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A First Order Non-Stationary Seasonal Autoregressive Model With a Random Coefficient Parameter

Lect. Dr. Jinan Abdullah Anber
Baghdad Technical College of Management
Dept. Of Information Techniques, Middle Technical University, Baghdad, Iraq
E-mail: jinanaa69@mtu.edu.iq

Prof. Mohammed Qadoury Abed
A l-Mansour University College , Dept. of Accounting & Banking Science, Baghdad, Iraq
E-mail: mohammed.qadaury@muc.edu.iq

Prof. Dr. Wadhah S. Ibrahim
College of Management and Economics , Dep. Of Statistics, Al_Mustansiriyah University, Baghdad, Iraq.
E-mail: dr_wadhah_stat@uomustansiriyah.edu.iq

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Abstract

This research aims to study a first-order seasonal autoregressive model with a random parameter taking different formulas and following the effect of the season on this parameter, assuming that the random errors of the model follows a standard normal distribution. Samples were selected (30, 60, 150, 240) and season lengths (4, 12) and the experiment was repeated 5000 times. One of the main findings is that the value of MSE decreases when increases in sample size and length of the season. Also, it is possible to estimate the parameter value of the model by traditional methods, even if it is a random coefficient. It is also noticed that the model (6) has the lowest MSE value for all sample sizes and different season lengths.

Keywords: Non-Stationary, Autoregressive Model, Seasonal, Random Coefficient, Exact Likelihood Method

1- Introduction

Seasonal time series are a set of observed values correlated to each other. These are generated in succession with the continuation of time to refer to the symmetric pattern of the movement of the time series in successive time periods. This means that the series repeats itself after fixed time periods. It is called the length of the season and is abbreviated by (s).

Many phenomena, whether economic, social, medical, and others, follow their behavior in time series. Therefore, there are many studies interested in studying this due to the possibility of understanding the nature of the changes occurring in the values of a particular phenomenon with time. It is to determine the causes and results, and to interpret and predict the future changes based on the past ones .The analysis of non - stationary autoregressive models with constantly-changing coefficients with time (t) is important in the field of time series. Also, these coefficients $\Phi_s(t)$ depend on time and are called *random coefficients*, which are abbreviated as RC. These can take different forms (linear, Quadratic, Exponential, etc.).In 1960, Winters(15) proposed a new method of exponential boot to treat seasonal time series, Then, till today, studies have continued and mainly concerned with seasonal time series from various aspects.

In 1986, Alena koubkova(1) discussed and examined, at the Tenth Prague Conference, both the non-seasonal autoregressive model when the parameter is followed by random coefficients and the covariance and spectrum functions of that model. In 1990, Klaus Potzelberger (12) gave a description of the autoregressive processes with first-order random coefficients by using the analytical properties of the transition probabilities, and then using them to find solutions to some differential integral equations. In 2002, Al-Nassir & Abed(5) studied the RCAR(1) model when the parameter of the model is a random parameter with different formulas. In 2006, Aue & Horath & Steinebach(7) estimated the random coefficients of a first-order autoregressive model.

In 2013, Salim(13) presented a comparison of some methods for estimating the parameters of a first-order stochastic autoregressive model. In 2014, Al-Nassir & Abed & Ibraheem(6) studied the RCAR (1) model when the random errors of the model follow a Non Gaussian distribution.

The present aims to study examne a first-order seasonal autoregressive model with a random parameter taking different formulas and following the effect of the season on this parameter.

2- Non-Stationary Seasonal Autoregressive Model SAR(1)(2),(3),(4)

In many cases, the time series are non - stationary. This may be due to the fact that the parameter of seasonal autoregressive, abbreviated as $\Phi_s(t)$, changes with time (t), or it is not fixed. The model can be written as follows:

$$Y_t = \Phi_s(t)Y_{t-s} + a_t \qquad \text{... (1)}$$

So, $\Phi_s(t)$ represents the seasonal autoregressive parameter (time-dependent random coefficient t).

a_t: Random Error and Distributed Normally.The model is RCSAR(1).

$$E(Y_t) = \left[\prod_{i=1}^{t} \Phi_s(i)\right] Y_0 \qquad \text{... (2)}$$

$$Var(Y_t) = [\textstyle\sum_{i=1}^{t}\{ \Phi_s(t) \ldots \Phi_s(t-i-1)\}2 + 1]\sigma_a^2 \qquad \text{... (3)}$$

From the above two formulas, it is important to note that the mean and the variance of the time series Y_t depend on time t; this indicates that it is non–stationary.

3- Estimation the Parameter $\boldsymbol{\Phi_s}$(9),(10),(11),(12)

The Exact Maximum Likelihood method was used to estimate the model parameter, which makes the function value the greatest as possible. After deriving the function:

$$L = (2\pi\,\sigma_a^2)^{-\frac{n}{2}} \left|M^{(1,0)}\right|^{\frac{1}{2}} e^{\frac{-S(\Phi_s)}{2\sigma_a^2}} \qquad \dots(4)$$

Estimated values for the model parameter can be obtained according to the following formula:

$$\hat{\Phi}_s = \frac{(n-s-1)\sum_{t=s+1}^{n} Y_t Y_{t-s}}{(n-s)\sum_{t=s+2}^{n} Y_{t-s}^2} \qquad \dots(5)$$

4- RCSAR(1) Models[(5),(6),(8),(13)]

Model (1) **:** $\Phi_s(t) = a_0 + a_1 t$ ... (6)

The RCSAR(1) model can be written as follows:

$$Y_t = (a_0 + a_1 t)Y_{t-s} + a_t \qquad \dots(7)$$

And by using the ordinary Least Squares Method to estimate (a_0) and (a_1) and consequently this accounts the sum of error S(a) as follows:

$$S(a) = \sum_{t=2}^{n}[Y_t - (a_0 + a_1 t)Y_{t-s}]^2 \qquad \dots(8)$$

By taking the first derivative with respect to each of a_0 and a_1 and solving the two equations, the following formulas can be obtained:

$$\hat{a}_1 = \frac{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} t\,Y_t Y_{t-s} - \sum_{t=2}^{n} Y_t Y_{t-s} \sum_{t=2}^{n} t\,Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} t^2 Y_{t-1}^2 - [\sum_{t=2}^{n} t\,Y_{t-s}^2]^2} \qquad \dots(9)$$

$$\hat{a}_0 = \frac{\sum_{t=2}^{n} Y_t Y_{t-s}}{\sum_{t=2}^{n} Y_{t-s}^2} - \frac{\sum_{t=2}^{n} t\,Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2}\,.(\hat{a}_1) \qquad \dots(10)$$

Model (2) **:** $\Phi_s(t) = a_0 + a_1 t + a_2 t^2$ ... (11)

The RCSAR(1) model can be written as follows:

$$Y_t = (a_0 + a_1 t + a_2 t^2)\,Y_{t-s} + a_t \qquad \dots(12)$$

and by using the ordinary Least Squares Method to estimate (a_0) and (a_1) as follows:

$$\hat{a}_2 = \frac{t^2 \sum_{t=2}^{n} Y_t Y_{t-s} + a_0\, t^2 \sum_{t=2}^{n} Y_{t-s}^2 + a_1 t^2 \sum_{t=2}^{n} Y_{t-s}^2}{t^4 \sum_{t=2}^{n} Y_{t-s}^2} \qquad \dots(13)$$

$$\hat{a}_1 = \frac{t \sum_{t=2}^{n} Y_t Y_{t-s} + a_0\, t \sum_{t=2}^{n} Y_{t-s}^2 + a_2 \sum_{t=2}^{n} Y_{t-s}^2}{t^2 \sum_{t=2}^{n} Y_{t-s}^2} \qquad \dots(14)$$

$$\hat{a}_0 = \frac{\sum_{t=2}^{n} Y_t Y_{t-s} + a_1\, t \sum_{t=2}^{n} Y_{t-s}^2 + a_2 t^2 \sum_{t=2}^{n} Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2} \qquad \dots(15)$$

Model (3) **:** $\Phi_s(t) = a_0 + a_1 t^2$ ... (16)

The RCSAR(1) model can be written as follows:

$$Y_t = (a_0 + a_1 t^2) Y_{t-s} + a_t \quad \text{... (17)}$$

And by using the ordinary Least Squares Method to estimate (a_0), (a_1) as follows :

$$\hat{a}_1 = \frac{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} t^2 Y_t Y_{t-s} - \sum_{t=2}^{n} Y_t Y_{t-s} \sum_{t=2}^{n} t^2 Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} t^4 Y_{t-s}^2 - [\sum_{t=2}^{n} t^2 Y_{t-s}^2]^2} \quad \text{... (18)}$$

$$\hat{a}_0 = \frac{\sum_{t=2}^{n} Y_t Y_{t-s}}{\sum_{t=2}^{n} Y_{t-s}^2} - \frac{\sum_{t=2}^{n} t\, Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2} \cdot (\hat{a}_1) \quad \text{... (19)}$$

Model (4): $\Phi_s(t) = \lambda_0 + + \lambda_1 e^{|t|/k}$ ... (20)

If k is a positive integer (k > 0).The RCSAR(1) model can be written as follows:

$$Y_t = (\lambda_0 + + \lambda_1 e^{|t|/k}) Y_{t-s} + a_t \quad \text{... (21)}$$

and by using the ordinary Least Squares Method to estimate (λ_0), (λ_1) as follows:

$$\hat{\lambda}_1 = \frac{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} e^{|t|/k} Y_t Y_{t-s} - \sum_{t=2}^{n} Y_t Y_{t-s} \sum_{t=2}^{n} e^{|t|/k} Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} e^{|t|/k} Y_{t-s}^2 - [\sum_{t=2}^{n} e^{|t|/k} Y_{t-s}^2]^2} \quad \text{... (22)}$$

$$\hat{\lambda}_0 = \frac{\sum_{t=2}^{n} Y_t Y_{t-s}}{\sum_{t=2}^{n} Y_{t-s}^2} - \frac{\sum_{t=2}^{n} e^{|t|/k} Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2} \cdot (\hat{\lambda}_1) \quad \text{... (23)}$$

Model (5): $\Phi_s(t) = \lambda_0 + + \lambda_1 e^{-zY_{t-s}^2}$ ... (24)

where z is a positive integer parameter (z > 0), and it is used to modify the effect of Y_t on the exponential term. The effect of the exponential term is smaller when z is large and its effect is greater when z is small.

The RCSAR(1) model can be written as follows:

$$Y_t = (\lambda_0 + + \lambda_1 e^{-zY_{t-s}^2}) Y_{t-s} + a_t \quad \text{... (25)}$$

It is a non-linear model, and it is known as an exponential autoregressive model of the first order.

And by using the ordinary Least Squares Method to estimate (λ_0), (λ_1) as follows:

$$\hat{\lambda}_1 = \frac{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} e^{-zY_{t-s}^2} Y_t Y_{t-s} - \sum_{t=2}^{n} Y_t Y_{t-s} \sum_{t=2}^{n} e^{-zY_{t-s}^2} Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2 \sum_{t=2}^{n} e^{-zY_{t-s}^2} Y_{t-s}^2 - \left[\sum_{t=2}^{n} e^{-zY_{t-s}^2} Y_{t-s}^2\right]^2} \quad \text{... (26)}$$

$$\hat{\lambda}_0 = \frac{\sum_{t=2}^{n} Y_t Y_{t-s}}{\sum_{t=2}^{n} Y_{t-s}^2} - \frac{\sum_{t=2}^{n} e^{-zY_{t-s}^2} Y_{t-s}^2}{\sum_{t=2}^{n} Y_{t-s}^2} \cdot (\hat{\lambda}_1) \quad \text{... (27)}$$

Model (6):

$$\Phi_s(t) = \begin{cases} \rho_1 & , \; Y_{t-s} \le 0 \\ \rho_2 & , \; Y_{t-s} > 0 \end{cases} \quad \text{... (29)}$$

The RCSAR(1) model can be written as follows:

$$Y_t = \begin{cases} \rho_1 Y_{t-s} + a_t & , \; Y_{t-s} \le 0 \\ \rho_2 Y_{t-s} + a_t & , \; Y_{t-s} > 0 \end{cases} \quad \text{... (30)}$$

Whereas: $\rho_k = \frac{n}{n-k} \cdot \frac{\sum_{t=k+1}^{n} Y_t Y_{t-k}}{\sum_{t=1}^{n} Y_t^2} \quad , k = 1,2$

5- Simulation

5.1- Description of the Experiments

The simulation method was used by designing six experiments, assuming that the Random Errors of the model follows a standard normal distribution. Samples were selected (30, 60, 150, 240) and season lengths (4, 12) and the experiment was repeated 5000 times. The results were compared by using a MSE scale for the parameter; this is because being considered the best, the most common, and the most efficient measure, according to the following formula:(14)

$$MSE(\Phi_s) = \frac{1}{R} \sum_{i=1}^{R} (\widehat{\Phi}_{s(i)} - \Phi_s)^2 \qquad \dots (31)$$

Initial values of the coefficients ($a_0,a_1,a_2,\lambda_0,\lambda_1$) were assumed according to the following table:

Table(1): Initial Values for RCSAR(1) Models

Model	Initial Values
1	a_0=0.04 , a_1=0.001
2	a_0=0.04 , a_1=0.001 , a_2= 0.00001
3	a_0= 0.04 , a_1=0.00001
4	λ_0=0.2 , λ_1=0.05 , k=115
5	λ_0=0.8 , λ_1=-0.8 , z=115
6	ρ_1=-0.3 , ρ_2=0.8

5.2- Results

By adopting simulation experiments, the results are summarized in the following table:

Table(2): MSE Values for RCSAR(1) Model Parameter

s	n	Model 1	Model 2	Model 3	Model 4	Model 5	Model 6
4	30	0.003782	0.003509	0.003787	0.003594	0.001896	0.000463
	60	0.001982	0.001984	0.001983	0.001906	0.000893	0.000049
	150	0.000956	0.000968	0.000959	0.000930	0.000454	0.000019
	240	0.000519	0.000637	0.000552	0.000459	0.000306	0.000286
12	30	0.000415	0.000385	0.000416	0.000395	0.000208	0.000214
	60	0.000218	0.000218	0.000218	0.000209	0.000098	0.000011
	150	0.000105	0.000106	0.000105	0.000102	0.000050	0.000046
	240	0.000057	0.000070	0.000061	0.000050	0.000034	0.000113

6- Conclusions

The most important conclusions were:

1- The larger the sample size, the lower the MSE value for the parameter.
2- The longer the season, the lower the MSE value for the parameter, and for all forms of random coefficients (models) that were used in the research.
3- The possibility of estimating the parameter of the model when it is a random parameter by using the known traditional methods.
4- The lowest value of MSE when n=4, s= 4,12 was for model 6,and the largest value for MSE was for model 3.
5- The lowest value of MSE when n = 60, s = 4.12 was for model 6, and the largest value for MSE was for model 2.
6- The lowest value for MSE when n = 150, s = 4.12 was for model 6, and the largest value for MSE it was for model 5 when s4 and for model 2 when s=12.
7- The lowest value of MSE was when n=240, s=4 was for model 6 and the largest value was for model 2.
8- The lowest value of MSE was when n = 240, s = 12 for model 5, and the largest value was for model 6.

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First A. Author: Lecturer. Dr . Jinan Abdullah Anber, Baghdad 1969. Hold MSc. Statistics (Baghdad Univ.2010) , PH.D Statistics (Baghdad Univ.2016). Published many papers in Iraqi and international journals. From 1994 and still staff Member in Baghdad Technical college of management, Dept. of Information Technique, Middle Technical University, Baghdad, Iraq. ORCID: 0000-0003-2377-9128

Second A. Author: Prof. Mohammed Qadoury Abed, Baghdad 1970. Hold MSc. Statistics (Baghdad Univ.1996). Published many papers in Iraqi and international journals. From 1997 and still staff Member in Al-Mansour University College. Baghdad, Iraq. M of QEAAs, M of Iraqi Academics Syndicate. ORCID: 0000-0001-5758-1369

Third A. Author: Prof. Dr . Wadhah Sabri Ibrahim, Basra 1969. Hold MSc. Statistics (Baghdad Univ. 1997), PH.D Statistics (Baghdad Univ. 2016). Published many papers in Iraqi and international journals. From 2006 and still staff Member in College of Management and Economics. Dept. of Statistics. Al-Mustansiriyah University. Baghdad, Iraq. ORCID: 0000-0003-1781-9621