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Some Fixed Point Theorems in Metric Space by using Altering Distance function

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Abstract
In this paper we prove some fixed point theorem in metric space by using altering distance function.
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1 Introduction and preliminaries:

In 1984 M.S. Khan, M Swalech and S. Sessa [10] expanded the research of metric fixed point theory to a new category by introducing a control function which they called an altering distance function.

Definition 1.1:([10])A function $\psi : R^+ \to R^+ = [0,1)$ is analtering distance function if the following properties are satisfied

(Ψ_1) $\psi(t) = 0 \Leftrightarrow t = 0$

(Ψ_2) ψ is monotonically non - decreasing

(Ψ_3) ψ is contineous

By Ψ we denote the set of all altering distance function

Definition 1.2: Let S be a self-mapping of a metric space $\langle M, d \rangle$ with a non-empty fixed point F(S), then S is said to satisfy the property P, $F(S) = F(S^n)$ if for each n є N

The following lemma given by G.Babu and P.P. Sailaja [3] will be used in the sequel in order to prove our main result.

Lemma 1.3: Let $\langle M, d \rangle$ be a metric space, let $\{x_n\}$ be a sequence in M such that

$$\lim_{n\to\infty}[d(x_n, x_{n+1})] = 0$$

If $\{x_n\}$ is not a Cauchy sequence in M then there is a constant $\in_0 > 0$, and the sequences of positive integers m(k) and n(k) with m(k) > n(k) > k such that

$d(x_{m(k)}, x_{n(k)}) \geq \in_0$ and $d(x_{m(k)-1}, x_{n(k)}) < \in_0$ and

1) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \in_0$
2) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \in_0$
3) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \in_0$.

Remark1.4: from lemma (1.3) it easy to get

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \in_0 .$$

Using these control functions the authors extend the Banach contraction principle by taking $\psi = Id$ (the identity mapping), in the inequality of contraction (1.5.1) of the following theorem.

Theorem 1.5: Let $\langle M, d\rangle$ be a metric space, let $\psi \in \Psi$ and let $S: M \to M$ be a mapping which satisfies the following inequality

$$\psi[d(Sx, Sy)] \le a\,\psi[d(x, y)] \qquad (1.5.1)$$

For all x, y ϵ M, and for some $0 < a < 1$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each x in M, $\lim_{n\to\infty} S^n x = z_0$.

Beside this Jaggi[7] introduced a new contraction mapping and a fixed point through rational expression for self-mapping which are

$$d(Tx, Ty) \le \alpha \frac{d(x, Tx)d(y, Ty)}{d(x, y)}$$

for all x, y ϵ X, $x \ne y$ and for some $\alpha \epsilon [0,1)$ then, T has a unique fixed point in X.

The above expansion is not valid if x = y this condition is removed by Das and Gupta[5] and proved a fixed point theorem for self-mapping on taking following expression.

$$d(Tx, Ty) \le \alpha \frac{d(x, Tx)[1 + d(y, Ty)]}{d(x, y)} + \beta d(x, y)$$

for all x, y ϵ X, and for some $\alpha, \beta\ \epsilon[0,1)$, $0 < \alpha + \beta < 1$ then, T has a unique fixed point in X.

Recently J. R. Morales and E M Rojas [16] proved altering distance function and fixed point theorem through rational expression, Manish Sharma and A.S. Saluja[15] proved fixed point theorem by altering distance

In the paper we prove some fixed point and common fixed point theorems in metric spaces by using altering distance function.

2 Main Results:

Theorem 2.1: Let $\langle M, d\rangle$ be a complete metric space. Let $\psi \in \Psi$ and let $S: M \to M$ be a mapping which satisfies the following condition:

$$\begin{aligned}\psi d(Sx, Sy) \le\ & \alpha\psi\left[\frac{d(x, Sx)d(y, Sy)d(x, Sy) + d(x, y)d(y, Sx)d(y, Sy)}{[d(x, y)]^2 + d(x, Sy)d(y, Sy)}\right] \\ & + \beta\psi[d(x, Sx) + d(y, Sy)] \\ & + \gamma\psi[d(x, Sy) + d(y, Sx)] \\ & + \eta\psi\left[\frac{d(x, Sx)d(y, Sy)}{1 + d(x, y)}\right] \\ & + \delta\psi[d(x, y)] \end{aligned} \qquad (2.1.1)$$

For all x, y ϵ M, x ≠ y, and for some α, β, γ, η, δ > 0 with $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each $x \in M$ $\lim_{n\to\infty} S^n x = z_0$.

Proof: Let $x \in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as follows

$$x_{n+1} = Sx_n = S^{n+1}x \text{ , for each } n \ge 0$$

Now $\psi d(x_n, x_{n+1}) = \psi[d(Sx_{n-1}, Sx_n)]$

$$\begin{aligned} & \le \alpha\psi\left\{\frac{d(x_{n-1}, Sx_{n-1})d(x_n, Sx_n)d(x_{n-1}, Sx_n) + d(x_{n-1}, x_n)d(x_n, Sx_{n-1})d(x_n, Sx_n)}{[d(x_{n-1}, x_n)]^2 + d(x_{n-1}, Sx_n)d(x_n, Sx_n)}\right\} \\ & + \beta\psi\{d(x_{n-1}, Sx_{n-1}) + d(x_n, Sx_n)\} + \gamma\psi\{d(x_{n-1}, Sx_n) + d(x_n, Sx_{n-1})\} \\ & + \eta\psi\left\{\frac{d(x_{n-1}, Sx_{n-1})d(x_n, Sx_n)}{1 + d(x_{n-1}, x_n)}\right\} + \delta\psi\{d(x_{n-1}, x_n)\}\end{aligned}$$

$$\leq \alpha\psi\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})d(x_{n-1},x_{n+1})+d(x_{n-1},x_n)d(x_n,x_n)d(x_n,x_{n+1})}{[d(x_{n-1},x_n)]^2+d(x_{n-1},x_{n+1})d(x_n,x_{n+1})}\right\}$$

$$+\beta\psi\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}+\gamma\psi\{d(x_{n-1},x_{n+1})+d(x_n,x_n)\}$$

$$+\eta\psi\left\{\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}\right\}+\delta\psi\{d(x_{n-1},x_n)\}$$

$$\leq \alpha\psi\{d(x_{n-1},x_n)\}+\beta\psi\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}+\gamma\psi\{d(x_{n-1},x_n)+d(x_n,x_{n+1})\}$$

$$+\eta\psi\{d(x_n,x_{n+1})\}+\delta\psi\{d(x_{n-1},x_n)\}$$

$$\Rightarrow\quad (1-\beta-\gamma-\eta)\psi d(x_n,x_{n+1})\leq(\alpha+\beta+\gamma+\delta)\psi d(x_{n-1},x_n)$$

$$\Rightarrow\quad \psi d(x_n,x_{n+1})\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\psi d(x_{n-1},x_n)$$

Therefore

$$\psi d(x_n,x_{n+1})\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)\psi d(x_{n-1},x_n)$$

$$\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^2\psi d(x_{n-2},x_{n-1})$$

....

....

$$\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^n\psi d(x_0,x_1)$$

$$\psi d(x_n,x_{n+1})\leq\left(\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\right)^n\psi d(x_0,x_1) \qquad (2.1.2)$$

Since $\frac{\alpha+\beta+\gamma+\delta}{1-\beta-\gamma-\eta}\in(0,1)$ from (2.1.2) we obtain $\lim_{n\to\infty}\psi[d(x_n,x_{n+1})]=0$

From the fact that $\psi\in\Psi$, we have $\lim_{n\to\infty}[d(x_n,x_{n+1})]=0$ (2.1.3)

Now we will show that $\{x_n\}$ is a Cauchy sequence in M. suppose that $\{x_n\}$ is not Cauchy sequence which means that there is a constant $\in_0$ such that for each positive integer k, there are positive integers m(k) and n(k) with m(k) >n(k) > k such that

$$d(x_{m(k)},x_{n(k)})\geq\in_0 \quad \text{and} \quad d(x_{m(k)-1},x_{n(k)})<\in_0$$

From lemma (1.3) and Remark (1.4) we obtain

$$\lim_{k\to\infty} d(x_{m(k)},x_{n(k)})=\in_0 \qquad (2.1.4)$$

$$\lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1})=\in_0 \qquad (2.1.5)$$

For $x=x_{m(k)}$ and $y=y_{n(k)}$ from (2.1.1) we have

$$\psi[d(x_{m(k)+1},x_{n(k)+1})]=\psi[d(Sx_{m(k)},Sx_{n(k)})]$$

$$\leq\alpha\psi\left\{\frac{d(x_{m(k)},Sx_{m(k)})d(x_{n(k)},Sx_{n(k)})d(x_{m(k)},Sx_{n(k)})+d(x_{m(k)},x_{n(k)})d(x_{n(k)},Sx_{m(k)})d(x_{n(k)},Sx_{n(k)})}{[d(x_{m(k)},x_{n(k)})]^2+d(x_{m(k)},Sx_{n(k)})d(x_{nk)},Sx_{n(k)})}\right\}$$

$$+\beta\psi\left[d\left(x_{m(k)},Sx_{m(k)}\right)+d\left(x_{n(k)},Sx_{n(k)}\right)\right]+\gamma\psi\left[d\left(x_{m(k)},Sx_{n(k)}\right)+d\left(x_{n(k)},Sx_{m(k)}\right)\right]$$
$$+\eta\psi\left\{\frac{d\left(x_{m(k)},Sx_{m(k)}\right)d\left(x_{n(k)},Sx_{n(k)}\right)}{1+d\left(x_{m(k)},x_{n(k)}\right)}\right\}+\delta\psi\left[d\left(x_{m(k)},x_{n(k)}\right)\right]$$
$$\le\alpha\psi\left\{\frac{d\left(x_{m(k)},x_{m(k)+1}\right)d\left(x_{n(k)},x_{n(k)+1}\right)d\left(x_{m(k)},x_{n(k)+1}\right)+d\left(x_{m(k)},x_{n(k)}\right)d\left(x_{n(k)},x_{m(k)+1}\right)d\left(x_{n(k)},x_{n(k)+1}\right)}{\left[d\left(x_{m(k)},x_{n(k)}\right)\right]^2+d\left(x_{m(k)},x_{n(k)+1}\right)d\left(x_{n(k)},x_{n(k)+1}\right)}\right\}$$
$$+\beta\psi\left[d\left(x_{m(k)},x_{m(k)+1}\right)+d\left(x_{n(k)},x_{n(k)+1}\right)\right]+\gamma\psi\left[d\left(x_{m(k)},x_{n(k)+1}\right)+d\left(x_{n(k)},x_{m(k)+1}\right)\right]$$
$$+\eta\psi\left\{\frac{d\left(x_{m(k)},x_{m(k)+1}\right)d\left(x_{n(k)},x_{n(k)+1}\right)}{1+d\left(x_{m(k)},x_{n(k)}\right)}\right\}+\delta\psi\left[d\left(x_{m(k)},x_{n(k)}\right)\right]$$

Using (2.1.3), (2.1.4) and (2.1.5) we obtain

$$\psi(\in)=\lim_{k\to\infty}\psi\left[d\left(x_{m(k)+1},x_{n(k)+1}\right)\right]$$
$$\le\gamma\lim_{k\to\infty}\psi\left[d\left(x_{m(k)},x_{n(k)+1}\right)+d\left(x_{n(k)},x_{m(k)+1}\right)\right]+\delta\psi\left[d\left(x_{m(k)},x_{n(k)}\right)\right]$$
$$\le\gamma\psi[\in+\in]+\delta\psi(\in)$$

Since $\in$ is arbitrary $\psi(\in)\le(\gamma+\delta)\psi(\in)$

Since $(\gamma+\delta)\in(0,1)$, we get a contradiction, then $\{x_n\}$ is a Cauchy sequence in the complete metric space M. Thus there exist $z_0\in M$ such that $\lim_{n\to\infty}x_n=z_0$

Setting $x=x_n$ and $y=z_0$ in (2.1.1) we get

$$\psi[d(x_{n+1},Sz_0)]=\psi[d(Sx_n,Sz_0)]$$
$$\le\alpha\psi\left\{\frac{d(x_n,Sx_n)d(z_0,Sz_0)d(x_n,Sz_0)+d(x_n,z_0)d(z_0,Sx_n)d(z_0,Sz_0)}{[d(x_n,z_0)]^2+d(x_n,Sz_0)d(z_0,Sz_0)}\right\}$$
$$+\beta\psi[d(x_n,Sx_n)+d(z_0,Sz_0)]+\gamma\psi[d(x_n,Sz_0)+d(z_0,Sx_n)]$$
$$+\eta\psi\left\{\frac{d(x_n,Sx_n)d(z_0,Sz_0)}{1+d(x_n,z_0)}\right\}+\delta\psi[d(x_n,z_0)]$$
$$\le\alpha\psi\left\{\frac{d(x_n,x_{n+1})d(z_0,Sz_0)d(x_n,Sz_0)+d(x_n,z_0)d(z_0,x_{n+1})d(z_0,Sz_0)}{[d(x_n,z_0)]^2+d(x_n,Sz_0)d(z_0,Sz_0)}\right\}$$
$$+\beta\psi[d(x_n,x_{n+1})+d(z_0,Sz_0)]+\gamma\psi[d(x_n,Sz_0)+d(z_0,x_{n+1})]$$
$$+\eta\psi\left\{\frac{d(x_n,x_{n+1})d(z_0,Sz_0)}{1+d(x_n,z_0)}\right\}+\delta\psi[d(x_n,z_0)]$$

$$\lim_{n\to\infty}\psi[d(x_{n+1},Sz_0)]=\lim_{n\to\infty}\psi[d(Sx_n,Sz_0)]$$
$$\le\beta\psi[d(z_0,Sz_0)]+\gamma\psi[d(z_0,Sz_0)]$$
$$\le(\beta+\gamma)\psi d(z_0,Sz_0)$$
$$\lim_{n\to\infty}\psi[d(x_{n+1},Sz_0)]\le(\beta+\gamma)\psi d(z_0,Sz_0)$$
$$\Rightarrow\quad \psi d(z_0,Sz_0)\le(\beta+\gamma)\psi d(z_0,Sz_0)$$

Since $\beta,\gamma\in(0,1)$ then $\psi[d(z_0,Sz_0)]=0$, Which implies that $d(z_0,Sz_0)=0$. Thus $z_0=Sz_0$

Now we are going to establish the uniqueness of fixed point theorem. Let y_0, z_0 be two fixed point of S such that $y_0 \neq z_0$, putting $x = y_0$ and $y = z_0$ in (2.1.1) we get

$$\psi[d(y_0,z_0)] = \psi[d(Sy_0,Sz_0)]$$

$$\leq \alpha\psi\left[\frac{d(y_0,Sy_0)d(z_0,Sz_0)d(y_0,Sz_0)+d(y_0,z_0)d(z_0,Sz_0)d(z_0,Sz_0)}{[d(y_0,z_0)]^2+d(y_0,Sz_0)d(z_0,Sz_0)}\right]$$

$$+\beta\psi[d(y_0,Sy_0)+d(z_0,Sz_0)]+\gamma\psi[d(y_0,Sz_0)+d(z_0,Sy_0)]$$

$$+\eta\psi\left[\frac{d(y_0,Sy_0)d(z_0,Sz_0)}{1+d(y_0,z_0)}\right]+\delta\psi[d(y_0,z_0)]$$

$$\leq \alpha\psi\left[\frac{d(y_0,y_0)d(z_0,z_0)d(y_0,z_0)+d(y_0,z_0)d(z_0,y_0)d(z_0,z_0)}{[d(y_0,z_0)]^2+d(y_0,z_0)d(z_0,z_0)}\right]$$

$$+\beta\psi[d(y_0,y_0)+d(z_0,z_0)]+\gamma\psi[d(y_0,z_0)+d(z_0,y_0)]$$

$$+\eta\psi\left[\frac{d(y_0,y_0)d(z_0,z_0)}{1+d(y_0,z_0)}\right]+\delta\psi[d(y_0,z_0)]$$

$$\leq (2\gamma+\delta)\delta\psi[d(y_0,z_0)]$$

$$\psi d(y_0,z_0) \leq (2\gamma+\delta)\delta\psi[d(y_0,z_0)]$$

⇨ $\psi[d(y_0,z_0)] = 0$

⇨ $d(y_0,z_0) = 0$

⇨ $y_0 = z_0$

Remark: In theorem (2.1) if $\alpha = \beta = \gamma = \eta = 0$ and $\psi(t) = t$, we get the result of Banach [1].

Col2.1.a: Let $\langle M, d\rangle$ be a complete metric space and let $S: M \to M$ be a mapping which satisfies the following condition:

$$d(Sx,Sy) \leq \alpha\left[\frac{d(x,Sx)d(y,Sy)d(x,Sy)+d(x,y)d(y,Sx)d(y,Sy)}{[d(x,y)]^2+d(x,Sy)d(y,Sy)}\right]$$
$$+\beta[d(x,Sx)+d(y,Sy)]$$
$$+\gamma[d(x,Sy)+d(y,Sx)]$$
$$+\eta\left[\frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}\right]$$
$$+\delta[d(x,y)] \qquad (2.1.a)$$

For all x, y ϵ M, x ≠ y, and for some α, β, γ, η, δ > 0 with $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each x in M. $\lim_{n\to\infty} S^n x = z_0$.

Proof: It is enough, if we consider $\psi(t) = t$ in theorem (2.1)

Col 2.1.b: Let $\langle M, d\rangle$ be a complete metric space and let $S: M \to M$ be a mapping which satisfies the following condition:

$$\int_0^{d(Sx,Sy)} \xi(t)dt \le \alpha \int_0^{\frac{d(x,Sx)d(y,Sy)d(x,Sy)+d(x,y)d(y,Sx)d(y,Sy)}{[d(x,y)]^2+d(x,Sy)d(y,Sy)}} \xi(t)dt$$
$$+\beta\int_0^{d(x,Sx)+d(y,Sy)} \xi(t)dt$$
$$+\gamma\int_0^{d(x,Sy)+d(y,Sx)} \xi(t)dt$$
$$+\eta\int_0^{\frac{d(x,Sx)d(y,Sy)}{1+d(x,y)}} \xi(t)dt \qquad (2.1.b)$$
$$+\delta\int_0^{d(x,y)} \xi(t)dt$$

For all x, y ϵ M, x ≠ y, and for some α, β, γ, η, δ > 0 with $\alpha + 2\beta + 2\gamma + \eta + \delta < 1$where $\xi : R^+ \to R^+$ is a Lesbesgue-integrable mapping which is summable on compact subset of R^+, non-negative and such that for each $\in > 0, \int_0^{\in} \xi(t)dt > 0$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each $x \in M$, $\lim_{n\to\infty} S^n x = z_0$.

Proof: If we take $\psi(t) = \int_0^t \xi(t)dt$ in theorem (2.1), we get our result.

Theorem 2.2: Let $\langle M, d\rangle$ be a complete metric space. Let$\psi \in \Psi$ and let $S : M \to M$ be a mapping which satisfies the following condition:

$$\psi[d(Sx,Sy)] \le \alpha\psi \max\{d(Sx,x), d(Sy,x)\} + \beta\psi \max\{d(Sx,y), d(Sy,y)\}$$
$$+\gamma\psi \max\{d(Sx,y), d(Sy,x)\} \qquad (2.2.1)$$

For all x, y ϵ M, x ≠ y, and for some α, β, γ, > 0 with $2\alpha + \beta + 2\gamma < 1$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each x in M, $\lim_{n\to\infty} S^n x = z_0$.

Proof: Let $x \in M$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as follows

$x_{n+1} = Sx_n = S^{n+1}x$, for each n ≥ 0

Now $\psi d(x_n, x_{n+1}) = \psi[d(Sx_{n-1}, Sx_n)]$

$$\le \alpha\psi \max\{d(Sx_{n-1}, x_{n-1}), d(Sx_n, x_{n-1})\} + \beta\psi \max\{d(Sx_{n-1}, x_n), d(Sx_n, x_n)\}$$
$$+\gamma\psi \max\{d(Sx_{n-1}, x_n), d(Sx_n, x_{n-1})\}$$
$$\le \alpha\psi \max\{d(x_n, x_{n-1}), d(x_{n+1}, x_{n-1})\} + \beta\psi \max\{d(x_n, x_n), d(x_{n+1}, x_n)\}$$
$$+\gamma\psi \max\{d(x_n, x_n), d(x_{n+1}, x_{n-1})\}$$
$$\le \alpha\psi d(x_{n+1}, x_{n-1}) + \beta\psi d(x_{n+1}, x_n) + \gamma\psi d(x_{n+1}, x_{n-1})$$

⇨ $$\psi d(x_n, x_{n+1}) \le \left(\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma}\right)\psi d(x_n, x_{n-1})$$

Therefore

$$\psi d(x_n, x_{n+1}) \le \left(\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma}\right)\psi d(x_{n-1}, x_n)$$
$$\le \left(\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma}\right)^2 \psi d(x_{n-2}, x_{n-1})$$
$$\le \ldots\ldots\ldots\ldots$$
$$\le \left(\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma}\right)^n \psi d(x_0, x_1)$$

$$\psi d(x_n, x_{n+1}) \le \left(\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma}\right)^n \psi d(x_0, x_1) \qquad (2.2.2)$$

Since $\frac{\alpha+\gamma}{1-\alpha-\beta-\gamma} \in (0,1)$ from (2.1.2) we obtain $\lim_{n\to\infty} \psi[d(x_n, x_{n+1})] = 0$ mn

From the fact that $\psi \in \Psi$, we have $\lim_{n\to\infty}[d(x_n, x_{n+1})] = 0$ (2.2.3)

Now we will show that $\{x_n\}$ is a Cauchy sequence in M. suppose that $\{x_n\}$ is not Cauchy sequence which means that there is a constant $\in_0$ such that for each positive integer k, there are positive integers m(k) and n(k) with m(k) > n(k) > k such that

$$d(x_{m(k)}, x_{n(k)}) \ge \in_0 \quad \text{and} \quad d(x_{m(k)-1}, x_{n(k)}) < \in_0$$

From lemma (1.3) and Remark (1.4) we obtain

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \in_0 \qquad (2.2.4)$$
$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \in_0 \qquad (2.2.5)$$

For $x = x_{m(k)}$ and $y = y_{n(k)}$ from (2.2.1) we have

$$\psi[d(x_{m(k)+1}, x_{n(k)+1})] = \psi[d(Sx_{m(k)}, Sx_{n(k)})]$$
$$\le \alpha\psi \max\{d(Sx_{m(k)}, x_{m(k)}), d(Sx_{n(k)}, x_{m(k)})\}$$
$$+ \beta\psi \max\{d(Sx_{m(k)}, x_{n(k)}), d(Sx_{n(k)}, x_{n(k)})\}$$
$$+ \gamma\psi \max\{d(Sx_{m(k)}, x_{n(k)}), d(Sx_{n(k)}, x_{m(k)})\}$$
$$\le \alpha\psi \max\{d(x_{m(k)+1}, x_{m(k)}), d(x_{n(k)+1}, x_{m(k)})\}$$
$$+ \beta\psi \max\{d(x_{m(k)+1}, x_{n(k)}), d(x_{n(k)+1}, x_{n(k)})\}$$
$$+ \gamma\psi \max\{d(x_{m(k)+1}, x_{n(k)}), d(x_{n(k)+1}, x_{m(k)})\}$$

Using (2.2.3), (2.2.4) and (2.2.5) we obtain

$$\psi(\in) = \lim_{k\to\infty} \psi[d(x_{m(k)+1}, x_{n(k)+1})]$$
$$\le \alpha \lim_{k\to\infty} \psi[d(x_{n(k)+1}, x_{m(k)})] + \beta \lim_{k\to\infty} \psi[d(x_{m(k)+1}, x_{n(k)})] + \gamma \lim_{k\to\infty} \psi[d(x_{m(k)+1}, x_{n(k)})]$$
$$\le \alpha\psi(\in) + \beta\psi(\in) + \gamma\psi(\in)$$
$$\le (\alpha+\beta+\gamma)\psi(\in)$$

Since $\in$ is arbitrary $\psi(\in) \le (\alpha+\beta+\gamma)\psi(\in)$

Since $(\alpha+\beta+\gamma) \in (0,1)$, we get a contradiction, and then $\{x_n\}$ is a Cauchy sequence in the complete metric space M. Thus there exist $z_0 \in M$ such that $\lim_{n\to\infty} x_n = z_0$

Setting $x = x_n$ and $y = z_0$ in (2.2.1) we get

$$\begin{aligned}\psi[d(x_{n+1},Sz_0)] &= \psi[d(Sx_n,Sz_0)]\\ &\le \alpha\psi\max\{d(Sx_n,x_n)+d(Sz_0,x_n)\}+\beta\psi\max\{d(Sx_n,z_0)+d(Sz_0,z_0)\}\\ &+\gamma\psi\max\{d(Sx_n,z_0)+d(Sz_0,x_n)\}\\ &\le \alpha\psi\max\{d(x_{n+1},x_n),d(Sz_0,x_n)\}+\beta\psi\max\{d(x_{n+1},z_0),d(Sz_0,z_0)\}\\ &+\gamma\psi\max\{d(x_{n+1},z_0),d(Sz_0,x_n)\}\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\psi[d(x_{n+1},Sz_0)] &= \lim_{n\to\infty}\psi[d(Sx_n,Sz_0)]\\ &\le \alpha\psi[d(Sz_0,z_0)]+\beta\psi[d(Sz_0,z_0)]+\gamma[d(Sz_0,z_0)]\\ &\le(\alpha+\beta+\gamma)\psi d(Sz_0,z_0)\end{aligned}$$

$$\lim_{n\to\infty}\psi[d(x_{n+1},Sz_0)]\le(\alpha+\beta+\gamma)\psi d(Sz_0,z_0)$$

$$\Rightarrow\quad \psi d(Sz_0,z_0)\le(\alpha+\beta+\gamma)\psi d(Sz_0,z_0)$$

Since $\alpha+\beta+\gamma\in(0,1)$ then $\psi[d(z_0,Sz_0)]=0$, Which implies that $d(Sz_0,z_0)=0$. Thus $z_0=Sz_0$ Now we are going to establish the uniqueness of fixed point theorem. Let y_0, z_0 be two fixed point of S such that $y_0\ne z_0$, putting $x=y_0$ and $y=z_0$ in (3.2.1) we get $\psi[d(y_0,z_0)]=\psi[d(Sy_0,Sz_0)]$

$$\begin{aligned}&\le \alpha\psi\max\{d(Sy_0,y_0),d(Sz_0,y_0)\}+\beta\psi\max\{d(Sy_0,z_0),d(Sz_0,z_0)\}\\ &+\gamma\psi\max\{d(Sy_0,z_0),d(Sz_0,y_0)\}\\ &\le \alpha\psi\max\{d(y_0,y_0),d(z_0,y_0)\}+\beta\psi\max\{d(y_0,z_0),d(z_0,z_0)\}\\ &+\gamma\psi\max\{d(y_0,z_0),d(z_0,y_0)\}\\ &\le \alpha\psi d(z_0,y_0)+\beta\psi d(y_0,z_0)+\gamma\psi d(z_0,y_0)\\ &\le(\alpha+\beta+\gamma)\psi d(y_0,z_0)\end{aligned}$$

$$\Rightarrow\quad \psi[d(y_0,z_0)]=0$$

$$\Rightarrow\quad d(y_0,z_0)=0$$

$$\Rightarrow\quad y_0=z_0$$

Hence proved

Col2.2.a: Let $\langle M,d\rangle$ be a complete metric space and let $s:M\to M$ be a mapping which satisfies the following condition:

$$\begin{aligned}[d(Sx,Sy)] &\le \alpha\max\{d(Sx,x),d(Sy,x)\}+\beta\max\{d(Sx,y),d(Sy,y)\}\\ &+\gamma\max\{d(Sx,y),d(Sy,x)\}\end{aligned}\qquad (2.2.a)$$

for all x, y ϵ M, x ≠ y, and for some α, β, γ, > 0 with $2\alpha+\beta+2\gamma<1$ then, S has a unique fixed point in $z_0\in M$ and moreover for each x in M. $\lim_{n\to\infty}S^n x=z_0$.

Proof: It is enough, if we consider $\psi(t)=t$ in theorem (2.2.1)

Col 2.2.b: Let $\langle M,d\rangle$ be a complete metric space and let $s:M\to M$ be a mapping which satisfies the following condition:

$$\begin{aligned}\int_0^{d(Sx,Sy)}\xi(t)dt &\le \alpha\int_0^{\max\{d(Sx,x),d(Sy,x)\}}\xi(t)dt+\beta\int_0^{\max\{d(Sx,y),d(Sy,y)\}}\xi(t)dt\\ &+\gamma\int_0^{\max\{d(Sx,y),d(Sy,x)\}}\xi(t)dt\end{aligned}\qquad (2.2.b)$$

For all x, y ϵ M, x ≠ y, and for some α, β, γ > 0 with $2\alpha+\beta+2\gamma<1$ where $\xi:R^+\to R^+$ is a Lesbesgue-integrable mapping which is summable on compact subset of R^+, non-negative and such that for each

$\in > 0, \int_0^{\in} \xi(t)dt > 0$ then, S has a unique fixed point in $z_0 \in M$ and moreover for each $x \in M$, $\lim_{n\to\infty} S^n x = z_0$.

Proof: If we take $\psi(t) = \int_0^t \xi(t)dt$ in theorem (2.2.1), we get our resul**t.**

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