An Efficient Three Step Method For finding the Root Of Non-linear Equation with Accelerated convergence.

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Abstract

We have made an effort to design an accurate numerical strategy to be applied in the vast computing domain of numerical analysis. The purpose of this research is to develop a novel hybrid numerical method for solving a nonlinear equation, That is both quick and computationally cheap, given the demands of today's technological landscape. Sixth-order convergence is demonstrated by combining the classical Newton method, on which this method is largely based, with another two-step third-order iterative process. The effectiveness index for this novel approach is close to 1.4309, and it requires only five evaluations of the functions without a second derivative. The findings are compared to standard practice. The provided technique demonstrates higher performance in terms of computational efficiency, productivity, error estimation, and CPU times. Moreover, its accuracy and performance are tested using a variety of examples from the existing literature.

Keywords: efficient scheme, nonlinear application, nonlinear functions, error estimation, computational cost.

1. Introduction

A non-linear phenomenon occurs not only in the discipline of mathematics but also in various field of science, such as Virial equation in Chemistry, Plank's Radiational law in Physics, Colebrook equation in fluid dynamics. Boltzmann transport equation (BTE) in physics and many more. Therefore, the significance of numerical schemes in the field of science and mathematics cannot be denied. Exact techniques are nearly impossible to drive for solving a non-linear phenomenon. So, the researchers working in applied field, are paying hard to drive a method which depict less computational cost and converges rapidly.

Numerous non-linear models using different techniques such as interpolation technique, quadrature rule, Taylor series, transformation of Signum and tangent technique, decomposition techniques, merging techniques of different orders have been derived for solving these non-linear phenomena for studying in detail, see [1-4]. In recent research (see[5]) cubically convergent alteration of Regulae Falsi method and Newton method have been presented for computing single root of non-linear equation $f(x) = 0$.further there has been another algorithm which uses techniques of difference operator and Taylor series to solve non-linear equation $f(x) = 0$ (see[6]).furthermore, a quartic derivative free algorithm utilizing the forward and finite difference schemes on well-known householder's method [7] presented for solving non-linear equations (see[8]). Mastoie [9] presented third order derivative method for solving non-linear equation and many more work has been in process now a days for details see article [10],[11],[12]. In these recent research, Author had tried his best to develop a method which either be derivative free or have greater convergence and take less computational cost in modeling the complicated engineering problems.

Keeping in view the prime stability features of the non-linear schemes, there are numbers of research has been done to obtain the best converging scheme which take a smaller number of function evaluation per iteration and give greater order of convergence. In [13] a new three step iterative method is suggested, establishing the convergence order five, however, the procedure brings extra evaluation of function in each step. In [14] Mani. S developed a three-step method with five evaluations of functions per iteration but having very complex structure. [15] Noor, M. A Noor presented three families of single, double and triple step schemes which are cubically convergent with best convergence shown by three step method which depicts the better performance than Chun's method [16].

The main purpose of this paper is to construct three step iterative method for modeling non-linear equation. Encouraged by currently proposed research, we developed a new three step iterative method by taking Newton classical method as first step due to its simplicity and merging it in well-known jaiswal J.P method [24]. The developed method is constructed for sixth order convergence with five evaluations of function with no double derivative per iteration.

This paper is composed as follows: Section2 presents some existing materials about modeling non-linear equations. Then, section3 presents the construction of the proposed method. Section4 demonstrates its convergence order. Some theoretical comparison with stability analysis based on existing same order method has been displayed in section5. In the end, section6 depicts the conclusion with crucial finding.

2. Existing materials and methods

In this section, we will have a brief review on some existing famous techniques which are commonly used by many researchers due to their simplicity and accuracy. It is obligatory to use numerical techniques for solving the complex structures of non-linear equation where the exact technique failed to yield result. However, when it comes to solve non-linear equations through numerical technique, one needs to investigate and check the technique under some favorable circumstances. In this regard, one of the oldest and simplest techniques is Newton Raphson technique, given as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(1)

Symbolized as NR in this paper. Its order of convergence is two with just two evaluations of function per iteration.

In [18] three step iterative method have been presented using Newton Raphson method as first step. This method possesses the eighth order of convergence with five evaluation of function and is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{1-(f(y_n))^2}, \\ x_{n+1} &= z_n - (y_n - z_n) - \frac{f(z_n)}{f(y_n)-2f(z_n)}, \end{aligned}\right\}$$

(2)

Where $h(y_n)$ can be calculated as:

$$h(y_n) = \frac{2f(y_n)}{f'(y_n) - \sqrt{\left(f'(y_n)\right)^2 - 4(f(y_n)^2}}, \tag{3}$$

In [19] With the given approximation of x_0 cubically convergent Halley's method is given by:

$$\left. x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}, \right\} \tag{4}$$

It is the single step method with third order of convergence symbolically written as Halley3.

By merging NR method and Halley3 in [20] a new modified Halley's method was proposed by Noor et al. in 2007 using finite difference approximation of second derivative. This Method possesses two steps with just four evaluation of function (two function and 2 its derivative).

$$\left.\begin{aligned} y_n &= x_p - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{2f(x_n)f(y_n)f'(y_n)}{2f(x_n)f'^2(y_n) - 2f'^2(x_n)f(y_n) + f'(x_n)f(y_n)f'(y_n)}, \end{aligned}\right\} \tag{5}$$

By approximating the strategy of Homotopy perturbation and Javidi's method [21] a fourth order root-finding algorithm was constructed in 2022, with just three evaluation of functions per iteration. See [22] for details. Method is shown below

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(y_n)\eta(x_n, y_n)}{2[f'^3(y_n) - f(y_n)f'(y_n)\eta(x_n, y_n)]}, \end{aligned}\right\} \tag{6}$$

Further, three step optimal algorithm having convergence order six with just four function evaluations per step were also presented in [22]. Algorithm is given below

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0,1,2, \dots, \\ z_n &= y_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(y_n)\eta(x_n, y_n)}{2[f'^3(y_n) - f(y_n)f'(y_n)\eta(x_n, y_n)]}, \\ x_{n+1} &= z_n - G\frac{f(z_n)}{f'(x_n)}, \end{aligned}\right\} \tag{7}$$

Currently, the authors in [23] combined the Bawzir and Ostrowiski method and finally adding Newton's method as last step, three step method has been proposed with eighth convergence order. Method is shown below

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$z_n = x_n + (\beta - 1)\left(\frac{f(x_n)[f(x_n) - f(y_n)]}{f'((x_n)[f(x_n) - 2f(y_n)])}\right) - \beta\left(\frac{f(x_n)}{f'^{(x_n)}} + \frac{f(y_n)\left(f(x_n)^3 + f(y_n)^2 f(x_n) + \frac{1}{2f(y_n)^3}\right)(f(x_n) + f(y_n))^2}{f'^{(x_n)} f(x_n)^5}\right),$$

$$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$

Noor et al. [31] have proposed three step methods of sixth order by decomposition techniques. We denoted it as M6. The schemes described in the following equation:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= -\frac{(y_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)} \\ x_{n+1} &= y_n - \frac{(y_n + z_n - x_n)^2}{2}\frac{f''(x_n)}{f'(x_n)} \end{aligned}\right\} \tag{9}$$

T lofti et al. [32] developed three step iterative method with four evaluation of the function but it requires evaluation of two vector and Jacobian matrix respectively per iteration. The developed scheme is presented as follow:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \\ x_{n+1} &= z_n - \left(\frac{7}{2} - 4\frac{f'(y_n)}{f'(x_n)} + \frac{3}{2}\left(\frac{f'(y_n)}{f'(x_n)}\right)^2\right)\frac{f(z_n)}{f'(x_n)} \end{aligned}\right\} \textit{where } n = 0,1,2,3,\ldots \tag{10}$$

3. Construction of new iterative method:

Getting motivation by current findings and presented literature, we concluded that many authors have proposed different types of schemes using Newton's method as first, intermediate or as last step due to its simplicity in order to improve the convergence order. Similarly, we took Newton Raphson method as our starting step and merging it with well-known Jaiswal J.P method [24]. A new construction has been shown below

As we know that newton method is given as:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{11}$$

And third order iterative method by Jaiswal J.P which is to be merged is given below:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)} \end{aligned}\right\} \textit{where } n = 0,1,2,3,\ldots., \tag{12}$$

The purpose was to construct a new efficient scheme with a higher order convergence. Whereas, the number of function evaluation per step was overcome as much as possible. Similar technique was constructed in [25] and [26] and further used by many authors. Therefore, blending (11) with (12), we obtain:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= y_n - \frac{2f(y_n)}{f'(y_n)+f'(z_n)}, \end{aligned}\right\} \quad where\ n = 0,1,2,3,\ldots \tag{13}$$

The presented scheme will be referred as S6 during the numerical and theoretical analysis in rest of the paper, as well as in convergence, simulation and in compared examples discussed in the end.

4. Convergence order for $f(x) = 0$

Theorem 1. *Assume that* β *is the root of a differentiable function* $f : \Re \subset R \to R$ for an open interval So *that, the three-step iterative technique S6, i.e.,* $Eq^n.(13)$, *then exhibits six-order convergence, and the resulting error term is,*

$$e_{k+1} = \frac{f'''[\beta]^5}{32f'[\beta]^5} e_k^6 + O(e_k^7)$$

where $e_k = x_k - \beta$

Proof. Suppose β is the root of $f(x_k)$, x_k *be* n^{th} nearly to the root by S6 and $e_k = x_k - \beta$ be the error term after n^{th} iteration. Utilizing Taylor's series for $f(x_k)$ about β, we have

$$f(x_k) = f'[\beta]e_k + \frac{1}{2}f''[\beta]e_k^2 + O[e_k]^3 \tag{14}$$

By Taylor's series for $\frac{1}{f'(x_k)}$ about β, we obtained

$$\frac{1}{f'(x_k)} = \frac{1}{f'[\beta]} - \frac{f''[\beta]e_k}{f'[\beta]^2} + O[e_k]^2 \tag{15}$$

Multiplying (14) and (15) we get

$$\frac{f(x_k)}{f'(x_k)} = e_k - \frac{e_k^2 f''[\beta]}{2f'[\beta]} - \frac{e_k^3 f''[\beta]^2}{2f'[\beta]^2} + O[e_k]^4 \tag{16}$$

Using (16) in first step of (13)

$$\sigma_k = \frac{e_k^2 f''[\beta]}{2f'[\beta]} + \frac{e_k^3 f''[\beta]^2}{2f'[\beta]^2} + O[e_k]^4 \tag{17}$$

By Taylor's series for $f(y_k)$ about β, we obtained

$$f(y_k) = f'[\beta]\sigma_k + \frac{1}{2}f''[\beta]\sigma_k{}^2 + O[\sigma_k]^3 \tag{18}$$

By Taylor's series for $\frac{1}{f'(y_k)}$ about β, we obtained

$$\frac{1}{f'(y_k)} = \frac{1}{f'[\beta]} - \frac{f''[\beta]\sigma_k}{f'[\beta]^2} + O[\sigma_k]^2 \tag{19}$$

Multiplying (18) and (19) we get

$$\frac{f(y_k)}{f'(y_k)} = \sigma_k - \frac{\sigma_k{}^2 f''[\beta]}{2f'[\beta]} + \frac{\sigma_k{}^3 f''[\beta]^2}{2f'[\beta]^2} + \frac{\sigma_k{}^4 f''[\beta]^3}{2f'[\beta]^3} + O[\sigma_k]^5 \tag{20}$$

Using (20) in Second step of (13)

$$\mathcal{E}_k = \frac{\sigma_k{}^2 f''[\beta]}{2f'[\beta]} - \frac{\sigma_k{}^3 f''[\beta]^2}{2f'[\beta]^2} - \frac{\sigma_k{}^4 f''[\beta]^3}{2f'[\beta]^3} + O[\sigma_k]^5 \tag{21}$$

By Taylor's series for $f(z_k)$ about β, we obtained

$$f(z_k) = f'[\beta]\mathcal{E}_k + \frac{1}{2}f''[\beta]\mathcal{E}_k{}^2 + O[\mathcal{E}_k]^3 \tag{22}$$

Differentiating (22), we have

$$f'(z_k) = f'[\beta] + f''[\beta]\mathcal{E}_k + O[\mathcal{E}_k]^2 \tag{23}$$

Differentiating (18), we have

$$f'(y_k) = f'[\beta] + f''[\beta]\sigma_k + O[\sigma_k]^2 \tag{24}$$

Now for $2f(y_n)/(f'(y_k) + f'(z_k))$ we have

$$\sigma_k - \frac{\sigma_k \mathcal{E}_k f''[\beta]}{2f'[\beta]} + \frac{\sigma_k^2 \mathcal{E}_k f''[\beta]^2}{4f'[\beta]^2} + \frac{\sigma_k \mathcal{E}_k^2 f''[\beta]^2}{4f'[\beta]^2} \tag{25}$$

Using (25) in third step of (13) we get

$$e_{n+1} = \frac{\sigma_k \mathcal{E}_k f''[\beta]}{2f'[\beta]} - \frac{\sigma_k^2 \mathcal{E}_k f''[\beta]^2}{4f'[\beta]^2} - \frac{\sigma_k \mathcal{E}_k^2 f''[\beta]^2}{4f'[\beta]^2} \tag{26}$$

By substituting the value of σ_k and $\mathcal{E}_k$, finally we get

$$e_{k+1} = \frac{f''[\beta]^5}{32f'[\beta]^5} e_k^6 + O(e_k^7) \tag{27}$$

That displays the proposed technique, i.e., S6 is sixth order convergent.

5. Numerical Evaluation and Applications:

To present the effectiveness and applicability of our newly constructed iterative scheme, we demonstrate some real life engineering problems in this section. The newly constructed iterative scheme is compared to the following existing sixth order three steps iterative algorithms:

5.1. Noor et al. [27] have proposed a family of some second derivative-free three step methods by merging techniques denoted as S1 and S2. The schemes are described in the following equations:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= y_n - \frac{f(z_n)}{f'(y_n) + f(y_n)} \end{aligned}\right\} \tag{28}$$

And

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= y_n - \frac{f(y_n)f(z_n)}{f'(y_n)f(y_n) - 2f(z_n)} \end{aligned}\right\} \tag{29}$$

Respectively, Which are the sixth order convergent methods for finding the simple root of non-linear equation $f(x) = 0$.

5.2 Abro et al. in 2019 [28] has proposed the efficient non-linear solver iterative method, cited by many authors and researchers in their papers. The iterative scheme is demonstrated as follow:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} \\ x_{n+1} &= y_n - \frac{f(y_n) + f(z_n)}{f'(y_n)} \end{aligned}\right\} \quad n = 0,1,2,3, \ldots. \tag{30}$$

Which is the sixth order method for evaluating the root of non-linear equations.

5.3. Numerical abbreviations:

A few sixth order numerical schemes were briefly introducing in section 5 as 5.1 and 5.2 and a well-known Newton Raphson method in section 2, whose numerical comparison can be made with newly proposed iterative scheme. The following abbreviation will be used hereupon.

S6 : Proposed numerical scheme with sixth order convergence in eq (13)
SA1: A numerical scheme with sixth order convergence in eq (28)
SA 2 : A numerical scheme with sixth order convergence in eq (29)
AE : Efficient iterative scheme with sixth order convergence in eq (30)
NR : A well-known Newton Raphson method (1)
NFE : Number of function evaluation
NI : Number of iteration
μ : Exact root of non-linear equation
x_0 : Initial guess
T : CPU time
η : error
COC : Computational cost.

5.4. Numerical Implementation:

This section is constructed in order to demonstrate the performance of the developed efficient iterative scheme. Firstly, we mention the exact zeroes (root) of each non-linear equation in front of the non-linear function correct up to fifteenth decimal place. Secondly, we manipulate the developed scheme to solve the non-linear equation using MATLAB version R2018a (9.4.0.813654)64-bit (win64). Finally, to show the efficiency of the developed scheme, we compared the scheme with the well-known Newton method, method of the Noor et al [27] and Abro et al [28]. Tolerance used in this section is $|x^{n+1} - x^n| < 10^{-15}$

And maximum number of iteration to be manipulate is n = 50. Where μ is the real root.

$f_1(x) = x^3 + 4x^2 - 10$ $\mu = 1.365230013414097$

$f_2(x) = cosx - x$ $\mu = 0.739085133215160$

$f_3(x) = x^2 - e^x - 3x + 2$ $\mu = 0.257530285438961$

$f_4(x) = cosx^2 - \frac{x}{5}$ $\mu = 1.085982678007472$

$f_5(x) = e^x - 1$ $\mu = 0.000000000000000$

$f_6(x) = lnx + \sqrt{x} - 5$ $\mu = 8.309432694423157$

$f_7(x) = x^5 - x - 10000$ $\mu = 2.499203570440139$

$f_8(x) = x - tanx$ $\mu = 0.0000181998985189$

$f_9(x) = \sin^2 x - x^2 + 1$ $\mu = 1.404491648215341$

$f_{10}(x) = tanhx - tanx$ $\mu = 7.068582745638732$

$f_{11}(x) = (1 + cosx)(e^x - 2)$ $\mu = 0.693147180559945$

$f_{12}(x) = 10xe^{-x^2} - 1$ $\mu = 1.679630610284499$

Results of the above mentioned non-linear equation are demonstrating in Table 6, Table 7 and in Table 8 that developed scheme with efficient property, namely, the error approximation of the developed scheme for the root of non-linear equations is almost zero in all examples, further, taking lesser CPU time and consuming less iteration than the newton method and rest of sixth order methods in some cases.

Table 6. Comparison of number of iteration and error estimation between Newton's method and developed scheme

Function $f_n(x)$ n=1,2,3,4	Initial guess (x_0)	Newton method NR	New Proposed S6
$x^3 + 4x^2 - 10$	1.0	NI = 6 η = 2.12 x 10^{-11}	NI = 4 η = 0
$cosx - x$	1.7	NI =5 η = 2.22 x 10^{-16}	NI = 3 η = 0
$x^2 - e^x - 3x + 2$	-0.5	NI = 5 η = 4.34 x 10^{-13}	NI = 3 η = 0
$cosx^2 - \frac{x}{5}$	0.5	NI = 5 η = 0	NI = 3 η = 0

Table 7. Comparison of number of iteration and error estimation between method of the Noor et al [27] and developed scheme

Function $f_n(x)$ (n=5,6,7,8)	Initial guess (x_0)	Noor et al [27] SA1	New Proposed S6
$e^x - 1$	10	NI = 50 η = Failed	NI = 7 η = 0
$lnx + \sqrt{x} - 5$	11.9	NI =4 η = 0	NI = 3 η = 0
$x^5 - x - 10000$	9.8	NI = 50 η = 4.9012	NI = 6 η = 0
$x - tanx$	0.10	NI = 18 η = 1.110×10^{-16}	NI = 17 η = 0

Table 8. Comparison of time taken and error estimation between Abro et al [28] and developed scheme

Function $f_n(x)$ (n=9,10,11,12)	Initial guess (x_0)	Abro et al AE	New Proposed S6
$\sin^2 x - x^2 + 1$	1.3	T = 0.071376 η = 2.22 x 10^{-16}	T = 0.000022 η = 0
$tanhx - tanx$	7.6	T =0.010107 η = 0	T= 0.000030 η = 0
$(1 + cosx)(e^x - 2)$	0.5	T = 0.011731 η = 4.34 x 10^{-13}	T = 0.000021 η = 0
$10xe^{-x^2} - 1$	1	T = 0.012348 η = 2.2204x10^{-16}	T = 0.0006024 η = 0

5.5. Numerical Examples:

Some of the real life examples have been chosen from recent articles. It can be seen that the devised scheme S6 is working in the same manner to that of the other schemes, but in few cases S6 is showing more convergence, taking less CPU time and show minimizing error fast. We can see that S6 reaches the approximate root within the specified tolerance, meanwhile using less function evaluation in some case and computational cost. Therefore, the mathematical results demonstrate that the devised scheme is efficient in comparison with their theoretical results.

Example 1: Plank's radiation law

Plank's radiation law is use to find energy-density of an Iso-thermal black body. The following non-linear expression is used for it:

$$f(\lambda) = \frac{8\pi ch\lambda^{-5}}{\exp\left(\frac{ch}{\lambda kT}\right) - 1}, \qquad (31)$$

Here, λ is the wavelength of the radiation, the absolute temperature is given by T is, k, h is the Boltzmann's constant and Plank's constant respectively, and c is the speed of light. By finding the derivative of above equation we get: $f'(\lambda) = \left[\frac{8\pi ch\,\lambda^{-6}}{\exp\left(\frac{ch}{\lambda kT}\right)-1}\right]\left[\frac{\left(\frac{ch}{\lambda kT}\right)\exp\left(\frac{ch}{\lambda kT}\right)}{\exp\left(\frac{ch}{\lambda kT}\right)-1} - 5\right] = 0$

$$\frac{\left(\frac{ch}{\lambda kT}\right)\exp\left(\frac{ch}{\lambda kT}\right)}{\exp\left(\frac{ch}{\lambda kT}\right)-1} = 5$$

Using substitution $x = \left(\frac{ch}{\lambda kT}\right)$, in above equation we have the non-linear equation as

$$f(x) = exp(-x) - 1 + \frac{x}{5} = 0$$

Taking the initial guess $x_0 = 8$ for the computer program.

Table 1. Plank's radiation law

Method	*NI*	*root* (μ)	*Error* (η)	*NFE*	*COC*	*time* (*T*)
NR	5	4.965114231744276	$8.881784197001 \times 10^{-16}$	2	10	0.0031×10^{-1}
AE	3	4.965114231744276	$8.881784197001 \times 10^{-16}$	3	9	0.0014×10^{-2}
SA1	3	4.965114231744276	$8.881784197001 \times 10^{-16}$	6	18	0.0511×10^{-4}
SA2	3	4.965114231744276	$8.881784197001 \times 10^{-17}$	6	18	0.0014×10^{-3}
NP	3	4.965114231744276	0	6	18	0.0006×10^{-3}

Example 2: Ideal and Non − Ideal Gas Law .

The ideal gas law is illustrated as

$$f(v) = \left(P + \frac{a}{v^2}\right)(v - b) - RT. \qquad (32)$$

Considering the values of all parameters as $v = 3$ is an initial guess, $R = 0.082054\ L\,atm/(mol\ K)$, for carbon dioxide $a = 3.592$, $b = 0.04267$, $temperature\ T = 300Kelvin$, $and\ atmospherric\ pressure\ p = 1\ atm$. Results are shown in table 2.

$Table\ 2. Ideal\ and\ Non-Ideal\ Gas\ Law$

$Method$	NI	$root(\mu)$	$Error(\eta)$	NFE	COC	$time(T)$
NR	5	0.24512588128441	0	2	10	0.0021×10^{-4}
AE	3	0.24512588128441	0	3	15	0.0012×10^{-4}
$SA1$	3	0.24512588128441	0	6	18	0.0009×10^{-4}
$SA2$	–	$Failed$	–	6	–	–
NP	4	0.24512588128441	0	6	24	0.0003×10^{-5}

$Example\ 3: Blood\ Rheology\ Model$

The blood rheology is the characteristic of blood concerned with fluid flow. Flow of Blood fluid is non-Newtonian in nature which is also known as Caisson. To investigate the plug flow of non-Newtonian fluids, the following nonlinear equation will be used:

$$H = 1 - \frac{16}{7}\sqrt{x} + \frac{4}{3}x - \frac{1}{21}x^4 \qquad (33)$$

Where H calculates the reduction in flow-rate. Substituting $H = 0.40$ in (33) we obtained:

$$f_{11}(x) = \frac{1}{441}x^8 - \frac{8}{63}x^5 - 0.05714285714x^4 + \frac{16}{9}x^2 - 3.624489796x + 0.3$$

Let the initial guess for calculating the value of the above mentioned equation Is $x = 0$.

Table 3. Blood Rheology Model

Method	*NI*	*root* (μ)	*Error* (η)	*NFE*	*COC*	*time* (*T*)
NR	5	0.0864332945918103	0	2	10	0.0021×10^{-3}
AE	3	0.0864332945918103	0	3	9	0.0012×10^{-3}
SA1	3	0.0864332945918103	0	6	18	0.0009×10^{-3}
SA2	–	*Failed*	–	6	–	–
NP		0.0864332945918103	$1.387778780781446e-17$	6	18	0.0001×10^{-3}

Example 4. Adiabatic Flame temperature equation:
Adiabatic flame temperature equation has the following non-linear form:

$$f_4(x) = \Delta H - b_1(298 - x) - \frac{b_2}{2}(298^2 - x^2) - \frac{b_3}{2}(298^3 - x^3)$$

Whereas we have taken $\Delta H = -57798, b_1 = 7.256, b_2 = 0.002298, b_3 = 0.00000283$ for details (see [32],[33]) the above mentioned function is third order polynomial function whose derivative when found by putting the values of unknown was: $f_4'(x) = 2.83 \times 10^{-6}x^2 + 0.002298x + 7.256$. it must has three unique roots. Starting with the initial guess $x_0 = 2000$ the numerical results are displayed in table 4.

Table 4.Adiabatic Flame Temperature Equation:

Method	*NI*	*root*(μ)	*Error*(η)	*NFE*	*COC*	*time*(*T*)
NR	6	3062.356864618161	4.5474×10^{-13}	2	12	0.143631
AE	50	*failed*		3	–	0.015850
*SA*1	50	*failed*		6	–	0.009940
*SA*2	50	*failed*		6	–	0.013496
NP	3	3062.356864618161	0	6	18	0.000017

Example 5. Model of beam designing:

Considering model of beam positioning from [29] whose nonlinear function is given as:

$f_4(x) = x^4 + 4x^3 - 24x^2 + 16x + 16$

In the light of algebra's fundamental theorem, the zeros (roots) of the above mentioned polynomial must be four as it is a fourth degree polynomial, choosing initial guess as $x_0 = -0.75$ and applying on devised scheme. We can see the approximate results in table 5.

Table 5. model of beam designing:

Method	*NI*	*root* (μ)	*Error* (η)	*NFE*	*COC*	*time* (*T*)
NR	5	−0.535898384862245	3.9968×10^{-15}	2	10	0.010380
AE	3	−0.535898384862245	0	3	9	0.008782
SA1	3	−0.535898384862245	0	6	18	0.008782
SA2	50	*Failed*	−	−	−	−
NP	3	−0.535898384862245	0	6	18	0.0003×10^{-3}

6. Conclusion:

In current study, we have analyzed and established efficient root finding iterative scheme to solve non-linear models by merging second order Newton method and third order Jasiwal J.P method. The order of convergence and asymptotic error term was verified via Taylor expansion with approximate efficiency index was calculated as 1.4302. The dynamical behavior of the presented scheme demonstrated the greater convergence with less CPU time in comparison with the other mentioned schemes. To manifest the applicability and solidity of the developed scheme, some real life models, applications and well known non-linear equations have been tasted and verified, whose numerical results depicts that the developed scheme is more reliable in term of absolute error, CPU time, consumption of iteration in univariate cases and shows fast convergence towards the exact root.

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