Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.12, No.2, 2022

On the Equivariance of Location Reparameterization of Quantile Regression Model using Cauchy Transformation

Nwabueze Joy Chioma[1], Onyegbuchulem Besta Okey[2] and Nwakuya Maureen Tobechukwu[3*]

1. Department of Statistics, Micheal Okpara University of Agriculture Umudike
2. Department of Maths/Statistics, Imo State Polytechnic Umuagwo
3. Department of Maths/Statistics, University of Port Harcourt

*Email of correspond author: maureen.nwakuya@uniport.edu.ng

The research is by the authors

Abstract

Often times fine-tuning the location parameter of original variables or reparameterizing a model in order to make the result obtained from such change to have an improved natural interpretation is desirable. Based on the regression output such changes are expected to affect either the qualitative and quantitative conclusion. This article tends to examine the equivariance to location reparameterization of quantile regression model. The analysis was done using real life data set on fuel consumption (in miles per gallon), in highway driving as the response variable while car weight, length, wheel base, width, Engine size and horse power are the explanatory variables with a sample size of 91. The general Cauchy distribution was used to transform the quantile regression model. The results show that mean square errors from the quantile regression model estimates are similar across different location parameters of our study model; this therefore shows that quantile regression model has the property of equivariance to location reparameterization.

Keywords: Quantile Regression Model, Cauchit Quantile Regression Model, location Reparameterization and Mean Square Error

1. Quantile Regression Model

A better alternative to conditional-mean modelling has the tendency of measuring the intercept of the median regression which obviously happened to be a key theorem about minimizing sum of the absolute deviation and a geometrical algorithm for constructing median regression and this was proposed by Ruder Josip Boskovic, a Jesuit Catholic Priest from Dubrovnik in 1790.

The quantile notion however generalizes some terms like the percentile, the decile, the quintile and the quartile. As for the P^{th} quantile, it is used to denote that value of the dependent variable which its proportion is below the part of population that is P. Thus, quantiles can specify and understudy any position of a distribution. For example, 7.5% of the population lies below the 0.075^{th} quantile.

Koenker & Bassett (1978) introduced the first order Quantile Regression model which has the form

$$\mathbb{Q}_{y_i}(\tau/x) = \beta_0 + \beta_i x_i + F_u^{-1}(\tau) \qquad (1.1)$$

where

$\mathbb{Q}_{y_i}$ is the conditional value of the dependent variable given τ in the i^{th} trial,

β_0 is the intercept parameter,

β_i is the slope parameter,

τ denotes the quantile (e.g., for $\tau = 0.50$ the median),

x_i is the value of the independent variable in the i^{th} trial,

F_u is the common distribution function of the error τ,

$E(F_u^{-1}(\tau)) = 0, for\ i = 1, \ldots, n.\ eg\ F^{-1}(0.50) is\ the\ median\ quantile.$

These model conditional quantiles are functions of explanatory variables. Therefore, it may not be out of context to say that modelling quantile regression is naturally, an extension of the linear-regression model. This is so because, the idea of modelling linear-regression is to determine the change(s) in the conditional mean of the response variable which inherently is associated with a change in the independent variables, but the idea in modelling quantile- regression on the other hand is to determine the changes in the conditional quantile. A lot of work has been done in quantile regression these include; Onyegbuchulam et al (2019) evaluated the assumptions of the Linear Regression Model on the Quantile Regression Model, Nwakuya et al (2019) carried out response variable transformation for Quantile Regression Model, Nwakuya et al (2020) proposed a B.Bounded Logistic Quantile Regression as an alternative to Logistic Quantile Regression also Onyegbuchulam et al (2019), considered a choice of appropriate power transformation for skewed distribution of a Quantile Regression model. There are some properties of quantile regression model proposed by Koenker (1978), which makes it unique among other regression models. Such properties include: equivariance to monotone transformations, equivariance to scale parameterization, equivariance to shift parameterization, equivariance to location parameterization and, Robustness to Outliers property. The equivariance property of quantile regression model enable us to fine turn the parameters of the original variables or reparametrizing a model in order to make the result obtained from such change to have an improved natural interpretation. Based on the regression output such changes are expected to affect either the qualitative and quantitative conclusions. This is what is meant by equivariance to reparameterization.

Therefore, this research intends to investigate the equivariance to location reparametrization of quantile regression model using the quantile function of Cauchy distribution.

1.1 Cauchy Distribution

The Cauchy distribution with a common notation as *X~cauchy* (θ,α) where θ is the location parameter and $\alpha > 0$ is the scale parameter is coined after Augustin Cauchy, and it belongs to the family of stable distributions that is closed under the formation of sums of independent random variables, its expected value, the variance, skewness and kurtosis do not exist but its median is given as θ (Alzaatreh et. al; 2016). Cauchy distribution has been applied in various fields like mechanical and electrical theory, physical anthropology, measurement problems, risk and financial analysis. Nwabueze et al (2021) applied a Cauchy transformation approach to the robustness of quantile regression model to outlier. It was used by Stigler (1989) to derive an explicit expression for $P(Z_1 < 0, Z_2 \leq 0)$, where (Z_1, Z_2)) follows the standard bivariate normal distribution, it was also applied to model the points of effect of a fixed straight line of particles released from a point source. It is called a Lorenzian distribution in physics, where it is defined as the energy of an unstable state distribution in quantum mechanics. Generally, the general probability density function (PDF) of the Cauchy distribution is defined as:

$$f(y) = (\pi\alpha)^{-1}[1 + \{(y-\theta)/\alpha\}^2]^{-1}, -\infty < y < \infty \qquad (1.2)$$

While the standard probability density function (PDF) of the Cauchy distribution is:

$$f(y) = \pi^{-1}(1+y^2)^{-1}, for\ \theta = 0, \alpha = 1, -\infty < y < \infty$$ (Hao and Norman, 2007) (1.3)

Major characteristics of the Cauchy distribution, includes the non-existence of its mean, variance, skewness and kurtosis. Norman et al. (2005) also stated that there is no standardized form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation which do not exist. In this case, however, a standard form is obtained by substituting θ = 0, α = 1 which makes it to coincide with the student's t distribution with one degree of freedom.

Let, $y - \theta = u$, then equation 1.2 becomes; $f(y) = (\pi\alpha)^{-1}[1 + \{(u)/\alpha\}^2]^{-1}, -\infty < y < \infty$ (1.4)

And the CDF becomes, $F(y) = \frac{1}{2} + \frac{1}{\pi} tan^{-1}((y-\theta)/\alpha)$ (1.5)

while $F(\infty) = \frac{1}{2} + \frac{1}{\pi} tan^{-1}(\infty) = 1$ (1.6)

If α = 1 and θ = 0 in equation 1.5, it gives us the standard cumulative density function (CDF) of the Cauchy distribution, which is;

$$F(y) = \frac{1}{2} + \frac{1}{\pi} tan^{-1}(y), \quad -\infty < y < \infty \tag{1.7}$$

Let, $y = h[\mathbb{Q}_y(\tau)]$, then the CDF inverse (F^{-1}) function of the general Cauchy distribution becomes:

$$h[\mathbb{Q}_y(\tau)] = tan\left[\pi\left\{\mathbb{Q}_y(\tau) - \frac{1}{2}\right\}\right] + \theta \tag{1.8}$$

2. Cauchit Quatile Regression

Cauchit quantile regression is from the family of Cauchy distribution. Eugene et al., (2002) introduced the beta – generated family of distribution where the authors used the beta distribution as the base line distribution, this was followed by Alshawarbeh et al; (2013) who developed the beta – Cauchy distribution which was extended to *T - R(W)* family by Alzaartreh et. al; (2013), where the authors gave the *T - R(W) c*umulative distribution function as:

$G(x) = \int_{\alpha}^{W(F)(x)} r(t)dt$, where *r(t)* denotes the probability density function of the random variable T with support (a,b) for $-\infty < a < b < \infty$. The authors used the random variable T as the transformer to transform the random variable R into a new family of the generalize distribution of R. In this work, Cauchit quantile regression is introduced where the quantile function of Cauchy distribution is used as the transformer to transform the quantile regression into a new model that can handle ordinal response data and binary response data, as well manages outliers in such distribution. The general probability density function of a Cauchy distribution is given in equation 1.4, the general CDF is given in equation 1.6 while the CDF inverse (F^{-1}) or the Probit function of the Cauchy distribution that will be used for data simulation is derived from the CDF of Cauchy distribution in equation (1.8). The next step is to equate the CDF inverse (F^{-1}) of the Cauchy function of equation (1.8) to the quantile regression model of equation (1.1) and solve simultaneously for the Cauchit quantile regression model. Since equation (1.1) is true for the quantile regression model, and equation (1.8) is true for the Cauchy quantile function, equation (1.8) is therefore equated to equation (1.1) to form a Cauchit quartile regression model.

$$h[\mathbb{Q}_y(\tau)] = (\beta_0^{(\tau)} + \beta_1^{(\tau)} x_i) \tag{1.9}$$

$$\tan[\pi\{\mathbb{Q}_y(\tau) - 1/2\}] + \theta = (\beta_0^{(\tau)} + \beta_1^{(\tau)} x_i) \tag{2.0}$$

$$\tan[\pi\{\mathbb{Q}_y(\tau) - 1/2\}] = (\beta_0^{(\tau)} + \beta_1^{(\tau)} x_i) - \theta \tag{2.1}$$

$$\pi\{\mathbb{Q}_y(\tau) - 1/2\} = tan^{-1}\{(\beta_0^{(\tau)} + \beta_1^{(\tau)} x_i) - \theta\}$$

$$\mathbb{Q}_y(\tau) = \pi^{-1} tan^{-1}\{(\beta_0^{(\tau)} + \beta_1^{(\tau)} x_i) - \theta\} + 1/2 \qquad (2.2)$$

$$-1 < \mathbb{Q}_y(\tau) < 1, -\infty < x_i < \infty, 0 < \tau < 1$$

$\mathbb{Q}_{y_i} \longrightarrow$ the response variable and the CDF inverse (F^{-1}) of the distribution to be estimated

$\beta_0 \longrightarrow$ is the intercept parameter,

$\beta_i \longrightarrow$ is the slope parameter,

$\tau \longrightarrow$ specified quantiles of the model. This research examines the following quantiles: 0.05, 0.25, 0.5, 0.75, 0.95

$x_i \longrightarrow$ the covariates to be simulated

Equation (2.2) is the proposed Cauchit quantile regression model that will be used to examine the location parameterization of the quantile regression model.

2.1 Linear Programing Formulation of Cauchit Quantile Regression Problem

The check function in equation can also be expressed as:

$$\sum_{i=1}^{n} \rho_i(\varepsilon_i) = \sum_{i=1}^{n} \tau u_i = +(1-\tau) v_i = \tau \mathbf{1}_N^T \boldsymbol{U} + (1-\tau)\mathbf{1}_M^T \boldsymbol{V}$$

Where, $u = (u_{1,\dots,}u_N)^T \, and \; v = (v_{1,\dots,}v_N)^T$

And I_N is a vector *N X 1* all coordinate equal to 1. The residual must satisfy the N constraints that:

$$\mathbb{Q}_y(\tau) - \pi \, tan^{-1}(X_i^T \beta^{(\tau)}) + (1/2) = -\varepsilon_i = u_i - v_i$$

This results in the formulation as a linear program (LP):

$$\beta \in R^k \min_{u \in R^k, v \in R^k} \left\{\tau \mathbf{1}_N^T \boldsymbol{U} + (1-\tau)\mathbf{1}_N^T \boldsymbol{V} \,/\, y = \pi \, tan^{-1}(X_i^T \beta^{(\tau)}) + \frac{1}{2} + u_i - v_i, i = 1, \dots, N\right\}$$

As required by the canonical form of linear program, it can be observed that β $\beta \in R^k$ is still not restricted to be positive. This lead into decomposing it into negative and positive parts such as:

$\beta = \beta^+ - \beta^-$ where $\beta^+ = \max(0, \beta)$ is positive part while $\beta^- = \max(0, -\beta)$ is the negative part. The N constraints can then be written as:

$$\mathbb{Q}_y(\tau) = \begin{bmatrix} \mathbb{Q}_{y1}(\tau) \\ \vdots \\ \mathbb{Q}_{yN}(\tau) \end{bmatrix} = \pi \begin{bmatrix} tan^{-1}(X_1) \\ \vdots \\ tan^{-1}(X_N) \end{bmatrix} (\beta^{+1} - \beta^{-1}) + \left[\tfrac{1}{2}\right]_N + \mathbf{1}_N \boldsymbol{U} - \mathbf{1}_N \boldsymbol{V} \qquad (2.3)$$

Where $\mathbf{1}_N = diag\, \mathbf{1}_N, \left[\tfrac{1}{2}\right]_N = disg \left[\tfrac{1}{2}\right]_N$ and π is a scalar. Then next is to define; $y = \mathbb{Q}_{y_i}(\tau)$ and the design matrix X is given by; $\begin{bmatrix} X_i^T \\ \vdots \\ X_N^T \end{bmatrix}$. To rewrite the constant:

$$\left.\begin{aligned} y &= \pi \tan^{-1}(X)(\beta^+ - \beta^-) + \mathbf{1}_N \boldsymbol{U} - \mathbf{1}_N \boldsymbol{V} + \left[\tfrac{1}{2}\right]_N \\ &= [\pi]_N^T \tan^{-1}[\boldsymbol{X}, -\boldsymbol{X}, \mathbf{1}_N, -\mathbf{1}_N] \begin{bmatrix} \beta^+ \\ \beta^- \\ u \\ v \end{bmatrix} + \left[\tfrac{1}{2}\right]_N \end{aligned}\right\} \qquad (2.4)$$

Define the $(N + 2k + 2N)$ matrix as; $\boldsymbol{A} = [\pi]_N^T \tan^{-1}[\boldsymbol{X}, -\boldsymbol{X}, \mathbf{1}_N, -\mathbf{1}_N] + \left[\tfrac{1}{2}\right]_N$

And introducing β^+ and β^- as variables over which to minimize so that they are part of Z to obtain:

$y = A\begin{bmatrix} \beta^+ \\ \beta^- \\ u \\ v \end{bmatrix} = \boldsymbol{AZ}$ Because β^+ and β^- only affect the minimization problem through the constraint, a 0 of dimension $2kX1$ must be introduced as part of the coefficient vector C which can then be appropriately define as: $C = \begin{bmatrix} 0 \\ \tau \mathbf{1}_N \\ (1-\tau)\mathbf{1}_N \end{bmatrix}$, thus ensuring that: $C^T Z = 0^T[\pi]\tan^{-1}(\beta^+ - \beta^-) + \frac{1}{2} + \tau \mathbf{1}_N^T \boldsymbol{U} + (\mathbf{1} - \tau)\mathbf{1}_N^T v_i$

$$= \sum_{i=1}^{N} \rho_\tau(\varepsilon_i) \qquad (1.2)$$

3.0 Data Presentation, Analysis, Results and Discussions

The experiment was done using real life data borrowed from logistic quantile regression by Efron (1979) on fuel consumption for 91 cars measured on six explanatory variable with the design matrix given as $(n\,X\,k)$, where n=91, and $x_i = (x_1, x_2, x_3, x_4, x_5, x_6)$ where x_1 is weight of the car, x_2 is length of the car, x_3 is width of the car, x_4 is Wheel base of the car, x_5 is Engine size of the car, and x_6 is the car's horse power, the response variable (y_i) is miles per gallon of fuel consumed by the car. The size of the sample is 91 but in order to estimate the standard error, the confidence interval and the p-values, the real life data were bootstrapped 200 times following Efron (1979). The data analysis for the experiments was done using the models in equations (2.2)

Table 3.1: Cauchity Quantile Regression Mean Square Error for Real Life Data n = 91, bootstrapped 200 times for different values of α and θ

	θ						
		Quantile 0.5					
α	-3	-2	-1	0	1	2	3
0.005	1.1e-09	1.1e-09	1.1e-09	1.1e-09	1.1e-09	1.1e-09	1.1e-09
0.01	4.5e-09	4.5e-09	4.5e-09	4.5e-09	4.5e-09	4.5e-09	4.5e-09
0.5	1.1e-05	1.1e-05	1.1e-05	1.1e-05	1.1e-05	1.1e-05	1.1e-05
1	4.5e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05
2	0.00018	0.00018	0.00018	0.00018	0.00018	0.00018	0.00018
3	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041	0.00041
		Quantile 0.25					
0.005	8.0e-10	8.0e-10	8.0e-10	8.0e-10	8.0e-10	8.0e-10	8.0e-10
0.01	3.2e-09	3.2e-09	3.2e-09	3.2e-09	3.2e-09	3.2e-09	3.2e-09
0.5	8.1 e-06	8.1 e-06	8.1 e-06	8.1 e-06	8.1 e-06	8.1 e-06	8.1 e-06
1	3.2e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05	3.2e-05
2	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013

3	0.00029	0.00029	0.00029	0.00029	0.00029	0.00029	0.00029
		Quantile 0.5					
0.005	7.6e-10	7.6e-10	7.6e-10	7.6e-10	7.6e-10	7.6e-10	7.6e-10
0.01	3.0e-09	3.0e-09	3.0e-09	3.0e-09	3.0e-09	3.0e-09	3.0e-09
0.5	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06
1	3.0e-05	3.0e-05	3.0e-05	3.0e-05	3.0e-05	3.0e-05	3.0e-05
2	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013	0.00013
3	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027	0.00027
		Quantile 0.75					
0.005	7.2e-10	7.2e-10	7.2e-10	7.2e-10	7.2e-10	7.2e-10	7.2e-10
0.01	2.9e-09	2.9e-09	2.9e-09	2.9e-09	2.9e-09	2.9e-09	2.9e-09
0.5	7.2e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06	7.6 e-06
1	2.9e-05	2.9e-05	2.9e-05	2.9e-05	2.9e-05	2.9e-05	2.9e-05
2	0.00012	0.00012	0.00012	0.00012	0.00012	0.00012	0.00012
3	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026	0.00026
		Quantile 0.95					
0.005	1.2e-09	1.2e-09	1.2e-09	1.2e-09	1.2e-09	1.2e-09	1.2e-09
0.01	4.8e-09	4.8e-09	4.8e-09	4.8e-09	4.8e-09	4.8e-09	4.8e-09
0.5	1.2 e-05	1.2 e-05	1.2 e-05	1.2 e-05	1.2 e-05	1.2 e-05	1.2 e-05
1	4.8e-05	4.8e-05	4.8e-05	4.8e-05	4.8e-05	4.8e-05	4.8e-05
2	0.00019	0.00019	0.00019	0.00019	0.00019	0.00019	0.00019
3	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043	0.00043

Table 3.3 is the result of Mean Square Error of Cauchity Quantile Regression for the real life data set for different values of α and θ. The results show that all the values of the mean square errors are within the range of zero and it maintained a significant reduction as the values of α reduces. The value of the Mean Square Error for location parameter θ remains equivariance for $-3 \leq \theta \leq 3$. In quantile 0.95 at $\alpha = 3$ the Mean Square Error is 0.00043 for all values of θ, at $\alpha = 2$ the Mean Square Error is 0.00019 for all values of θ, at $\alpha = 1$ the Mean Square Error is 4.8e-05 for all values of θ. This continues in that order for all quantiles at each $\alpha = 0$ for all values of $-3 \leq \theta \leq 3$ as can be seen in Table 3.1

Table 3.2: Estimated Parameters for the Cauchit QR Model Computed using the Real Life Data

Quantiles	parameters	coefficient	Std error	T - value	Pr(>|t|)
0.05	**intercept**	0.01890	0.01875	1.00813	0.31629
	weight	0.00000	0.00000	0.09876	0.92157
	Length	0.00006	0.00004	1.63622	0.10554
	Wheel base	-0.00018	0.00015	-1.19987	0.23356
	width	-0.00040	0.00024	-1.66296	0.10005
	Engine Size	0.00125	0.00068	1.83661	0.06980
	Horse Power	-0.00002	0.00001	-1.39614	0.16635
0.25	**intercept**	-0.01699	0.00858	-1.98008	0.05097
	weight	-0.00001	0.00000	-4.16024	0.00008
	Length	0.00008	0.00002	3.21769	0.00184
	Wheel base	0.00002	0.00007	0.22930	0.81919
	width	0.00006	0.00009	0.65323	0.51539
	Engine Size	0.00024	0.00045	0.54877	0.58462
	Horse Power	0.00000	0.00001	-0.28229	0.77842
0.5	**intercept**	-0.01548	0.00636	-2.43319	0.01709
	weight	-0.00001	0.00000	-5.37093	0.00000
	Length	0.00006	0.00003	1.85380	0.04728
	Wheel base	0.00003	0.00007	0.38048	0.70455
	width	0.00009	0.00009	1.02585	0.30791
	Engine Size	0.00013	0.00042	0.31063	0.75685
	Horse Power	0.00000	0.00001	0.20961	0.83448
0.75	**intercept**	-0.00517	0.00772	-0.66995	0.50472
	weight	-0.00001	0.00000	-5.77556	0.00000
	Length	0.00001	0.00004	0.36953	0.01267
	Wheel base	0.00005	0.00009	0.55679	0.57915
	width	0.00006	0.00009	0.67536	0.50130
	Engine Size	0.00062	0.00058	1.07008	0.28765
	Horse Power	0.00000	0.00001	0.25664	0.79808
0.95	**intercept**	0.03252	0.11513	0.28248	0.77827
	weight	-0.00001	0.00001	-0.52314	0.60225
	Length	0.00014	0.00028	0.49358	0.62289
	Wheel base	-0.00037	0.00099	-0.36906	0.71301
	width	-0.00016	0.00103	-0.15139	0.88003
	Engine Size	0.00366	0.00424	0.86169	0.39131
	Horse Power	-0.00004	0.00008	-0.51268	0.60952

Table 3.3: Descriptive Analysis for the Residuals of the Cauchit Quantile Regression Model Computed with Real Life Data

quantiles	Skewness	kurtosis	mean	Median	RMSE	MSE	SD
0.05	4.58190	26.6010	0.00369	0.00250	0.00671	0.00005	0.00564
0.25	5.134375	31.4077	0.00167	0.00046	0.00568	0.00004	0.00546
0.5	5.141661	31.4474	0.00101	0.00000	0.00550	0.00003	0.00544
0.75	5.009144	30.3745	0.00000	-0.0007	0.00537	0.00003	0.00561
0.95	3.716878	21.1170	-0.0041	-0.0049	0.00694	0.00005	0.00607

3.1 Conclusion

The values of the Mean Square Error for location parameter θ in Table 3.1 remain equivariance for $-3\leq \theta \leq 3$. In quantile 0.95 at $\alpha = 3$ in table 3.1 the Mean Square Error is 0.00043 for all values of θ, at $\alpha = 2$ the Mean Square Error is 0.00019 for all values of θ, at $\alpha = 1$ the Mean Square Error is 4.8e-05 for all values of α. This continues in that order for all quantiles at each $\alpha > 0$ for all values of $-3\leq \theta \leq 3$.

We therefore conclude that quantile regression model has the property of equivariance to location parametrization.

3.2 Recommendation

Following our conclusion, we therefore recommend for the researcher to reparameterized quantile regression model when necessary.

References

Alshawarbeh, E, Famoye, F, Lee, C; (2013), Beta-Cauchy distribution: some properties and applications. *J. Stat. Theory Appl. 12(4), 378–391*

Alzaatreh, A, Lee, C, Famoye, F: A & Ghash, I (2016), The generalized Cauchy family of distributions with applications, *Statistical Distributions and Applications, 3:12, DOI 10.1186/s40488-016-0050-3*

Efron, B. (1979), Bootstrap methods: Another look at the jackknife. *Annals of Statistics, 7*, 1-26.

Eugene, N, Lee, C, Famoye, F; (2002), Beta-normal distribution and its applications. *Commun. Stat. Theory Methods 31(4), 497–512*

Hao L. &Naiman,D.Q., (2007). Quantile Regression*; 01-Hao.qxd. 3/13/2007.3.28*

Koenker, R., & Bassett, J., G. (1978). *Regression quantiles. Econometrica, 46, 33-50.*

Nwabueze, J.C, Onyegbuchulem, B.O; Nwakunya M.T& Onyegbuchulem, C.A. (2021), A Cauchy transformation approach to the robustness of quantile regression model to outliers, *Royal statistical society Nigeria local group book proceedings.*

Nwakunya M.T, Onyegbuchulem, B.O & Nwabueze, J.C. (2020), B.Bounded Logistic Quantile Regression an alternative to Logistic Quantile Regression, *Academic Journal of Statistics and Mathematics, Vol 6, No.7, July, 2020*

Nwakuya M.T, Nwabueze, J.C, Onyegbuchulem, B.O & Imo J.C. (2019), Response Variable Transformation for Quantile Regression Model, *International Journal of Scientific & Engineering Research Vol 10, Issue 8, August-2019*

Onyegbuchulem, B.O; Nwakunya M.T, Nwabueze, J.C.& Otu, A.O (2019) Evaluation of the Assumptions of the Linear Regression Model on the Quantile Regression Model, *International Journal of Advancements in Research & Technology, Vol 2, Issue 9, September-2019*

Onyegbuchulem, B.O; Nwakunya M.T, Nwabueze, J.C.& Otu, A.O (2019) Choice of Appropriate Power Transformation of Skewed Distribution for Quantile Regression Model, *American Journal of Applied Mathematics and Statistics, 2019, Vol.7 no3,105-111;*

Stigler, SM; (1989), Letter to the editor: normal orthant probabilities. *Am. Stat*. 43(4), 291