d_{n-1} \le h.d_{n-2}$

Therefore $d_n \le h.^2 d_{n-2}$, by repeating we get

$$d_n \le hd_{n-1} \le h.^2 d_{n-2} \le \dots\dots\dots\dots \le h^n d_0 \tag{3.1.7}$$

This implies that $\lim_{n\to\infty} d_n = 0$

Thus $\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} d(y_{n+1}, y_n) = 0$

For each m ≥ n, we obtain by (3.1.7)and the repeated application of triangular inequality that

$d(x_n, x_m) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + ... + d(x_{m-1}, x_m)$ and

$d(y_n, y_m) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + ... + d(y_{m-1}, y_m)$

Adding these we get

$$\begin{aligned} d(x_n, x_m) + d(y_n, y_m) \le & [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \\ & + [d(x_{n+1}, x_{n+2}) + d(y_{n+1}, y_{n+2})] \\ & + [d(x_{n+2}, x_{n+3}) + d(y_{n+2}, y_{n+3})] \\ & + [d(x_{n+3}, x_{n+4}) + d(y_{n+3}, y_{n+4})] \\ & + ... \end{aligned}$$

$$+ [d(x_{m-1}, x_m) + d(y_{m-1}, y_m)]$$

$$\le [h^n + h^{n+1} + h^{n+2} + h^{n+3} + ... + h^{m-1}] d_0$$

$$\le \frac{h^n}{1-h^n} d_0 \to 0, \text{ as } n \to \infty$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. Since X is complete metric space, there exist x, y ϵ X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$

Thus by taking limit $n \to \infty$ in equation(3.1.2), we get

$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = F \lim_{n\to\infty} (x_{n-1}, y_{n-1}) = F(x, y)$ and

$$y = \lim_{n\to\infty} y_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = F \lim_{n\to\infty} (y_{n-1}, x_{n-1}) = F(y, x)$$

Therefore $x = F(x, y) \,\&\, y = F(y, x)$

Thus F has a coupled fixed point in X.

Theorem 3.2: Let $(X, \le)$ be a partially ordered metric set and suppose that there exist a metric d on X such that (X; d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X, such that for some $\alpha, \beta,, \eta, \delta \in [0,1)$ for all x, y, u, v in X, $x \ne u$ with

$$\begin{aligned} d(F(x,y), F(u,v)) \le & \alpha \left\{ \frac{d(x, F(x,y)) d(u, F(u,v)) d(x, F(u,v)) + d(x,u) d(u, F(x,y)) d(u, F(u,v))}{[d(x,u)]^2 + d(x, F(x,y)) d(u, F(u,v))} \right\} \\ & + \beta \{ d(x, F(x,y)) + d(u, F(u,v)) \} \\ & + \eta \left\{ \frac{d(x, F(x,y)) d(u,, F(u,v))}{d(x,u)} \right\} \\ & + \delta \{ d(x,u) \} \end{aligned} \tag{3.2.1}$$

, $2\alpha + 2\beta + \eta + \delta < 1$ then F has a coupled fixed point in X.

Proof: choose $x_0, y_0 \in X \times X$ and set

$x_1 = F(x_0, y_0)$ *and* $y_1 = F(y_0, x_0)$ and in general

$$x_{n+1} = F(x_n, y_n) \text{ and } y_{n+1} = F(y_n, x_n) \tag{3.2.2}$$

With $x_0 \le F(x_0, y_0) = x_1$ (say) and $y_0 \ge F(y_0, x_0) = y_1$ (say) (3.2.3)

By iterative process above

$$x_2 = F(x_1, y_1) \text{ and } y_2 = F(y_1, x_1)$$

Therefore $F^2(x_0,y_0)=F(F(x_0,y_0),F(y_0,x_0))=F(x_1,y_1)=x_2$ and $F^2(y_0,x_0)=F(F(y_0,x_0),F(x_0,y_0))=F(y_1,x_1)=y_2$

Due to the mixed monotone property of F; we obtain

$$x_2=F^2(x_0,y_0)=F(x_1,y_1)\geq F(x_0,y_0)$$
$$y_2=F^2(y_0,x_0)=F(y_1,x_1)\leq F(y_0,x_0)$$

In general, we have for n ϵ N

$$x_{n+1}=F^{n+1}(x_0,y_0)=F(F^n(x_0,y_0),F^n(y_0,x_0))$$
$$y_{n+1}=F^{n+1}(y_0,x_0)=F(F^n(y_0,x_0),F^n(x_0,y_0))$$

It is obvious that

$$x_0\leq F(x_0,y_0)=\mathrm{x}_1\leq F^2(x_0,y_0)=x_2\leq\ldots\ldots\ldots\leq F^n(x_0,y_0)=x_n\leq\ldots\ldots\ldots$$

and

$$y_0\geq F(y_0,x_0)=y_1\geq F^2(y_0,x_0)=y_2\geq\ldots\ldots\ldots\geq F^n(y_0,x_0)=y_n\geq\ldots\ldots\ldots\ldots$$

Thus by mathematical induction principle, we have for nϵ N

$$\left.\begin{array}{l} x_0\leq \mathrm{x}_1\leq x_2\leq\ldots\ldots\ldots\leq x_n\leq x_{n+1}\leq\ldots\ldots\ldots \\ y_0\geq y_1\geq y_2\geq\ldots\ldots\ldots\geq y_n\geq y_{n+1}\geq\ldots\ldots \end{array}\right\}\quad\ldots\ldots\ldots\ldots \tag{3.1.4}$$

Therefore we have by condition (3.2.1) that

$$d(x_{n+1},x_n)=d(F(x_n,y_n),F(x_{n-1},y_{n-1}))$$
$$\leq\alpha\left\{\frac{\begin{array}{l}d(x_n,F(x_n,y_n))d(x_{n-1},F(x_{n-1},y_{n-1}))d(x_n,F(x_{n-1},y_{n-1}))\\+d(x_n,x_{n-1})d(x_{n-1},F(x_n,y_n))d(x_{n-1},F(x_{n-1},y_{n-1}))\end{array}}{[d(x_n,x_{n-1})]^2+d(x_n,F(x_{n-1},y_{n-1}))d(x_{n-1},F(x_{n-1},y_{n-1}))}\right\}$$
$$+\beta\{d(x_n,F(x_n,y_n))+d(x_{n-1},F(x_{n-1},y_{n-1}))\}$$
$$+\eta\left\{\frac{d(x_n,F(x_n,y_n))d(x_{n-1},F(x_{n-1},y_{n-1}))}{d(x_n,x_{n-1})}\right\}+\delta\{d(x_n,x_{n-1})\}$$
$$\leq\alpha\left\{\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)d(x_n,x_n)+d(x_n,x_{n-1})d(x_{n-1},x_{n+1})d(x_{n-1},x_n)}{[d(x_n,x_{n-1})]^2+d(x_n,x_n)d(x_{n-1},x_n)}\right\}$$
$$+\beta\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}$$
$$+\eta\left\{\frac{d(x_n,x_{n+1})d(x_{n-1},x_n)}{d(x_n,x_{n-1})}\right\}+\delta\{d(x_n,x_{n-1})\}$$
$$\leq\alpha\{d(x_{n-1},x_{n+1})\}+\beta\{d(x_n,x_{n+1})+d(x_{n-1},x_n)\}+\eta\{d(x_n,x_{n+1})\}+\delta\{d(x_n,y_n)\}$$
$$d(x_{n+1},x_n)\leq\frac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}d(x_n,x_{n-1}) \tag{3.2.5}$$

Similarly since $y_{n-1}\geq y_n$ and $x_{n-1}\leq x_n$, from (3.2.1) we have

$$d(y_n,y_{n+1})\leq\left(\frac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}\right)d(y_n,y_{n-1}) \tag{3.2.6}$$

Adding (3.2.5) and (3.2.6) we get

$$d(x_n,x_{n+1})+d(y_n,y_{n+1})\le\left(\frac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}\right)d(x_n,x_{n-1})+\left(\frac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}\right)d(y_n,y_{n-1})$$
$$=\left(\frac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}\right)\left[d(x_n,x_{n-1})+d(y_n,y_{n-1})\right]$$

Let us denote $h=\dfrac{\alpha+\beta+\delta}{1-(\alpha+\beta+\eta)}$ and $d(x_n,x_{n+1})+d(y_n,y_{n+1})$ by d_n then $d_n\le h.d_{n-1}$

Similarly it can be proved that $d_{n-1}\le h.d_{n-2}$

Therefore $d_n\le h.^2 d_{n-2}$, by repeating we get

$$d_n\le hd_{n-1}\le h.^2 d_{n-2}\le \dots\dots\dots\dots\le h^n d_0 \tag{3.2.7}$$

This implies that $\lim\limits_{n\to\infty} d_n=0$

Thus $\lim\limits_{n\to\infty} d(x_{n+1},x_n)=\lim\limits_{n\to\infty} d(y_{n+1},y_n)=0$

For each m ≥ n, we obtain by (3.2.7)and the repeated application of triangular inequality that

$d(x_n,x_m)\le d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3})+d(x_{n+3},x_{n+4})+\dots\dots+d(x_{m-1},x_m)$ and

$d(y_n,y_m)\le d(y_n,y_{n+1})+d(y_{n+1},y_{n+2})+d(y_{n+2},y_{n+3})+d(y_{n+3},y_{n+4})+\dots\dots+d(y_{m-1},y_m)$

Adding these we get

$$\begin{aligned}d(x_n,x_m)+d(y_n,y_m)&\le[d(x_n,x_{n+1})+d(y_n,y_{n+1})]\\&+[d(x_{n+1},x_{n+2})+d(y_{n+1},y_{n+2})]\\&+[d(x_{n+2},x_{n+3})+d(y_{n+2},y_{n+3})]\\&+[d(x_{n+3},x_{n+4})+d(y_{n+3},y_{n+4})]\\&+\dots\dots\\&+[d(x_{m-1},x_m)+d(y_{m-1},y_m)]\\&\le\left[h^n+h^{n+1}+h^{n+2}+h^{n+3}+\dots+h^{m-1}\right]d_0\\&\le\frac{h^n}{1-h^n}d_0\to 0,\ \text{ as } n\to\infty\end{aligned}$$

Therefore $\{x_n\}$and $\{y_n\}$are Cauchy sequences in X. Since X is complete metric space, there exist x, y ϵ X such that $\lim\limits_{n\to\infty} x_n=x$ and $\lim\limits_{n\to\infty} y_n=y$

Thus by taking limit $n\to\infty$ in equation (3.2.2), we get

$$x=\lim_{n\to\infty}x_n=\lim_{n\to\infty}F(x_{n-1},y_{n-1})=F\lim_{n\to\infty}(x_{n-1},y_{n-1})=F(x,y)\text{ and}$$
$$y=\lim_{n\to\infty}y_n=\lim_{n\to\infty}F(y_{n-1},x_{n-1})=F\lim_{n\to\infty}(y_{n-1},x_{n-1})=F(y,x)$$

Therefore $x=F(x,y)\,\&\,y=F(y,x)$

Thus F has a coupled fixed point in X.

Theorem 3.3: Let the hypothesis of theorem (**3.2**) holds. In addition suppose that there exist z ϵ X, which is comparable to x and y $\forall\ x,y\in X$ then F has a unique fixed point

Suppose that there exist (x^*,y^*) and $(x',y')\in X\times X$ are coupled fixed point of F

Case I: If x^* and x' are comparable & y^* and y' are also comparable and $x^*\ne x',y^*\ne y'$ then by contractive condition we have

$$d(x^*,x')=d(F(x^*,y^*),F(x'\,y'))$$

$$\leq \alpha\left\{\frac{d(x^*,F(x^*,y^*)),d(x',F(x',y'))d(x^*,F(x',y'))+d(x^*,x')d(x',F(x^*,y^*))d(x',F(x',y'))}{[d(x^*,x')]^2+d(x^*,F(x^*,y^*))d(x',F(x',y'))}\right\}$$
$$+\beta\{d(x^*,F(x^*,y^*))+d(x',F(x',y'))\}$$
$$+\eta\left\{\frac{d(x^*,F(x^*,y^*))d(x',F(x',y'))}{d(x^*,x')}\right\}+\delta[d(x^*,x')]$$
$$\leq \alpha\left\{\frac{d(x^*,x^*),d(x',x')d(x^*,x')+d(x^*,x')d(x',x^*)d(x',x')}{[d(x^*,x')]^2+d(x^*,x^*)d(x',x')}\right\}+\beta\{d(x^*,x^*)+d(x',x')\}$$
$$+\eta\left\{\frac{d(x^*,x^*)d(x',x')}{d(x^*,x')}\right\}+\delta[d(x^*,x')]$$
$$d(x^*,x')\leq \delta.d(x^*,x')$$

This is contradiction, since $\delta<1$ as $2\alpha+2\beta+\eta+\delta<1$. Thus $x^*=x'$ .Also

$$d(y^*,y')=d(F(y^*,x^*),F(y',x'))$$
$$\Rightarrow \quad d(y^*,y')\leq \delta.d(y^*,y')$$

This is contradiction, since $\delta<1$ as $2\alpha+2\beta+2\gamma+\eta+\delta<1$. Thus $y^*=y'$

Therefore (x^*,y^*) is unique coupled fixed point of F

Case II: If x^* is not comparable to x' & y^* is not comparable to y' then by contractive condition there exist w comparable to x^* & x' and there exist $v$comparable to y^* and y'

Monotonicity implies that w_n is comparable to $x_n^*=F(x_{n-1}^*,y_{n-1}^*)=x^*$ and w_n is comparable to w_1 . Also monotonicity implies that y_n^* is comparable to v and y_n^* is also comparable to w_2 .

On the other hand if $x_n^*\neq w_1, x'_n\neq w_1$ then by contractive condition we get

$$d(w_1,x_n^*)=d(F(w_1,w_2),F(x_{n-1}^*,y_{n-1}^*))$$

Case III: If (x^*,y^*) is not comparable to (x',y') then there exist (w,v) comparable to (x^*,y^*) & (x',y') . Monotonicity implies that

$$d\left(\begin{pmatrix}x^*\\y^*\end{pmatrix},\begin{pmatrix}x'\\y'\end{pmatrix}\right)=d\left(\begin{pmatrix}F^n(x^*,y^*)\\F^n(y^*,x^*)\end{pmatrix},\begin{pmatrix}F^n(x',y'\\F^n(y',x')\end{pmatrix}\right)$$
$$\leq d\left(\begin{pmatrix}F^n(x^*,y^*)\\F^n(y^*,x^*)\end{pmatrix},\begin{pmatrix}F^n(w,v)\\F^n(v,w)\end{pmatrix}\right)+d\left(\begin{pmatrix}F^n(w,v)\\F^n(v,w)\end{pmatrix},\begin{pmatrix}F^n(x',y'\\F^n(y',x')\end{pmatrix},\right)$$
$$\leq d(F^n(x^*,y^*),F^n(w,v))+d(F^n(y^*,x^*),F^n(v,w))+d(F^n(w,v),F^n(x',y'))+d(F^n(v,w),F^n(y',x'))$$
$$\leq \alpha^n\left\{\frac{d(x^*,F^n(x^*,y^*))d(w,F^n(w,v))d(x^*,F^n(w,v))+d(x^*,w)d(w,F^n(x^*,y^*))d(w,F^n(w,v))}{[d(x^*,w)]^2+d(x^*,F^n(w,v))d(w,F^n(w,v))}\right\}$$
$$+\beta^n\{d(x^*,F^n(x^*,y^*))+d(w,F^n(w,v))\}$$
$$+\eta^n\frac{d(x^*,F^n(x^*,y^*))+d(w,F^n(w,v))}{d(x^*,w)}$$
$$+\delta^n.d(x^*,w)$$

$$+\alpha^n\left\{\frac{d(y^*,F^n(y^*,x^*))d(v,F^n(v,w))d(y^*,F^n(v,w))+d(y^*,v)d(v,F^n(y^*,x^*))d(v,F^n(v,w))}{[d(y^*,v)]^2+d(y^*,F^n(v,w))d(v,F^n(v,w))}\right\}$$
$$+\beta^n\{d(y^*,F^n(y^*,x^*))+d(v,F^n(v,w))\}$$
$$+\eta^n\frac{d(y^*,F^n(y^*,x^*))d(v,F^n(v,w))}{d(y^*,v)}$$
$$+\delta^n.d(y^*,v)$$
$$+\alpha^n\left\{\frac{d(w,F^n(w,v))d(x',F^n(x',y'))d(w,F^n(x',y'))+d(w,x')d(x',F^n(w,v))d(x',F^n(x',y'))}{[d(w,x')]^2+d(w,F^n(x',y'))d(x',F^n(x',y'))}\right\}$$
$$+\beta^n\{d(w,F^n(w,v))+d(x',F^n(x',y'))\}$$
$$+\eta^n\frac{d(w,F^n(w,v))d(x',F^n(x',y'))}{d(w,x')}$$
$$+\delta^n.d(w,x')$$
$$+\alpha^n\left\{\frac{d(v,F^n(v,w))d(y',F^n(y',x'))d(v,F^n(y',x'))+d(v,y')d(y',F^n(v,w))d(y',F^n(y',x'))}{[d(v,y')]^2+d(v,F^n(y',x'))d(y',F^n(y',x'))}\right\}$$
$$+\beta^n\{d(v,F^n(v,w))+d(y',F^n(y',x'))\}$$
$$+\eta^n\frac{d(v,F^n(v,w))d(y',F^n(y',x'))}{d(v,y')}$$
$$+\delta^n.d(v,y')$$
$$\le\alpha^n\left\{\frac{d(x^*,x^*)d(w,w)d(x^*,w)+d(x^*,w)d(w,x^*)d(w,w)}{[d(x^*,w)]^2+d(x^*,w)d(w,w)}\right\}$$
$$+\beta^n\{d(x^*,x^*)+d(w,w)\}++\eta^n\frac{d(x^*,x^*)+d(w,w)}{d(x^*,w)}+\delta^n.d(x^*,w)$$
$$+\alpha^n\left\{\frac{d(y^*,y^*)d(v,v)d(y^*,v)+d(y^*,v)d(v,y^*)d(v,v)}{[d(y^*,v)]^2+d(y^*,v)d(v,v)}\right\}$$
$$+\beta^n\{d(y^*,y^*)+d(v,v)\}+\eta^n\frac{d(y^*,y^*)d(v,v)}{d(y^*,v)}+\delta^n.d(y^*,v)$$
$$+\alpha^n\left\{\frac{d(w,w)d(x',x')d(w,x')+d(w,x')d(x',w)d(x',x')}{[d(w,x')]^2+d(w,x')d(x',x')}\right\}$$
$$+\beta^n\{d(w,w)+d(x',x')\}+\eta^n\frac{d(w,w)d(x',x')}{d(w,x')}+\delta^n.d(w,x')$$
$$+\alpha^n\left\{\frac{d(v,v)d(y',y')d(v,y')+d(v,y')d(y',v)d(y',y')}{[d(v,y')]^2+d(v,y')d(y',y')}\right\}$$
$$+\beta^n\{d(v,v)+d(y',y')\}+\eta^n\frac{d(v,v)d(y',y')}{d(v,y')}+\delta^n.d(v,y')$$
$$\le\delta^n[d(x^*,w)+d(y^*,v)+d(w,x')+d(v,y')]$$
$$\to 0 \quad \text{as n} \to\infty$$

Hence F has unique fixed point.

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