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Special Example of the Finite Dimension in Local Cohomology Modules

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Abstract

Let, R be an total local ring, I an Ideal of R, and let M be a finitely generated R-module. Then $H_j^i(M)$ Is I-cofinite in two cases. 1) I be a none zero principle Ideal. 2) I be a prime Ideal with dimension 1. In this paper we produce special example of the cofiniteness problem in local cohomology respect to a system of ideals. In fact we construct an ideal system with tridiagonal matrices of tridiagonal subsets which are true in the finite dimension in local cohomology modules.

Keywords: Local cohomology, Cofinite, Ideal system, Tridiagonal matrix

1. Introduction

Throughout this paper, R denotes a commutative Noetherien ring with non-zero identity and M is a finitely generated R-module.

1.1 Definition (See [5, Definition 3.3.2 and Definition 9.1.3])

Let φ be a system of ideals of R. The arithmetic rank of φ, denoted by $ara(\phi)$ is defined as follows $ara(\varphi) = \max\{ara(I) : I \in \varphi\}$. Note that $ara(\varphi)$ is either a positive integer, whenever φ is non-empty, or infinite. In particular, if $\varphi = \{I^i : i \geq 0\}$ then $ara(\varphi) = ara(I)$.

1.2 Definition(See[1, Definition5])

Let M be a finitely generated R-module. We define the finiteness dimension $f_\phi(M)$ of M relative to φ by $f_\varphi(M) = \inf\{i \geq 0; H_\varphi^i(M) \text{ is not finitely generated}\} = \inf\{i \geq 0; IH_\varphi^i(M) \neq 0, \forall I \in \varphi\}$.

1.3 Theorem

Let φ be a system of ideals of R and M be a finitely generated R–module. Suppose that $n \geq 0$ is such that $H_\varphi^j(M)$ is finitely generated for all $j < n$ and that $H_J^j(M) = 0$ for all $j > n$ and all $J \in \varphi$. Then $H_\varphi^j(M)$, is φ–cofinite for all i.

Proof. See [2, Theorem 4.8].

1. 4 Corollary

Let φ be a system of ideals of R. Let M be a finitely generated R–module and I be an arbitrary ideal of R.

$i)$ If $ara(\varphi) \leq f_\varphi(M)$, then $H_\varphi^i(M)$ is φ–cofinite for all i.

$ii)$ If $H_I^i(M) = 0$ for all $i \geq f_I(M)$, then $H_I^i(M)$ is I–cofinite for all i.

$iii)$ If $ara(I) \leq f_I(M)$, then $H_I^i(M)$ is I–cofinite for all i.

Proof. $i)$ If $f_\varphi(M)$ is infinite then there is nothing to do any more. Hence we may assume that $f_\varphi(M)$ is finite. So $f_\varphi(M) = ara(\varphi)$ by[5, Corollary 3.3.3]. Now the result follows easily from Theorem 1.3. Let $\varphi = \{I^i; i > 0\}$ the statement $ii)$ is immediate consequence of Theorem 1.3. and $iii)$ is consequence of $i)$.

2. Special example

2.1 Remark

Let n be an integer and $D_n(A)$ be all bellow tridiagonal matrices $n \times n$ which it's elements is in A. Also we show determination H by $|H|$ and transpose H by $(-)^T$ for each H in $D_n(A)$.

2.2 Definition

Non-empty subset of U in $A^n = A \times \ldots \times A$ is called, tridiagonal subset of A^n if

$i) \forall (u_1,\ldots,u_n) \in U$ then $(u_1^{\alpha_1},\ldots,u_n^{\alpha_n}) \in U$ for all $\alpha_1,\ldots,\alpha_n, n \in N$.

$ii)$ There exist $(w_1,\ldots,w_n) \in U$ and tridiagonal matrices $H, K \in D_n(A)$ that $H(u_1,\ldots u_n)^T = (w_1,\ldots,w_n)^T = K(v_1,\ldots,v_n)^T$ for each $(u_1,\ldots,u_n),(v_1,\ldots,v_n) \in U$.

2.3 Theorem

Let $n \in N$ and $U = \{(x_1,\ldots x_n); x_1,\ldots x_n$ is an –regular sequence} then U is a tridiagonal set.

Proof. See [5, Theorem 7.2].

2.4 Theorem

Let U be a tridiagonal subset of A^k and let $\varphi = \{\sum_{i=1}^{n} Rx_i, (x_1,\ldots,x_n) \in U\}$ then φ is a system of ideals.

Proof. Let W and Z be in φ. Then there exist $(u_1,\ldots,u_n),(v_1,\ldots,v_n) \in U$ which $W = Ru_1 + \ldots + Ru_n$ and $Z = Rv_1 + \ldots + Rv_n$. Since U is tridiagonal then there exist $(w_1,\ldots,w_n) \in U$ that $w_i \in (Ru_1 + \ldots + Ru_i) \cap (Rv_1 + \ldots + Rv_i)$ for each $1 \le i \le n$. Then $w_i^2 \in (Ru_1 + \ldots + Ru_n).(Rv_1 + \ldots + Rv_n) = W.Z$ for each $1 \le i \le n$. Also $(w_1^2,\ldots,w_n^2) \in U$.(U is tridiagonal) then $X = Rw_1^2 + \ldots + Rw_n^2 \subseteq W.Z$, it means $X \subseteq W.Z$ then φ is a system of ideals.

2.5 Theorem

Let M be a finitely generated R –module and $I \trianglelefteq R$ which $IM \neq M$ .Then $grad_M(I) = \inf\{i; Ext_R^i(R/I, M) \neq 0\}$ or $H_I^i(M) \neq 0$.

Proof. Let $g = grad_M(I)$ .It will be proved by induction on g.

3. Conclusion

The following corollary is the main result of this paper.

3.1 Corollary

Let n be a positive integer and let $\varphi = \{\sum_{i=1}^{n} Rx_i, x_1,\ldots,x_n$ is an M –regular sequence} then $H_\varphi^j(M)$ is φ –cofinite for all j.

Proof. Note that by Theorem 2.4. φ is a system of ideals. Now using Theorem 2.5. $I = (x_1,\ldots,x_n)$ and $IM \neq M$. Then $H_\varphi^i(M) = \varinjlim_{i \in \varphi} H_I^i(M) = 0$. Now by using Theorem 1.3. it will be proved.

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