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Some Bounds for the Spectral Radius (SR)

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Abstract

The spectral radius $\rho(A)$ of a square matrix ($A \in M_n$) is the largest absolute value of any eigenvalue of A. I divided this research into two parts, the first part in which I mention the definition of the spectral radius and the spectral norm, then I mention some of the characteristics related to the spectral radius and the spectral norm that I need in my research, along with proof of some of them.

In the second part of the research, I discuss a group of inequalities and provide proof of them. I will rely on some well-known inequalities and some well-known characteristics about spectral radius and spectral norm.

Keywords: Nonnegative matrix (~M), Positive matrix (+M), Spectral radius (SR), Pinching inequalities

1. Introduction

Bounds for the spectral radius norm play a crucial role in various areas of mathematics, particularly in the study of matrices and linear operators. The spectral radius norm, often denoted as $\rho(A)$ for a matrix A, represents the largest absolute eigenvalue of A. Understanding bounds for this norm helps in assessing the behavior and properties of matrices in numerical analysis, control theory, and other fields.

And we now the spectral norm of a matrix A, denoted as $\|A\|$, is the largest singular value of A. It measures how much a matrix can stretch a vector, and it is equal to the square root of the largest eigenvalue of A^*A (where A^* denotes the conjugate transpose of A).

From the previous two definitions we obtain the following inequality $\rho(A) \leq \|A\|$, for which I provided a clear proof other than the definition, as I used it in the proof the spectral radius formula $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.

And I discuss and prove a group of inequalities and equalities based on the properties

$\rho(AB) = \rho(BA)$, and $\rho\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \rho\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right)$ for every $A, B, C, D \in Mn$

Where, AB and AB have the same eigenvalues, so $\rho(AB) = \rho(BA)$. And it is known to us that the $\rho(AB) \leq \rho(A)\rho(B)$ and this has been proven, as well as our inequality $\rho(A) \leq \|A\|$, and through that we get $\rho(AB) \leq \|A\|.\|B\|$.

These inequalities leverage the properties of spectral radius norms and matrix norms to establish relationships between matrix products and their norms. They are foundational in matrix theory and find applications in various fields such as numerical analysis, control theory, and optimization. Understanding these inequalities helps in analyzing the stability and performance of algorithms involving matrix computations, ensuring robustness and efficiency in practical applications.

2. Properties of SR

2.1 Definition: The SR $\rho(A)$ of a matrix $A \in M_n$ is

$r(A)$ =Max$\{|\lambda|: \lambda \in \sigma(A),\ \sigma(A)\ eigenvalues\ set\ \}$.

2.2 Theorem: if $A \in M_n$, for each matrix norm N, then $\rho(A) \leq N(A)$.

Proof:

Let $x \in C^n$ be a nonzero vector such that $Ax = \lambda x$.

Let $X = [x: x: \dots : x]$, then $|\lambda| N(X) = N(\lambda X) = N(AX) \leq N(A)N(X)$. known $N(X) \neq 0$, We have $|\lambda| \leq N(A)$, then $\rho(A) \leq N(A)$.

2.3 Result: for each $A \in M_n$,we have $\{\ \rho(A) \leq \|A\|, where\ \|A\| = \max_{\|x\|=1} \|AX\|\ \}$, ($\|A\|$ called spectral norm)

2.4 Note: $\rho(AB) = \rho(BA)$, for all $A, B \in M_n$.

To prove the previous note, we use known information $\sigma(AB) = \sigma(BA)$

2.5 Theorem: If A, B *are square matrices* and $AB = BA$, Then

(1) $\rho(A + B) \leq \rho(A) + \rho(B)$.
(2) $\rho(AB) \leq \rho(A)\rho(B)$.

Proof:

(1) *there is a unitary matrix* $U \in M_n$, such that U^*AU and U^*BU are both upper triangular , (by schur's theorem)

$$T_1 = U^*AU = \begin{bmatrix} \lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & \lambda_2 & a_{23} & \dots & a_{2n} \\ 0 & 0 & \lambda_3 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_n \end{bmatrix} \lambda_i \in \sigma(A), \text{ and}$$

$$T_2 = U^*BU = \begin{bmatrix} \mu_1 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & \mu_2 & b_{23} & \dots & b_{2n} \\ 0 & 0 & \mu_3 & \dots & b_{3n} \\ \vdots & \vdots & \square & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \mu_n \end{bmatrix}, \ \mu_i \in \sigma(B)$$

We have $\sigma(A) = \sigma(T_1) = \{\lambda_i : i = 1, \ldots, n\}$, and $\sigma(B) = \sigma(T_2) = \{\mu_i : i = 1, \ldots n\}$.

And, $\sigma(A + B) \subseteq \sigma(A) + \sigma(B)$, and $\sigma(AB) \subseteq \sigma(A)\sigma(B)$.

So, $\rho(A + B) \leq \rho(A) + \rho(B)$ and $\rho(AB) \leq \rho(A)\rho(B)$.

2.6 Note: when $AB \neq BA$, then Theorem 2.5 is false.

Example: Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

We have $\rho(A) = 0$ and $\rho(B) = 0$, and $\rho(A + B) = 1$.

Then, $\rho(A + B) = 1 > \rho(A) + \rho(B) = 0$.

2.7 Note: If $A \in M_n$, then $\rho(A) = \lim_{k\to\infty} \|A^k\|^{1/k}$.

Proof:

Since $\rho(A)^k = \rho(A^k) \leq \|A^k\|$, it follows that $\rho(A) \leq \|A^k\|^{1/k}$ for all $k = 1,2, \ldots$

Given > 0 , Let $\hat{A} = A/(\rho(A) + \varepsilon)$. Then $(\hat{A}) < 1$, and so by ($\lim_{k\to\infty} A^k = 0 \text{ if and only if } \rho(A) < 1$)

$\|\hat{A}^k\| \to 0$ as $k \to \infty$

So there is $k_0 \geq 1$ such that $\|\hat{A}^k\| < 1$ for all $k \geq k_0$.

Thus ,

$$\|A^k\| < (\rho(A) + \varepsilon)^{\wedge}k \text{ for all } k \geq k_0 ,$$

And so

$$\|A^k\|^{1/k} < \rho(A) + \varepsilon \text{ for all } k \geq k_0.$$

Since $\rho(A) \leq \|A^k\|^{1/k}$ for all k , it follows that

$$\rho(A) - \varepsilon < \|A^k\|^{1/k} < \rho(A) + \varepsilon \text{ for all } k \geq k_0,$$

$$\lim_{k\to\infty} \|A^k\|^{1/k} = \rho(A).$$

3. Some bounds for the SR

By $\rho(AB) = \rho(BA)$, and $\rho\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \rho\left(\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right)$ for every $A, B, C, D \in Mn$

The following inequalities will be presented.

3.1 Theorem: for every $A_1, A_2, B_1, B_2 \in M_n$, then

$$\rho(A_1B_1 + A_2B_2) \leq \frac{1}{2}(\|A_1B_1\| + \|A_2B_2\| + \sqrt{(\|B_1A_1\| - \|B_2A_2\|)^2 + 4\|B_1A_2\|\|B_2A_1\|})$$

Proof:

We have $\rho(A_1B_1 + A_2B_2) = \rho\left(\begin{bmatrix} A_1B_1 + A_2B_2 & 0 \\ 0 & 0 \end{bmatrix}\right)$

$$= \rho\left(\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}\right)$$

$$= \rho\left(\begin{bmatrix} B_1 & 0 \\ B_2 & 0 \end{bmatrix}\begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix}\right) \quad (\text{by } \rho(AB) = \rho(BA))$$

$$= \rho\left(\begin{bmatrix} B_1A_1 & B_1A_2 \\ B_2A_1 & B_2A_2 \end{bmatrix}\right)$$

$$\leq \rho\left(\begin{bmatrix} \|B_1A_1\| & \|B_1A_2\| \\ \|B_2A_1\| & \|B_2A_2\| \end{bmatrix}\right)$$

$$= \tfrac{1}{2}(\|A_1B_1\| + \|A_2B_2\| + \sqrt{(\|B_1A_1\| - \|B_2A_2\|)^2 + 4\|B_1A_2\|\|B_2A_1\|})$$

as required.

3.2 Note: If $A, B \in Mn$, then

$$\rho(A+B) \leq \tfrac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4min(\|AB\|, \|BA\|)}\right) \cdots\cdots\cdots\cdots\cdots (*)$$

Proof:

By Theorem 3.1, it is assumed that $A_1 = A$, $A_2 = B_1 = I$, and $B_2 = B$ we can have

$$\rho(A+B) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|BA\|}\right) \cdots\cdots\cdots\cdots\cdots (1)$$

and letting $B_1 = A$, $B_2 = A_1 = I$ and $A_2 = B$, we have

$$\rho(A+B) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|AB\|}\right) \cdots\cdots\cdots\cdots\cdots (2)$$

From the inequality (1and 2) we get

$$\rho(A+B) \leq \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4min(\|AB\|, \|BA\|)}\right) \cdots\cdots\cdots\cdots\cdots (*)$$

Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

We will find $\rho(A+B) = 2$, $\|A\| = 2$, $\|B\| = 2$

$$\rho(A+B) \leq \tfrac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4min(\|AB\|, \|BA\|)}\right)$$

$$= 2 \leq \tfrac{1}{2}\left(2 + 2 + \sqrt{(2-2)^2 + 4min(0, 4)}\right) = 2$$

And we calculated the $\|A+B\|$, And it was $\|A+B\| = 2\sqrt{2}$

Therefore, it is not permissible to apply the note 4.2 to spectral norm.

3.3 Note: for every $, B \in Mn$**,** we have

$$\rho(A+B) \leq \|A\| + \|B\|$$

Proof:

$$\rho(A+B) \leq \tfrac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4min(\|AB\|, \|BA\|)}\right) \quad (\text{ by note 3.2})$$

$$\leq \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4\max(\|AB\|,\|BA\|)}\right)$$

$$\leq \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2+4(\|A\|,\|B\|)}\right)$$ (By submultiplicative of the spectral norm)

$$= \frac{1}{2}\left(\|A\|+\|B\|+\sqrt{(\|A\|-\|B\|)^2}\right) = \|A\|+\|B\|$$

3.4 Note: If $A, B \in Mn$, then

$$\rho(AB \pm BA) \leq \frac{1}{2}\left(\|AB\|+\|BA\|+\sqrt{(\|AB\|-\|BA\|)^2+4min(\|A^2\|,\|B^2\|)}\right) \cdots\cdots (3)$$

Proof:

Suppose that $A_1 = B_2 = A$, $B_1 = B$, and $A_2 = \pm B$. then by theorem 3.1

$$\rho(A_1B_1 + A_2B_2) \leq \frac{1}{2}(\|A_1B_1\|+\|A_2B_2\|+\sqrt{(\|B_1A_1\|-\|B_2A_2\|)^2+4\|B_1A_2\|\|B_2A_1\|})$$

We get $\rho(AB \pm BA) \leq \frac{1}{2}\left(\|AB\|+\|BA\|+\sqrt{(\|AB\|-\|BA\|)^2+4min(\|A^2\|,\|B^2\|)}\right)$

3.5 Note: for every $A, B \in Mn$, we have

$$\rho(AB \pm BA) \leq \left(\|AB\|+\sqrt{min(\|A\|\,\|AB^2\|,\ \ \|B\|\,\|A^2B\|)}\right) \cdots\cdots (4)$$

And

$$\rho(AB \pm BA) \leq \left(\|BA\|+\sqrt{min(\|A\|\,\|B^2A\|,\ \ \|B\|\,\|BA^2\|)}\right) \cdots\cdots (5)$$

Proof:

Suppose that $A_1 = I,\ A_2 = B,\ B_1 = AB$, and $B_2 = \pm A$, then by theorem 3.1 we have

$$\rho(AB \pm BA) \leq \left(\|AB\|+\sqrt{\|A\|\,\|AB^2\|}\right) \cdots\cdots (6)$$

And Suppose that $A_1 = AB, A_2 = B,\ B_1 = I$, and $B_2 = \pm A$ Apply it to Theorem 3.1 to get

$$\rho(AB \pm BA) \leq \left(\|AB\|+\sqrt{\|B\|\,\|A^2B\|}\right) \ldots\ldots\ldots (7)$$

First result: We obtain the inequality (4) by considering the previous inequalities (6) and (7)

To obtain the fifth inequality, we assume the following $A_1 = \pm BA,\ B1 = I,\ B_2 = B$, and $A_2 = A$ and apply it to the Theorem 3.1,

$$\rho(AB \pm BA) \leq \left(\|AB\|+\sqrt{\|A\|\,\|B^2A\|}\right) \cdots\cdots (8)$$

Similarly, letting $A_1 = I,\ B_1 = \pm BA, A_2 = A$, and $B_2 = B$ in Theorem 3.1, we have

$$\rho(AB \pm BA) \leq \left(\|BA\|+\sqrt{\|B\|\,\|BA^2\|}\right) \cdots\cdots (9)$$

3.6 Note: if $A, U \in Mn$ where U is unitary, then

$$\rho(AU \pm UA) \leq \|A\|+\sqrt{\|A^2\|}$$

Proof:

We will use the fourth inequality with the following assumption $B = U$

$$\rho(AU \pm UA) \leq \|AU\|+\sqrt{min(\|A\|\,\|AU^2\|,\ \ \|U\|\,\|A^2U\|)}$$

$$= \|A\|+\sqrt{min(\|A\|^2,\|A^2\|)}$$

$= \|A\| + \sqrt{\|A^2\|}$ (By submultiplicative of the spectral norm)

3.7 Note: the pinching inequalities for the usual operator norm assert that if $A, B, C, D \in M_n$, then

$$\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\| \le \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \cdots\cdots (11)$$

And

$$\left\| \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \right\| \le \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \cdots\cdots (12)$$

To see this, let

$$T = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \text{ and } U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

Then U is unitary,

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \frac{1}{2}(T + UTU^*), \quad and \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \frac{1}{2}(T - UTU^*)$$

Now,

$$\left\| \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \right\| = \frac{1}{2}\|T + UTU^*\| \le \frac{1}{2}(\|T\| + \|UTU^*\|)$$

$$= \frac{1}{2}(\|T\| + \|T\|) = \|T\| = \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|$$

And

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \frac{1}{2}(T - UTU^*)$$

$$\le \frac{1}{2}\|T + UTU^*\| = \frac{1}{2}(\|T\| + \|UTU^*\|)$$

$$= \frac{1}{2}(\|T\| + \|T\|) = \|T\| = \left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\|$$

It should be noted that the previous result cannot be applied to the spectral radius, See this example

$$T = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Then

$$\rho\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = 1 > \rho(T) = 0$$

Also,

$$\rho\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 1 > \rho(T) = 0$$

3.8 Note: if $A, B, C, D \in M_n$, then

1) $\rho\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \max(\rho(A), \rho(D))$
2) $\rho\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) = \sqrt{\rho(BC)}$

Proof:

1) Let $L = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$

Since $\sigma(L) = \sigma(A) \cup \sigma(D)$, it follows that

$$\rho(L) = max(\rho(A),\ \rho(D))$$

2) We will use this property $\rho(A^k) = (\rho(A))^k \ for\ k = 1,2,\ldots$
$and\ by\ \{\ \rho(AB) = \rho(BA)\}$, to get

$$\left(\rho\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right)\right)^2 = \rho\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}^2\right) = \rho\left(\begin{bmatrix} BC & 0 \\ 0 & CB \end{bmatrix}\right) = \rho(BC)$$

Then,

$$\rho\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \sqrt{\rho(BC)}.$$

Example:

Consider $L = \left(\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}\right) Then$

$\rho(L) = \sqrt{2}$, but $\rho(B = 2) = 2$, and $\rho\ (C = 1) = 1$.

So

$$max\ (\rho(B), \rho\ (C)) = 2 > \sqrt{2}$$

Thus

$$\rho(L) = max(\rho(B),\ \rho(C))$$

3.9 Theorem: for all $A, B, C, D \in M_n$, and if $T = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$, then

$$max(\rho(A), \rho(D)\ ) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right) \cdots\cdots (13)$$

And

$$\sqrt{\rho(BC)} \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right) \cdots\cdots (14)$$

Proof:

1) Prove the inequality 13

Let $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, then $U \in M_n$ is unitary,

$$2\begin{bmatrix} A & 0 \\ 0 & -D \end{bmatrix} = TU + UT, \qquad and\ \ 2\begin{bmatrix} 0 & -B \\ C & 0 \end{bmatrix} = TU - UT$$

Thus

$$\begin{aligned} 2max(\rho(A), \rho(D)\ ) &= 2\rho\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \\ &= \rho\left(\begin{bmatrix} 2A & 0 \\ 0 & -2D \end{bmatrix}\right) \\ &= \rho(TU + UT) \end{aligned}$$

$$\leq \|T\| + \|T^2\|^{\frac{1}{2}} \qquad \text{(By note 3.6)}$$

And

$$max(\rho(A), \rho(D)) \leq \frac{1}{2}\left(\|T\| + \|T^2\|^{\frac{1}{2}}\right)$$

2) prove the inequality (14), we have

$$2\sqrt{\rho(BC)} = 2\rho\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) = \rho\left(\begin{bmatrix} 0 & -2B \\ 2C & 0 \end{bmatrix}\right)$$

$$= \rho(TU - UT) \leq \|T\| + \|T^2\|^{\frac{1}{2}} \qquad \text{(By note 3.6)}$$

3.10 Theorem: if $A, B \in M_n$, then

$$\rho(AB) \leq \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\min(\|A\|\|BAB\|, \|B\|\|ABA\|)}\right) \cdots (15)$$

Proof:

Letting $A_1 = I,\ A_2 = A,\ B_1 = AB \ and \ B_2 = B$ in the theorem 3.1, we have

$$2\rho(AB) \leq \frac{1}{2}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\,\|B\|\|ABA\|}\right) \cdots\cdots (16)$$

And hence

$$\rho(AB) \leq \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\,\|B\|\|ABA\|}\right) \cdots\cdots (17)$$

Now, by symmetry, letting $B_1 = I,\ B_2 = B,\ A_1 = AB \ and \ A_2 = A$ in the theorem 3.1,

we have

$$2\rho(AB) \leq \frac{1}{2}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\,\|A\|\|BAB\|}\right) \cdots\cdots (18)$$

And so,

$$\rho(AB) \leq \frac{1}{4}\left(\|AB\| + \|BA\| + \sqrt{(\|AB\| - \|BA\|)^2 + 4\,\|A\|\|BAB\|}\right) \cdots\cdots (19)$$

Thus, the inequality (15) now follows from (17) and (19)

3.11 Note: if $A \in M_n$, then for every $k = 1,2, \ldots$, we have

$$(\rho(A))^k \leq \frac{1}{2}(\|A^k\| + \sqrt{\min(\|A\|\|A^{2k-1}\|, \|A^{k-1}\|\|A^{k+1}\|)})$$

Proof:
Letting $B = A^{k-1}$ in the theorem 3, we have

$$\rho(A^k) \leq \frac{1}{4}\left(\|A^k\| + \|A^k\| + \sqrt{(\|A^k\| - \|A^k\|)^2 + 4\min(\|A\|\|A^{2k-1}\|, \|A^{k-1}\|\|A^{k+1}\|)}\right)$$

And hence

$$\rho(A)^k = \rho(A^k) \leq \frac{1}{2}\left(\|A^k\| + \sqrt{\min(\|A\|\|A^{2k-1}\|, \|A^{k-1}\|\|A^{k+1}\|)}\right)$$

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