$\delta - L -$ Paracompact and $\delta - L_2 -$ Paracompact

M. Kareem[1,a], H. Hussain[1,b]

[1] *Department of Mathematics, College of Education, Alzahraa University, Iraq.*

[a)] Corresponding author:maryam Khenyab @ alzahraa. edu.iq

[b)] Corresponding author:[)] huda.ismail@alzahraa.edu.iq

Abstract

In this paper , we shall generalize the definitions and the results of the work[4], from topological spaces to topological vector spaces by using the $\delta - open\ sets$ structures and define another types of δTVS which we will call $\delta - L -$Paracompact ($\delta - L_2 -$Paracompact) topological vector spaces, A $\delta -$Topological vector space $(\delta TVS\ \mathcal{V}_{(K)})$ is called $\delta - L -$paracompact if there exist a $\delta -$paracompact space $\mathcal{U}_{(K)}$ and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$ such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each $\delta -$Lindelöf subspace $A \subseteq \mathcal{V}_{(K)}$. A $\delta TVS\ \mathcal{V}_{(K)}$ is called $\delta - L_2 -$paracompact if there exist a $\delta - T_2 -$ paracompact space $\mathcal{U}_{(K)}$ and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each $\delta -$Lindelöf subspace $A \subseteq \mathcal{V}_{(K)}$. We investigate these two properties.

Keywords: Lindelöf, $\delta -$paracompact, countably normal, $\delta - C -$paracompact, $\delta - C_2 -$paracompact, $\delta - L -$paracompact, $\delta - L_2 -$paracompact, $\delta - L -$normal.

Introduction

The purpose of this paper is to investigate two new properties, $\delta - L -$paracompactness and $\delta - L_2 -$paracompactness. Some of their aspects are similar to L-normality, and some are distinct. Throughout this paper, we denote an ordered pair by(v, u), the set of positive integers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. A $\delta - T_4 -$ space is a $\delta - T_1\ \delta -$ normal space and a Tychonoff space ($\delta - T_3$) is a $\delta - T_1$ completely regular space. Int A and $\bar{A}$ denote the interior and the closure of A, respectively. An ordinal Υ is the set of all ordinal α such that $\alpha < \Upsilon$. The first infinite ordinal is w_0 .

1. Definition [9]

A $\delta TVS\ \mathcal{V}_{(K)}$ is called $\delta - C -$Paracompact if there exist a $\delta -$ Paracompact space $\mathcal{U}_{(K)}$and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each $\delta -$compact subspace $A \subseteq \mathcal{V}_{(K)}$.

A $\delta TVS\ \mathcal{V}_{(K)}$is called $\delta - C_2 -$ Paracompact if there exist a $\delta - T_2 -$ Paracompact space $\mathcal{U}_{(K)}$and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$ such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each $\delta -$compact subspace $A \subseteq \mathcal{V}_{(K)}$.

We use the idea of Arhangel'skii's, Maryam khenyab and Zahir Dobeas Al-Nafie definition above and give the following definition:

2. Definition

A δTVS $\mathcal{V}_{(K)}$is called $\delta - L$ −Paracompact if there exist a δ −Paracompact space $\mathcal{U}_{(K)}$ and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$ such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each δ −Lindelöf subspace $A \subseteq \mathcal{V}_{(K)}$. A δTVS $\mathcal{V}_{(K)}$ is called $\delta - L_2$ −Paracompact if there exist a $\delta - T_2 -$ paracompact space $\mathcal{U}_{(K)}$ and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each δ −Lindelof subspace $A \subseteq \mathcal{V}_{(K)}$.

(Recall that a space X is of countable tightness if for each subset A of X and each $x \in X$ with $x \in \overline{A}$ there exists a countable subset $B \subseteq A$ such that $x \in \overline{B}$.)

3. Theorem

If $\mathcal{V}_{(K)}$ is a $\delta - L -$ paracompact ($\delta - L_2 -$ paracompact) and of countable tightness $f: \mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$ is a witness function of the $\delta - L -$ paracompact ($\delta - L_2 -$ paracompact) of $\mathcal{V}_{(K)}$, then f is $\delta -$ continuous.

Proof. Let $A \subseteq \mathcal{V}$ be arbitrary. We have$f(\bar{A}) = f\left(\bigcup_{B \in [A] \le w_0} \bar{B}\right) = \bigcup_{B \in [A] \le w_0} f(\bar{B}) \subseteq \bigcup_{B \in [A] \le w_0} \overline{f(B)} \subseteq \overline{f(A)}$.

Therefore, f is continuous

(Since any first countable space is Fréchet, any Fréchet space is sequential, and any sequential space is of countable tightness, we conclude that a witness function of the L-paracompactness (L_2-paracompactness) first countable (Fréchet, sequential) space X is continuous) . The following corollary is also clear.

4. Corollary

Any $\delta - L_2$ −paracompact space which is of countable tightness must be at least $\delta - T_2$.

Since any δ −compact space is δ −Lindelöf, then any $\delta - L$ −paracompact space is $\delta - C$ −paracompact and any $\delta - L_2$ −paracompact space is $\delta - C_2$ −paracompact. The converse is not true in general. Obviously, no Lindelöf non-paracompact space is $\delta - L$ −paracompact. So, no uncountable set $\mathcal{V}_{(K)}$ with countable complement topology is $\delta - L$ −paracompact, but it is $\delta - C_2$ −paracompact, hence $\delta - C$ −paracompact, because the only compact subspaces are the finite subspaces, and the countable complement topology is $\delta - T_1$, so compact subspaces are discrete. Hence the discrete topology on $\mathcal{V}_{(K)}$ and the identity function will witness $\delta - C_2$ −paracompactness.

Any δ −paracompact space is $\delta - L$ −paracompact, just by taking $\mathcal{U} = \mathcal{V}$ and the identity function. It is clear from the definitions that any $\delta - L_2$ −paracompact is $\delta - L$ −paracompact. In general, the converse is not true. Assume that $\mathcal{V}$ is δ −Lindelöf and $\delta - L_2$ −paracompact, then the witness function is a homeomorphism which gives that $\mathcal{V}$ is Hausdorff. Thus, any paracompact Lindelöf space which is not Hausdorff is an $\delta -$

$L-$paracompact space that cannot be $\delta - L_2-$paracompact . In particular, any compact space which is not Hausdorff cannot be $\delta - L_2-$paracompact . There is a case when the $\delta - L-$paracompactness implies $\delta - L_2-$paracompatness given in the next theorem .

5. Theorem

If $\mathcal{V}_{(K)}$ is $\delta - T_3-$Separable $\delta - L-$paracompactand countable tightness, then $\mathcal{V}_{(K)}\delta-$Paracompact $\delta - T_4$.

Proof:

Let $\mathcal{U}_{(K)}$ be a $\delta-$Paracompact space and $f: \mathcal{V}_{(K)} \rightarrow \mathcal{U}_{(K)}$ be a bijective witness to $\delta - L-$paracompactness of $\mathcal{V}_{(K)}$. Then f is continuous because $\mathcal{V}_{(K)}$ is of countable tightness . Let $\mathcal{D}$ be contable dense subset of $\mathcal{V}_{(K)}$. We show that f is $\delta-$closed. Let $\mathcal{H}$be any non-empty $\delta-$closed proper subset of $\mathcal{V}_{(K)}$. suppose that $f(p) = q \in \mathcal{U} \backslash f(\mathcal{H})$; then $p \notin \mathcal{H}$. Using regularity , let $A \; and \; B$ be disjoint $\delta-$open subset of $\mathcal{V}_{(K)}$ containing $p \; and \; \mathcal{H}$, respectively. Then $A \cap (\mathcal{D} \cup \{p\})$ is $\delta-$open in the $\delta-$Lindelof subspace $\mathcal{D} \cup \{p\}$ containing p , so $f(A \cap (\mathcal{D} \cup \{p\}))$ is $\delta-$open in the subspace$f(\mathcal{D} \cup \{p\})$ of $\mathcal{U}_{(K)}$ containing q . Thus $f\big(A \cap (\mathcal{D} \cup \{p\})\big) = f(A) \cap f(\mathcal{D} \cup \{p\}) = W \cap f(\mathcal{D} \cup \{p\})$ for some $\delta-$open subset $W \; in \; \mathcal{U}_{(K)} with \; q \in W$. We claim that $W \cap f(\mathcal{H}) = \emptyset$. Suppose otherwise, and take $u \in W \cap f(\mathcal{H})$. Let $v \in \mathcal{H}$ such that $f(v) = u$. Not that $v \in B$. Since $\mathcal{D}$ is dense in $\mathcal{V}_{(K)}$,$\mathcal{D}$ is also dense in the $\delta-$open set B . Thus $v \in \overline{B \cap \mathcal{D}}$. Now since W is $\delta-$open in $\mathcal{U}_{(K)}$ and f is continuous , $f^{-1}(W)$is an $\delta-$open set in $\mathcal{V}_{(K)}$; it also contains v . Thus we can choose $d \in f^{-1}(W) \cap B \cap \mathcal{D}$. Then $f(d) \in W \cap f(B \cap \mathcal{D}) \subseteq W \cap f(\mathcal{D} \cup \{p\}) = f(A \cap \mathcal{D} \cup \{p\})$. So $f(d) \in f(A) \cap f(B)$, a contradiction. Thus $W \cap f(\mathcal{H}) = \emptyset$. Not that $q \in W$. As $q \in \mathcal{U} \backslash f(\mathcal{H})$ was arbitrary , $f(\mathcal{H})$ is $\delta-$closed . So f is homeomorphism and $\mathcal{V}_{(K)}$ is $\delta-$paracompact . Since $\mathcal{V}_{(K)}$ is also $\delta - T_2$ is $\delta-$normal . Not that $\mathcal{V}_{(K)}$ is also $\delta-$Lindelof being $\delta-$separable and $\delta-$Paracompact.

6. Theorem

$\delta - L-$paracompactness($\delta - L_2-$paracompactness) is a topological property .

Proof:

Let $\mathcal{V}_{(K)}$ be an $\delta - L-$Parncompact space and $\mathcal{V}_{(K)} \cong \mathcal{Z}_{(K)}$. Let $\mathcal{U}_{(K)}$ be a $\delta-$paracompact space and $f: \mathcal{V}_{(K)} \rightarrow \mathcal{U}_{(K)}$ be a bijection such that $f|_C: C \rightarrow f(C)$ is a homeomorphism for each $\delta-$Lindelöf subspace $C$of $\mathcal{V}_{(K)}$. Let $g: \mathcal{Z}_{(K)} \rightarrow \mathcal{V}_{(K)}$ be a homeomorphism Then $f \; o \; g : \mathcal{Z}_{(K)} \rightarrow \mathcal{U}_{(K)}$ satisfies all requirements.

7. Theorem

$\delta - L-$paracompactness($\delta - L_2-$paracompactness) is an additwe property.

Proof:

Let $\mathcal{V}_\delta$ be an $\delta - L-$Paracompact space for each $\delta \in \Lambda$. We show that their sum $\bigoplus_{\delta \in \Lambda} \mathcal{V}_\delta$ $\delta-$Paracompact. For each $\delta \in \Lambda$. pick a $\delta-$Paracompact space $\mathcal{U}_\delta$ and a bijective function $f_\delta: \mathcal{V}_\delta \rightarrow \mathcal{U}_\delta$ such that $f|_{\alpha_{|C\alpha}} : C_\delta \rightarrow$

$f_\delta(C_\delta)$ is a homeomorphism for each δ −Lindelof subspace $C_\delta\ of\ \mathcal{V}_\delta$. Since $\mathcal{U}_\delta$ is δ −Paracompact for each $\delta \in \Lambda$, then the sum $\oplus_{\delta\in\Lambda}\mathcal{U}_\delta$cay is δ −Paracompact. Consider the function sum $\oplus_{\delta\in\Lambda}f_\delta : \oplus_{\delta\in\Lambda}\mathcal{V}_\delta \to \oplus_{\delta\in\Lambda}\mathcal{U}_\delta$ defined by $\oplus_{\delta\in\Lambda}f_\delta\ (v) = f_\beta\ (v)\quad if\ v\ \in \mathcal{V}_\beta, \beta \in\ \Lambda$. Now, a subspace $C\ \subseteq \oplus_{\delta\in\Lambda}\mathcal{V}_\delta$ is Lindelöf if and only if the set $\Lambda_0 = \{\delta \in \Lambda: C\ \cap \mathcal{V}_\delta\ \neq \emptyset$ is countable and $C\ \cap \mathcal{V}_\delta$is Lindelöf in $\mathcal{V}_\delta$, for each$\delta\ \in \Lambda_0$. If $C\ \subseteq\ \oplus_{\delta\in\Lambda}\mathcal{V}_\delta$, is Lindelöf, then $(\oplus_{\delta\in\Lambda}f_\delta\ )|_c$ is a homeomorphism because $f_{\delta|C\cap\mathcal{V}_\delta}$, is a homeomorphism for each $\delta\ \in \Lambda_0$.

8. Theorem

Every second countable $\delta - L_2$ −Paracompact space is metrizable.

Proof:

If $\mathcal{V}_{(K)}$ is a second countable space, then $\mathcal{V}_{(K)}$ is δ −Lindelöf. If $\mathcal{V}_{(K)}$ is also $\delta - L_2$ −paracompact, then $\mathcal{V}_{(K)}$ will be homeomorphic to a $\delta - T_2$ paracompact space $\mathcal{U}_{\ (K)}$ and, in particular, $\mathcal{U}_{\ (K)}$ is $\delta - T_4$. Thus $\mathcal{V}_{(K)}$ is second countable and regular, hence metrizable .

9. Corollary

Every $\delta - T_2$ second countable $\delta - L$ −paracompact space is metrizable.

10. Definition:

A $\delta TVS\ \mathcal{V}_{(K)}$is called $\delta - L$ −normal if there exist a δ −normal space $\mathcal{U}_{\ (K)}$ and a bijective function $f: \mathcal{V}_{(K)} \to \mathcal{U}_{\ (K)}$ such that the restriction $f|_A: A \to f(A)$ is a homeomorphism for each δ −Lindelöf subspace $A\ \subseteq\ \mathcal{V}_{(K)}$. Since any $\delta - T_2 -$ Paracompact space is δ −normal, it is clear that any $\delta - L_2$ −Paracompact space is $\delta - L$ −normal. In general, $\delta - L$ −Paracompactness does not imply $\delta - L$ −normality. Observe that any finite space which is not discrete is compact, hence Paracompact, thus $\delta - L$ −paracompact. So, any finite space which is not normal will be an example of an $\delta - L$ −paracompact which is neither $\delta - L_2$ −paracompact nor $\delta - L$ −normal. In general, $\delta - L$ −normality does not imply $\delta - L$ −Paracompactness. Here is an example.

11. Example

Let $\mathcal{V} = [0, \infty)$. Define Ꞇ $= \{\emptyset, \mathcal{V}\}\ \cup \{[0, v): v \in \mathbb{R}, 0\ < v\}$. Observe that $(\mathcal{V}, Ꞇ)$ is δ −normal because there are no two non-empty δ −closed disjoint subsets. Thus $(\mathcal{V}, Ꞇ)$is $\delta - L$ −normal. Observe that $(\mathcal{V}, Ꞇ)$ is second countable, hence hereditarily Lindelöf. $(\mathcal{V}, Ꞇ)$ cannot be δ −Paracompact because Ꞇ is coarser than the particular point topology on $\mathcal{V}$, where the particular point is 0. That's because any non-empty δ −open set contains 0. Therefore, $\mathcal{V}$ is $\delta - L$ −normal but not $\delta - L$ −paracompact.

Conclusions

this study gives a new view of the topology through the vector spaces. This work has many new results that can be summarized in the following facts:

1. If $\mathcal{V}_{(K)}$ is a $\delta - L -$ paracompact ($\delta - L_2 -$ paracompact) and of countable tightness $f:\mathcal{V}_{(K)} \to \mathcal{U}_{(K)}$ is a witness function of the $\delta - L -$ paracompact ($\delta - L_2 -$ paracompact) of $\mathcal{V}_{(K)}$, then f is $\delta -$ continuous.
2. If $\mathcal{V}_{(K)}$ is $\delta - T_3$ −Separable $\delta - L$ −paracompactand countable tightness, then $\mathcal{V}_{(K)}\delta$ −Paracompact $\delta - T_4$.
3. $\delta - L$ −paracompactness($\delta - L_2$ −paracompactness) is a topological property .
4. $\delta - L$ −paracompactness($\delta - L_2$ −paracompactness) is an additwe property.
5. Every second countable $\delta - L_2$ −Paracompact space is metrizable.
6. Every $\delta - T_2$ second countable $\delta - L$ −paracompact space is metrizable.

Reference:

[1] van Douwen, E.K. The Integers and Topology, in: Handbook of Set-Theoretic Topology,North-Holland, Amsterdam, 111-167, 1984.

[2] Engelking. R. General Topology, PWN, Warszawa, 1977.

[3] Juhász, I., Kunen, K. and Rudin, M.E. Two More Hereditarily Separable non-Lindelöf spaces, Cand. J. Math., 28. 998-1005, 1976.

[4] Kalantan, M.M. L- Parocompactness and L_2- Paracompactness to appear.

[5] Kalantan, L. and Szeptycki, P. k-normality and products of ordinals. Topology and its Applications, vol 123, 3, 537-545. 2002.

[6] Kalantan, L. Results about k-normality Topology and its Applications 125. 47-62. 2002.

[7] Kalantan, L., Alzumi, H. and Saeed, M.M. C-Parocompactness and C_2-Paracompactness to appear.

[8] Kalantan, L. and Saeed. M. L- Normalitu. Topology Proceedings. 50, 141-119. 2017.

[9] Kareem K. Maryam and AL-Nafie Z.D. δ −C-Parocompactness and δ −C_2-Paracompactness 2021.

[10] Rudin, M.E. A Separable Dowker space, Symposta Mathematica, Instituto Nazionale di Alta Mathematica, 1973.

[11] Saeed, M.M. Countable Normality, to appear.

[12] Singal, M.K. and Singal. A.R. Mildly Normal Spaces, Kyungpook Math J. 13, 29-31, 1973.

[13] Shchepin, E.V. Real Valued Functions and Spaces Close to Normal, Sib. J. Math. 13, 1182-1196, 1972.

[14] Steen, L. and Seebach, J.A. Counteresamples in Topology. Dover Publications, INC. 1995.

[15] Weiss, W. Small Docker Spaces, Pacific J. Math. 94. 185-192, 1981.