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Some Properties on Skew Polynomials Ring $\mathbb{R}[x, \theta, \Delta_\theta]$

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Abstract

Derivatives in polynomials were first defined by Ore. Since then, it has been used to construct many things, such as skew polynomials code with derivatives, constructed by Suprijanto and Tang in 2021. In that paper, the skew polynomial ring is defined explicitly for $\mathbb{Z}_4 + v\mathbb{Z}_4$, where $v^2 = v$, and automorphism $\theta(a + bv) = a + b - bv$, and the θ-derivative is $\Delta_\theta(x) = \theta(x) - x$. This paper finds some of the properties of the θ-derivative Δ_θ for any ring and automorphism θ.

Keywords: ring, automorphism, derivatives, skew-polynomials ring, construction

1. Introduction

In 1993, for any ring A, O. Ore defined a structure of a ring on the set $\{\sum_{i=0}^{n} a_i x_i \mid n \in N, a_i \in A\}$ with ordinary addition of polynomials and multiplication is defined by $xa = \theta(a)x + \Delta_\theta(a)$. The skew polynomial ring over A with a θ-derivative is denoted as $A[x;\ \theta, \Delta_\theta]$ where θ is an automorphism in A and Δ_θ depends on θ.

In 2018, Irwansyah et al. studied the structure of θ −cyclic codes over the ring B_k, including its connection to quasi-θ −cyclic codes over finite field $\mathbb{F}_{p^r}$ and skew polynomial rings over B_k. They also characterized Euclidean self-dual θ −cyclic codes over the rings. Suprijanto and Tang also show that a skew polynomial with θ-derivative could construct some linear codes. It has been demonstrated that the skew polynomial ring is significant in building a code. To provide a generalization, the main focus of this paper is to see some of the properties of θ-derivatives.

2. Preliminaries

In this section, we will give some definitions and theorems about automorphism and derivatives on the finite ring R;also, in this section, we will describe the skew polynomial ring over R.

Definition 2.1(Wahyuni, et al., 2021). Let R, S be two rings, a map $\theta: R \to S$ is a ring homomorphism if $\theta(x + y) = \theta(x) + \theta(y)$ and $\theta(xy) = \theta(x).\theta(y)$, for all $x, y \in R$. Also, if θ is a bijection from $R \to S$, it is a ring isomorphism.

Note that an isomorphism from R to itself is called an automorphism.

Definition 2.2 (Boucher and Ulmer, 2014). Let R be a finite ring and $\theta : R \to R$ be an automorphism on R. A map $\Delta_\theta: R \to R$ is called a θ-derivative on R if the following conditions are satisfied:

- $\Delta_\theta(x + y) = \Delta_\theta(x) + \Delta_\theta(y)$
- $\Delta_\theta(xy) = \Delta_\theta(x)y + \theta(x)y$

for all $x, y \in R$.

A ring that is equipped with an automorphism, such as in Definition 2.1, and a θ-derivative such as in Definition 2.2, is called a skew-polynomial ring $R[x, \theta;\ \Delta_\theta]$.

Definition 2.3 (Boucher and Ulmer, 2014). Let R be a finite ring with an automorphism θ and a θ-derivative $\Delta\theta$. The skew-polynomial ring $R[x, \theta;\ \Delta\theta]$ is the set of all polynomials over R with ordinary addition of polynomials and multiplication that is defined by $xa = \theta(a)x + \Delta\theta(a)$ for any $a \in R$.

3. Result

In this section, we will give some results about skew-polynomials rings. We will show the result in two parts: the first part presents the general result for $\Delta\theta(x) = k(\theta(x) - x)$, where $k \in R$ a unit, and the second part when we specify

$$R = \mathbb{Z}_4 + \mathbb{Z}_4 u_1 + \ldots + \mathbb{Z}_4 u_{2m}$$

1.1 General Formula for Skew Polynomials Ring with Derivative $\Delta_\theta(x) = k(\theta(x) - x)$.

The first result is the relation between the derivative Δθ and the automorphism, i.e., θ. The result is contained in this following lemma.

Lemma 3.1. Let $\Delta\theta = k(\theta(x) - x)$ with $k \in R$ is a unit. Then the following statements hold:

- $\Delta_\theta (x) = 0 \Longleftrightarrow \theta(x) = x.$
- $If\ k\ is\ a\ fixed\ element, i.e\ \theta(k) = k, then\ \Delta_\theta^n = k^n \sum_{i=1}^{n} \binom{n}{i} \theta^{n-i}(-1)^i$
- $\Delta_\theta(\theta(x)) = \theta(\Delta_\theta(x)).$

Proof: For the first part, it is clear. For the second part using induction, consider for $n = 2$ and $n = 3$.

$$\Delta_\theta \Delta_\theta (x) = \Delta_\theta(k(\theta(x) - x))$$
$$= k\left(\theta(k(\theta(x) - x)) - k(\theta(x) - x)\right)$$
$$= k^2(\theta^2(x) - 2\theta(x) + x)$$

and

$$\Delta_\theta \Delta_\theta \Delta_\theta (x) = \Delta_\theta(\Delta_\theta \Delta_\theta (x))$$
$$= \Delta_\theta(\theta^2(x) - 2\theta(x) + x)$$
$$= k\left(\theta((\theta^2(x) - 2\theta(x) + x)) - (\theta^2(x) - 2\theta(x) + x)\right)$$
$$= k^3(\theta^3 - 2\theta^2(x) + \theta(x) - \theta^2(x) + 2\theta(x) + x)$$
$$= k^3(\theta^3(x) - 3\theta^2(x) + 3\theta(x) - x).$$

Assuming the statement is true for r – 1, then the induction hypothesis is

$\Delta_\theta^{r-1} = k^{r-1} \sum_{i-0}^{r-1} \binom{r-1}{i} (\theta^{r-i}(a)(-1)^i$. We will verify the statement is true for $n = r$.

Consider:

$$\Delta_\theta^r (a) = \Delta_\theta\left(\Delta_\theta^{r-1} (a)\right)$$

$$= \Delta_\theta\left(k^{r-1}\sum_{i-0}^{r-1}\binom{r-1}{i}(\theta^{r-1-i}(a)(-1)^i\right)$$

$$= k(\theta(k^{r-1}\sum_{i-0}^{r-1}\binom{r-1}{i}(\theta^{r-1-i}(a)(-1)^i) - k^{r-1}\sum_{i-0}^{r-1}\binom{r-1}{i}(\theta^{r-i}(a)(-1)^i)$$

$$= k^r \sum_{i-0}^{r} \binom{r}{i} (\theta^{r-i}(a)(-1)^i$$

Next, we will show that $\Delta_\theta \theta = \theta\Delta_\theta$.

We see that

$$\Delta_\theta \theta(x) = k\Big(\theta\big(\theta(x)\big) - \theta(x)\Big)$$

$$= k(\theta(\theta(x) - x))$$

$$= \theta(k(\theta(x) - x))$$

$$= \theta\Delta_\theta (x). \blacksquare$$

In Definition (2.3), we know that $xa = \theta(a)x + \Delta_\theta(a)$. Using Definition (2.3) and Lemma (3.1), we get the description of $x^n a$ as follows.

Lemma 3.2. For any $a \in R$,

$$x^n a = \sum_{i=0}^{n} \binom{n}{i} \Delta_\theta^{n-i} \left(\theta^i(a)\right) x^i$$

Proof. Using mathematical induction, we will prove that $x^n a = \sum_{i=0}^{n} \binom{n}{i} \Delta_\theta^{n-i} \left(\theta^i(a)\right) x^i$.

For $n = 2$,

$$x^2 a = x(xa)$$

$$= x\big(\theta(a)x + \Delta\theta (a)\big)$$

$$= x\theta(a)x + x\Delta\theta (a)$$

$$= \Big(\theta\big(\theta(a)\big)x + \Delta\theta\big(\theta(a)\big)\Big)x + \theta\big(\Delta\theta (a)\big)x + \Delta\theta\big(\Delta\theta (a)\big)$$

$$= \theta^2(a)x^2 + \Delta\theta\big(\theta(a)\big)x + \theta\big(\Delta\theta (a)\big)x + \Delta\theta^2(a)$$

$$=\sum_{i=0}^{2}\binom{2}{i}\Delta_\theta^{2-i}(\theta i(a))x^i$$

Assume that the statement is true for $n = r - 1$, and verify the statement is

true for $n = r$. Consider that:

$$x^r a = x(x^{r-1}a)$$

$$= x\left(\sum_{i=0}^{r-1}\binom{r-1}{i}\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)x^i\right)$$

$$=\sum_{i=0}^{r-1}\binom{r-1}{i}\left(x\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)x^i\right)$$

$$=\sum_{i=0}^{r-1}\binom{r-1}{i}\left(\theta\left(\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)x^i\right)x+\Delta_\theta\left(\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)x^i\right)\right)$$

$$=\sum_{i=0}^{r-1}\binom{r-1}{i}\left(\theta\left(\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)x^{i+1}\right)+\left(\Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i\right)\right)$$

$$=\binom{r-1}{r-1}\Delta_\theta^0(\theta^r(a))x^r+\sum_{i=1}^{r-2}\binom{r-1}{i}\theta\left(\Delta_\theta^{r-1-i}\left(\theta^i(a)\right)\right)x^{i+1}$$

$$+\binom{r-1}{0}\Delta_\theta^r(\theta^0(a))+\sum_{i=1}^{r-1}\binom{r-1}{i}\Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i$$

$$=\theta^r(a)x^r+\sum_{i=1}^{r-1}\binom{r-1}{i-1}\theta\left(\Delta_\theta^{r-i}\left(\theta^{i-1}(a)\right)\right)x^i+\Delta_\theta^r(a)$$

$$+\sum_{i=1}^{r-1}\binom{r-1}{i}\Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i$$

$$= \theta^r(a)x^r + \sum_{i=1}^{r-1} \binom{r-1}{i-1} \Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i$$

$$+ \sum_{i=1}^{r-1} \binom{r-1}{i} \Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i + \Delta_\theta^r(a)$$

$$= \theta^r(a)x^r + \sum_{i=1}^{r-1} \left(\binom{r-1}{i-1} + \binom{r-1}{i}\right) \Delta_\theta^{r-i}\left(\theta^i(a)\right)x^i + \Delta_\theta^r(a)$$

$$= \sum_{i=0}^{r} \binom{r}{i} \Delta\left(\theta^i(a)\right)x^i$$

■

1.2 Some Properties of Skew Polynomials Ring with Derivatives for $R = \mathbb{Z}_4 + \mathbb{Z}_4 u_1 + \ldots + \mathbb{Z}_4 u_{2m}$ and $\Delta_\theta(x) = k(\theta(x) - x)$

In this section, we show an example that certain conditions in the previous section don't hold. We use

$$R = \mathbb{Z}_4 + \mathbb{Z}_4 u_1 + \cdots + \mathbb{Z}_4 u_{2m} = \frac{\mathbb{Z}_4[u_1, u_2, \ldots, u_{2m}]}{\langle u_1^2, -u_1, \ldots, u_i^2 - u_i, \ldots, u_{2m}^2 - u_{2m}, u_i u_j - u_j u_i \rangle}.$$

Lemma 3.3. Any element $a = a_0 + a_1 u_1 + \cdots + a_{2m} u_{2m} \in R$ is a unit if and only if a_0 and $a_0 + a_i$ are units in $\mathbb{Z}_4$, for all $i \in \{1, 2, \ldots, 2m\}$.

Proof. $\Longrightarrow$ Take any $a = a_0 + a_1 u_1 + \cdots + a_{2m} u_{2m}$ and $b = b_0 + b_1 u_1 + \cdots + b_{2m} u_{2m}$ such that

$ab = 1$

$(a_0 + a_1 u_1 + \cdots + a_{2m} u_{2m})(b_0 + b_1 u_1 + \cdots + b_{2m} u_{2m}) = 1$

$(a_0 b_0 + a_0 b_1 u_1 + a_1 b_1 u_1 + \cdots + a_0 b_{2m} u_{2m} + a_1 b_0 u_1 + a_2 b_0 u_2 + a_2 b_2 u_2 + \cdots + a_{2m} b_0 u_{2m} + a_{2m} b_{2m} u_{2m}) = 1$

$a_0 b_0 + u_1(a_0 b_1 + a_1 b_0 + a_1 b_1) + \cdots + u_{2m}(a_0 b_{2m} + a_{2m} b_0 + a_{2m} b_{2m}) = 1.$

Consider that

$a_0 b_i + a_i b_0 + a_i b_i = 0$

$a_0 b_i + a_i b_0 + a_i b_i + a_0 b_0 = 1$

$(a_0 + a_i)(b_0 + b_i) = 1$

So $a_0 + a_i$ and $b_0 + b_i$ are both units.

$\Longleftarrow$ Observe that, if $a_0, a_0 + a_1, a_0 + a_2, \ldots, a_0 + a_{2m}$ are units in $\mathbb{Z}_4$, then there exist $b_0, b_1, \ldots, b_{2m} \in \mathbb{Z}_4$ such that $a_0 b_0 = 1, (a_0 + a_1)b_1 = 1, \ldots, (a_0 + a_{2m})b_{2m} = 1$.

Now we will prove that for $a = (a_0 + a_1 u_1 + \cdots + a_{2m} u_{2m}) \in R$ there exists $c = (c_0 + c_1 u_1 + \cdots + c_{2m} u_{2m}) \in R$ such that $ac = 1$.

Choose $c_0 = b_0$ such that $a_0c_0 = 1$.

We have,

$a_0c_i + a_ic_0 + a_ic_i = 0$

$a_0c + a_ic_0 + a_ic_i + a_0c_0 = 1$

$(a_0 + a_i)(c_0 + c_i) = 1 = (a_0 + a_1)b_1$.

And we choose

$c_0 + c_1 = b_1$

$c_1 = b_1 - c_0$

$= b_1 - b_0$

Similarly $c_i = b_i - b_0$, for all $i \in \{1, 2, \cdots, 2m\}$. So there exists $c = b_0 + (b_1 - b_0)u_1 + \cdots + (b_{2m} - b_0)u_{2m}$ such that $ac = 1$.

■

Lemma 3.4. Let $\theta : R \to R$ with $\theta(a_0 + a_1u_1 + a_2u_2 + \cdots + a_ku_k \ldots + a_{2m}u_{2m}) = a_0 + a_k - (a_1u_1 + a_2u_2 + \cdots + a_{2m}u_{2m})$. Then θ is an automorphism with order 2, that is $\theta^2(a) = a$ for all $a \in R$.

Proof:

Take any $a, b \in R$ $where\ a = a_0 + a_1u_1 + a_2u_2 + \cdots + a_ku_k + \cdots + a_{2m}u_{2m}$ and $b = b_0 + b_1u_1 + b_2u_2 + \cdots + b_ku_k + \cdots + b_{2m}u_{2m}$.

Consider that

$\theta(a + b) = \theta(a_0 + b_0 + (a_1b_1)u_1 + (a_2 + b_2)u_2 + \cdots + (a_k + b_k)u_k + \cdots + (a_{2m} + b_{2m})u_{2m}$

$= a_0 + a_k - (a_1u_1 + a_2u_2 + \cdots + a_{2m}u_{2m}) + b_0 + b_k - (b_1u_1 + b_2u_2 + \cdots + b_ku_k + \cdots + (a_{2m} + b_{2m})u_{2m}$
$+ b_{2m}u_{2m}$

$= \theta(a) + \theta(b)$.

And

$\theta(ab) = \theta((a_0 + a_1u_1 + a_2u_2 + \cdots + a_ku_k + \cdots + a_{2m}u_{2m})(b_0 + b_1u_1 + b_2u_2 + \cdots + b_ku_k + \cdots + b_{2m}u_{2m}$
$) = \theta(a_0b_0 + a_0(b_1u_1 + b_2u_2 + \cdots + b_ku_k + \cdots + b_{2m}u_{2m}) + a_1b_1u_1 + \cdots + a_kb_ku_k + \cdots$
$+ a_{2m}b_{2m}u_{2m}$

$= a_0b_0 + a_0b_k + a_kb_0 + a_kb_k - (a_0(b_1u_1 + b_2u_2 + \cdots + b_ku_k + \cdots + b_{2m}u_{2m})$
$+ b_0(a_1u_1 + a_2u_2 + \cdots + a_ku_k + \cdots + a_{2m}u_{2m}) + a_1b_1u_1 + \cdots + a_kb_ku_k + \cdots$
$+ a_{2m}b_{2m}u_{2m}$

$\theta(a)\theta(b) = (a_0 + a_k - (a_1u_1 + a_2u_2 + \cdots + a_{2m}u_{2m}))(b_0 + b_k - (b_1u_1 + b_2u_2 + \cdots + b_{2m}u_{2m}))$

So that $\theta(ab) = \theta(a)\theta(b)$, also it is clear that $\theta^2(a) = a$.

Lemma 3.5. Let θ be the automorphism as in Lemma (3.4) and let $\Delta_\theta : R \to R$ with $\Delta_\theta(x) = (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x) - x)$ for all $x \in R$. Then Δ_θ is a θ −derivative on R.

Proof. For any x, y ∈ R, we observe that:

1. $\Delta_\theta(x + y) = (1 + 2u_k)(\theta(x + y) - (x + y))$
   $= (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x) + \theta(y) - (x + y))$
   $= (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x) - x)$
   $+ (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(y) - y)$
   $= \Delta\theta(x) + \Delta\theta(y)$
2. $\Delta_\theta(xy) = (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(xy) - xy)$
   $= (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x)\theta(y) - xy)$

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$= (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x)\theta(y) - xy - \theta(x)y + \theta(x)y)$
$= (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})\big(\theta(x)(\theta(y) - y)\big)$
$+ (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})(\theta(x) - x)y$
$= \Delta_\theta(x)y + \theta(x)\Delta_\theta(y)$
■

Lemma 3.6. Let $k = (1 + 2u_1 + 2u_2 + \ldots + 2u_{2m})$, and take θ the automorphism and Δ_θ the derivative as in Lemma (3.4) dan Lemma (3.5), respectively, then the following statements hold:

1. $\Delta_\theta \theta + \theta \Delta_\theta = 0$
2. $\Delta_\theta \Delta_\theta = 0$
3. $\Delta_\theta(x) = 0 \Leftrightarrow \theta(x) = x$

Proof.

1. For any $a = a_0 + a_1u_1 + \cdots + a_iu_i + \cdots + a_{2m}u_{2m} \in R$, observe that

$\Delta_\theta\big(\theta(a)\big) = \Delta_\theta\big((a_0 + a_i) - (a_1u_1 + \ldots + a_{2m}u_{2m})\big)$
$= \Delta_\theta(a_0 + a_i) + \Delta_\theta(a_1u_1 + \ldots + a_{2m}u_{2m})$
$= 0 + k(-a_i + 2(a_1u_1 + \ldots + a_{2m}u_{2m})$
$= k(-a_i + 2(a_1u_1 + \ldots + a_{2m}u_{2m})$.

$\theta\big(\Delta_\theta(a)\big) = \theta\big(k(\theta(a) - a)\big)$
$= \theta(k)\theta(\theta(a) - a)$
$= \theta(k)\big(\theta^2(a) - \theta(a)\big)$
$= \theta(k)(a_1u_1 + \ldots + a_{2m}u_{2m}) - \big(a_0 + a_i - (a_1u_1 + \ldots + a_{2m}u_{2m})\big)$
$= \theta(k)(-a_i + 2(a_1u_1 + \ldots + a_{2m}u_{2m})$.

Note that $k = -\theta(k)$, so that $\Delta_\theta \theta + \theta\Delta_\theta = 0$.

2. Let $a = a_0 + a_1u_1 + \cdots + a_iu_i + \cdots + a_{2m}u_{2m}$ and $k = 1 + 2u_1 + \ldots + 2u_{2m}$.

$\Delta_\theta\big(\Delta_\theta(a)\big) = \Delta_\theta\big(k(\theta(a) - a)\big)$
$= \Delta_\theta(k\theta(a) - ka)$
$= \Delta_\theta\big(k\theta(a)\big) + \Delta_\theta(-ka)$
$= k\Big(\theta\big(k\theta(a)\big) - k\theta(a)\Big) + k(\theta(-ka) + ka)$
$= k\theta(k)\theta^2(a) - k^2\theta(a) - k\theta(k)\theta(a) + k^2a$.

Consider that $k^2 = (1 + 2u_1 + \ldots + 2u_{2m})^2 = 1 + 4u_1 + \ldots + 4u_{2m} = 1$ and
$k\theta(k) = (1 + 2u_1 + \ldots + 2u_{2m})\big(1 + 2 - (2u_1 + \ldots + 2u_{2m})\big) = (1 + 2u_1 + \ldots + 2u_{2m})(3 - (2u_1 + \ldots + 2u_{2m})) = -1$

. So that $\Delta_\theta\big(\Delta_\theta(a)\big) = -1.a - \theta(a) + \theta(a) + a = 0$.

3. It is celar that 1 and $1 + 2 = 3$ are units in $\mathbb{Z}_4$. So k is a unit in R.

Therefore $\Delta_\theta(a) = 0 \Leftrightarrow \theta(x) = x$.
■

Lemma 3.7. For every $a \in R, ax^2 = x^2a$.

Proof. Since $xa = \theta(a)x + \Delta_\theta(a)$, then we have
$x^2a = x\big(\theta(a)x + \Delta_\theta(a)\big)$
$= x\big(\theta(a)\big)x + x\Delta_\theta(a)$
$= (\theta^2(a)x + \Delta_\theta\big(\theta(a)\big)x + \theta\big(\Delta_\theta(a)\big)x + \Delta^2\theta(a)$
$= \theta^2(a)x^2 + \Big(\Delta_\theta\big(\theta(a)\big) + \theta\big(\Delta_\theta(a)\big)\Big)x + \Delta^2\theta(a)$

$= ax^2.$

■

Corollary 3.8. For every $a \in R, n \in \mathbb{N}$, we have

$$x^n a = \begin{cases} (\theta(a)x + \Delta_\theta(a))x^{n-1}, if\ n\ is\ odd \\ ax^n\ , if\ n\ is\ even \end{cases}$$

Proof. We know that,

$$x^1 a = \theta(a)x + \Delta_\theta(a)x^0$$
$$= \theta(a)x + \Delta\Delta_\theta(a)$$
$$x^2 a = ax^2. (from\ Lemma\ 3.7)$$

Suppose the above statement holds for all $n \in \mathbb{N}$. We will prove for $n + 1$.
Consider two cases.

1. If $n + 1$ is even, then n is odd. So we get
$$x^{n+1}a = x(x^n a)$$
$$= x(\theta(a)x + \Delta_\theta(a))x^{n-1}$$
$$= x(\theta(a))x^n + x\Delta_\theta(a)x^{n-1}$$
$$= \left(\theta(a)^2 x + \Delta_\theta(\theta(a))\right)x^n + (\theta(\Delta_\theta(a)x + \Delta_\theta\Delta_\theta(a))x^{n-1}$$
$$= ax^{n+1} + \Delta_\theta(\theta(a))x^n + \theta(\Delta_\theta(a))x^n + \Delta_\theta{}^2(a)x^{n-1}$$
$$= ax^{n+1}.$$
2. If $n + 1$ is odd, then n is even and so we have
$$x^{n+1}a = x(x^n a)$$
$$= x(ax^n)$$
$$= (xa)x^n$$
$$= (\theta(a)x + \Delta_\theta(a))x^n$$
$$= \theta(a)x^{n+1} + \Delta_\theta(a)x^n .$$

■

5. Conclusion

In earlier results, we know that the form of the derivative with order one in form $\Delta_\theta = k(\theta(x) - x)$ also the product of skew-cyclic polynomials, i.e, $xa = \theta(a)x + \Delta_\theta(a)$. In this paper we give the generalization for any order , i.e , $\Delta_\theta^n = k^n \sum_{i=1}^{n} \binom{n}{i} \theta^{n-i}(-1)^i$ and $x^n a = \sum_{i=0}^{n} \binom{n}{i} \Delta_\theta^{n-i}\left(\theta^i(a)\right)x^i$..

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