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Expanding Generalized Gamma Distribution (EGGD)

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Abstract
In this paper we introduce a new distribution that is dependent on the Pearson type three distribution (gamma distribution)and we discuss some of properties like that the rth moment, , rth moment about the origin , mean ,moment generated , reliability , hazard functions , coefficient of (skeweness, variance , kurtosis) ,with estimation the parameter by using the moment estimation and maximum likelihood estimation .
Keyword: maximum likelihood estimation, moment estimation , gamma distribution ,incomplete gamma function

1. Introduction

The Pearson distribution is a family of continuous probability distributions . It was first published by Karl Pearson in 1895 and subsequently extended by him in 1901 and 1916 in a series of articles on biostatistics **[1]**. BOWMAN, and SHENTON(2007), mentioned that there are many types of this distribution like form type I to the type VI distribution, Pearson type I and type II distribution, is called beta distribution of the four parameter. We use Pearson type three distribution that is also called three parameter gamma distribution which is widely used in various scientific fields such as in reliability and hydrology.
The probability density function (p.d.f) of Pearson type three distribution is given by:

$$f(x,v,\lambda,\beta)=\frac{\beta^{\lambda}(x-v)^{\lambda-1}}{\Gamma(\lambda)}e^{-\beta(x-v)}I_{(0,\infty)}(x).$$

Cohen(1950) introduced estimation parameters of Pearson type III population from truncated samples , Nestor and George Karagiannidis constructed sum of gamma –gamma variates and applications in RF and optical wireless communications, George gerad den broeder,Jr drive introduce on parameter estimation of truncated Pearson type III distribution and introduction a new family of distribution based on the generalized Pearson differential equation with some application by Shakil , galam kivria and jai narain singh.

2. Another Distribution on the Person Type III .

Consider the random variable X with the standard Gamma distribution**, The Person Type III,** that is replaced by the random variable Y defined as $Y=\frac{sX-\epsilon}{\alpha}$where s,α,ϵ are the parameters, then the distribution of this variable is Expanded Generalized Gamma Distribution with four parameters ,**EGGD**, which can defined by :

$$f(x;s,\propto,\beta,\epsilon)=\frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1}e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}I_{\left(\frac{\epsilon}{s},\infty\right)}(x) \quad (1)$$

where : $s,\alpha$are the scale parameters, ϵ is the location parameter and $\beta$is the shape parameter ; $s,\alpha,\beta>0$, $\epsilon>0$
The plot of the p.d.f. of **EGGD** is given in **Figure 1**
The shapes of the p.d.f of EGGD based on change parameters is explained in the following **Figure 2**.
Now we can prove that **EGGD** is p.d.f.

$$\int_{\frac{\epsilon}{s}}^{\infty}\frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1}e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}dx=\int_0^{\infty}\frac{1}{\Gamma(\beta)}u^{\beta-1}e^{-u}du$$

, where $u=\frac{sx-\epsilon}{\alpha}\ du=\frac{s}{\alpha}dx$

$$=\frac{1}{\Gamma(\beta)}\int_0^{\infty}u^{\beta-1}e^{-u}du$$

$$=\frac{1}{\Gamma(\beta)}\Gamma(\beta)=1$$

<u>The c.d.f. of EGGD is given as</u>

$$F(x)=\int_{\frac{\epsilon}{s}}^{x}\frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1}e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}dx,\text{where } u=\frac{sx-\epsilon}{\alpha}\Rightarrow du=\frac{s}{\alpha}dx$$

$$F(X)=\int_0^{\frac{\alpha x-\epsilon}{s}}\frac{1}{\Gamma(\beta)}u^{\beta-1}e^{-u}du$$

$$\therefore F(X)=\frac{1}{\Gamma(\beta)}\gamma\left(\beta,\frac{sx-\epsilon}{\alpha}\right) \quad (2)$$

where$\gamma(\beta,\frac{sx-\epsilon}{\alpha})$ is the **incomplete Gamma function** that is define by :

$\gamma\left(\beta,\frac{sx-\epsilon}{\alpha}\right)=\int_0^x u^{\beta-1}e^{-u}du$

Plot of the c.d.f of the EGGD in **Figure 3**.

3. Properties of EGGD

Proposition 1:

The r-th central moment about the origin of **EGGD** is as

$$E(X^r)=\frac{\sum_{j=0}^{r}C_{j=0}^{r}\left(\frac{\epsilon}{s}\right)^{r-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)} \quad (3)$$

And then mean, variance , coefficient of variation, coefficient of skewness and the coefficient of kurtosis of **EGGD** are as follows:

$$E(X)=\frac{\alpha\beta+\epsilon}{s} \quad (4)$$

$$Var(X)=\left(\frac{\alpha}{s}\right)^2\beta \quad (5)$$

$$CV=\frac{\alpha\sqrt{\beta}}{\alpha\beta+\epsilon} \quad (6)$$

$$CS=\frac{\left(\frac{\alpha x-\epsilon}{s}\right)^3+3\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^2\beta+3\left(\frac{\alpha}{s}\right)^3\beta^2+2\left(\frac{\alpha}{s}\right)^3\beta-3\left(\frac{\alpha\beta+\epsilon}{s}\right)\frac{1}{n}\sum_{i=0}^{n}\left(x_i-\left(\frac{\alpha\beta+\epsilon}{s}\right)\right)^2+\left(\frac{\alpha\beta+\epsilon}{s}\right)^3}{\left(\frac{\alpha}{s}\right)^2\beta} \quad (7)$$

and $CK=\frac{E(x^4)-4\mu E(x^3)+6\mu^2E(x^2)-4\mu^3E(x)+\mu^4}{\left(\left(\frac{\alpha}{s}\right)^2\beta\right)^2}-3$ (8)

Proof :-

$$E(X^r)=\int_{\frac{\epsilon}{s}}^{\infty}x^r\frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1}e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}dx$$

Let $u=\frac{sx-\epsilon}{\alpha}\Rightarrow x=\frac{\alpha u+\epsilon}{s}$, $du=\frac{s}{\alpha}dx$,

So that we get that :

$$E(X^r)=\int_0^{\infty}\left(\frac{\alpha u+\epsilon}{s}\right)^r\frac{1}{\Gamma(\beta)}u^{\beta-1}e^{-u}du$$

We can use the Binomial theory **[8]**

$$\left(\frac{\alpha u+\epsilon}{s}\right)^r=\left(\frac{\alpha u}{s}+\frac{\epsilon}{s}\right)^r=\sum_{j=0}^{\infty}C_j^{r-j}\left(\frac{\alpha u}{s}\right)^j\left(\frac{\epsilon}{s}\right)^{r-j}$$

So that we have:

$$E(X^r)=\int_0^{\infty}\left[\frac{\alpha u+\epsilon}{s}\right]^r\frac{1}{\Gamma(\beta)}u^{\beta-1}e^{-(u)}du=\int_0^{\infty}\sum_{j=0}^{\infty}C_j^{r-j}\left(\frac{\alpha u}{s}\right)^j\left(\frac{\epsilon}{s}\right)^{r-j}\frac{1}{\Gamma\beta}u^{\beta-1}e^{-(u)}du$$

$$E(X^r)=\frac{\sum_{j=0}^{r}C_{j=0}^{r}\left(\frac{\epsilon}{s}\right)^{r-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)}$$

Now by using (2.3) for $r=1,2,3$ and 4

$$E(X)=\frac{\sum_{j=0}^{1}C_{j=0}^{1}\left(\frac{\epsilon}{s}\right)^{1-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)}=\frac{\frac{\epsilon}{s}\Gamma(\beta)+\frac{\alpha}{s}\Gamma(\beta+1)}{\Gamma(\beta)}=\frac{\alpha\beta+\epsilon}{s}$$

$$E(X^2)=\frac{\sum_{j=0}^{2}C_{j=0}^{1}\left(\frac{\epsilon}{s}\right)^{2-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)}=\frac{C_0^2\left(\frac{\epsilon}{s}\right)^2\Gamma(\beta)}{\Gamma(\beta)}+\frac{C_1^2\left(\frac{\epsilon}{s}\right)^1\left(\frac{\alpha}{s}\right)^1\overbrace{\Gamma(\beta+1)}^{\beta\Gamma(\beta)}}{\Gamma(\beta)}+\frac{C_2^2\left(\frac{\alpha}{s}\right)^2\overbrace{\Gamma(\beta+2)}^{(\beta^2+\beta)\Gamma(\beta)}}{\Gamma(\beta)}=\left(\frac{\alpha x+\epsilon}{s}\right)^2+\left(\frac{\alpha}{s}\right)^2\beta$$

$$E(X^3)=\frac{\sum_{j=0}^{3}C_{j=0}^{3}\left(\frac{\epsilon}{s}\right)^{3-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)}$$

$$=\left(\frac{\alpha\beta+\epsilon}{s}\right)^3+3\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^2\beta+3\left(\frac{\alpha}{s}\right)^3\beta^2+2\left(\frac{\alpha}{s}\right)^3\beta$$

$$E(X^4)=\frac{\sum_{j=0}^{4}C_{j=0}^{4}\left(\frac{\epsilon}{s}\right)^{4-j}\left(\frac{\alpha}{s}\right)^{j}\Gamma(\beta+j)}{\Gamma(\beta)}=\frac{\left(\frac{\epsilon}{s}\right)^4\Gamma(\beta)}{\Gamma(\beta)}+\frac{4\left(\frac{\epsilon}{s}\right)^3\left(\frac{\alpha}{s}\right)\Gamma(\beta+1)}{\Gamma(\beta)}+\frac{6\left(\frac{\epsilon}{s}\right)^2\left(\frac{\alpha}{s}\right)^2\Gamma(\beta+2)}{\Gamma(\beta)}+\frac{4\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^3\Gamma(\beta+3)}{\Gamma(\beta)}+\frac{\left(\frac{\alpha}{s}\right)^4\Gamma(\beta+4)}{\Gamma(\beta)}$$

$$\vdots$$

$$E(X^4)=\left(\frac{\alpha\beta+\epsilon}{s}\right)^4+6\left(\frac{\epsilon}{s}\right)^2\left(\frac{\alpha}{s}\right)^2\beta+4\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^3[3\beta^2+2\beta]+\left(\frac{\alpha}{s}\right)^4[6\beta^3+11\beta^2+6\beta]$$

$$V(X)=E(X^2)-(E(X))^2=\left(\frac{\alpha\beta+\epsilon}{s}\right)^2+\left(\frac{\alpha}{s}\right)^2\beta-\left(\frac{\alpha\beta+\epsilon}{s}\right)^2=\left(\frac{\alpha}{s}\right)^2\beta$$

$$CV = \frac{\sigma}{\mu} = \frac{\sqrt{\left(\frac{\alpha}{s}\right)^2 \beta}}{\frac{\alpha\beta+\epsilon}{s}} = \frac{\alpha\sqrt{\beta}}{\alpha\beta+\epsilon}$$

$$CS = \frac{\mu_3}{\sigma^3} = \frac{E(X-\mu)^3}{\sigma^3} = \frac{2\left(\frac{\alpha}{s}\right)^3 \beta}{\left(\frac{\alpha}{s}\right)^3 (\beta)^{\frac{3}{2}}} = \frac{2}{\sqrt{\beta}} \qquad \blacksquare$$

The plot of the coefficient of skeweness in **Figure 4** and coefficient of variance in **Figure 5** and coefficient of kurtosis in **Figure 6**.

Proposition 2:

The moment generated function of **EGGD** is as

$$E(e^{tx}) = \frac{s\sum_{j=0}^{\beta-1} C_j^{\beta-1}\left(\frac{t}{\alpha(s-t)}\right)^{\beta-j-1} \sum_{l=0}^{\beta-j-1} C_l^{\beta-j-1}\alpha^l \epsilon^{\beta-j-l-1}\Gamma(j+l+1)}{(s-t)\Gamma(\beta)}$$

Proof :-

$$E(e^{tx}) = \int_{\frac{\epsilon}{s}}^{\infty} e^{tx} \frac{s}{\alpha\Gamma\beta}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}dx$$

$$= \int_{\frac{\epsilon}{s}}^{\infty} \frac{s}{\alpha\Gamma\beta}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{sx-\epsilon}{\alpha}\right)+tx}dx$$

$$= \int_{\frac{\epsilon}{s}}^{\infty} \frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{(s-t)x-\epsilon}{\alpha}\right)}dx$$

let $u = \frac{(s-t)x-\epsilon}{\alpha} \Rightarrow x = \frac{\alpha u+\epsilon}{s-t}, du = \frac{s-t}{\alpha}dx$,

$$= \int_0^{\infty} \frac{s}{(s-t)\Gamma(\beta)}\left(u + \frac{t\left(\frac{\alpha u+\epsilon}{s-t}\right)}{\alpha}\right)^{\beta-1} e^{-u}du$$

$$= \frac{s}{(s-t)\Gamma(\beta)}\int_0^{\infty}\left(u + \frac{t(\alpha u+\epsilon)}{\alpha(s-t)}\right)^{\beta-1} e^{-u}du$$

$$= \frac{s}{(s-t)\Gamma(\beta)}\int_0^{\infty}\sum_{j=0}^{\beta-1} C_j^{\beta-1}u^j\left(\frac{t(\alpha u+\epsilon)}{\alpha(s-t)}\right)^{\beta-j} e^{-u}du$$

$$= \frac{s}{(s-t)\Gamma(\beta)}\sum_{j=0}^{\beta-1} C_j^{\beta-1}\left(\frac{t}{\alpha(s-t)}\right)^{\beta-1-j}\int_0^{\infty} u^j(\alpha u+\epsilon)^{\beta-1-j}e^{-u}du$$

$$= \frac{s\sum_{j=0}^{\beta-1} C_j^{\beta-1}\left(\frac{t}{\alpha(s-t)}\right)^{\beta-1-j}\int_0^{\infty} u^j \sum_{l=0}^{\beta-1-j} C_l^{\beta-1-j}(\alpha u)^l\epsilon^{\beta-1-j-l}e^{-u}du}{(s-t)\Gamma(\beta)}$$

$$\therefore E(e^{tx}) = \frac{s\sum_{j=0}^{\beta-1} C_j^{\beta-1}\left(\frac{t}{\alpha(s-t)}\right)^{\beta-j-1} \sum_{l=0}^{\beta-j-1} C_l^{\beta-j-1}\alpha^l \epsilon^{\beta-j-l-1}\Gamma(j+l+1)}{(s-t)\Gamma(\beta)} \qquad \blacksquare$$

Proposition 3:

The mode of EGGD is follows as :

$$x = \frac{\alpha(\beta-1)+\epsilon}{s}$$

Proof :-

$$\because f(x) = \frac{s}{\alpha\Gamma(\beta)}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}$$

$$\ln(f(x)) = \ln s - \ln\alpha - \ln\Gamma(\beta) + (\beta-1)\ln\left(\frac{sx-\epsilon}{\alpha}\right) - \frac{sx-\epsilon}{\alpha}$$

$$\frac{\partial \ln(f(x))}{\partial x} = \left(\frac{(\beta-1)}{\left(\frac{sx-\epsilon}{\alpha}\right)}\right)\left(\frac{s}{\alpha}\right) - \frac{s}{\alpha}$$

$$\frac{\partial \ln(f(x))}{\partial x} = 0$$

$$\left[\frac{(\beta-1)}{\left(\frac{x-\epsilon}{\alpha}\right)}\right]\left(\frac{s}{\alpha}\right) - \frac{s}{\alpha} = 0$$

$$\frac{(\beta-1)}{\left(\frac{sx-\epsilon}{\alpha}\right)} = 1$$

$$x = \frac{\alpha(\beta-1)+\epsilon}{s} \qquad \blacksquare$$

4. Hazard and reliability Functions:

$$R(x) = 1 - F(X) = 1 - \frac{1}{\Gamma_\beta}\gamma\left(\beta, \frac{sx-\epsilon}{\alpha}\right)$$

Also that the hazard function is :

$$h(x) = \frac{f(x)}{R(x)} = \frac{\frac{s}{\alpha\Gamma\beta}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1} e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}}{1-\frac{\mathbf{1}}{\Gamma(\beta)}\gamma\left(\beta,\frac{sx-\epsilon}{\alpha}\right)}$$

he plot of the R(x) and h(x) is given in **Figure 7 and 8** respectively .

5. Lemma[9]

Let T be a continuous random variable with twice differentiable density function f (t). Define the quantity $\eta(t) = -\frac{\acute{f}(t)}{f(t)}$ where $\acute{f}(t)$ denote the first derivative of the density function with respect to t. suppose that the first derivative of $\eta(t)$ - name $\acute{\eta}(t)$-exists. Glaser gave the following result (for more details see Glaser (1980)) If $\acute{\eta}(t) < 0$, for all t >0 , then the hazard rate is monotonically decreasing failure rate (DFR).

1. If $\acute{\eta}(t) > 0$, for all t > 0 , then the hazard rate is monotonically increasing failure rate (IFR).
2. If there exists t_0 , such that $\acute{\eta}(t) > 0$ for all $(0 < t < t_0)$; $\acute{\eta}(t_0) = 0$ and $\acute{\eta}(t) < 0$ for all $(t > t_0)$. in addition to that $\lim_{t\to 0} f(t) = 0$; then the hazard rate is upside down bathtub shaped(UPT).
3. If there exists t_0 , such that $\acute{\eta}(t) < 0$ for all $(0 < t < t_0)$; $\acute{\eta}(t_0) = 0$ and $\acute{\eta}(t) > 0$ for all (t > t0 .addition to that limt→∞f(t)=∞ ; it consequence that the hazard rate is bathtub shaped (BT).

For the distribution we begin by computing the quantity $\eta(t)$; by first taking the derivative of the density function given in (2.1) with respect to t which is by :

$\acute{f}(t) = \frac{\partial f(t)}{\partial t} = \frac{s}{\alpha\Gamma(\alpha)}\left(\frac{st-\epsilon}{\alpha}\right)^{\beta-1}\frac{s}{\alpha}\ e^{-\left(\frac{st-\epsilon}{\alpha}\right)}\left(\frac{(\beta-1)\alpha}{st-\epsilon}-1\right)$.......(1)

Divided both side of equation (1) by the measure (- f(t)) we obtain :

$\eta(x) = -\frac{s}{\alpha}\left(\frac{(\beta-1)\alpha}{st-\epsilon}-1\right)$ take the derivative with respect to t yield :

$\acute{\eta}(x) = \frac{s^2(\beta-1)}{(st-\epsilon)^2}$

According to the values of the shape parameter β :

1. $\acute{\eta}(x) < 0$ if $\beta < 1$
2. $\acute{\eta}(x) > 0$ if $\beta > 1$

We can show that the hazard function does not dependent on the value x and it mixture on the part one and two of the above lemma.

6. Estimate the parameter

We used two method of estimation to estimate the parameter that is :

1. Method of moment
2. Method of maximum likelihood estimation

So that the estimate of parameters is follows as:

a. Estimate the parameters by using method of moment

Let $x_1, x_2, \dots, x_n \sim EGGD(.)$then

$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\alpha\beta+\epsilon}{s}$ ....*(1)*

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 = \left(\frac{\alpha\beta+\epsilon}{s}\right)^2 + \left(\frac{\alpha}{s}\right)^2\beta$ ………………*(2)*

$\frac{1}{n}\sum_{i=1}^{n} {x_i}^3 = \left(\frac{\alpha\beta+\epsilon}{s}\right)^3 + 3\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^2\beta + 3\left(\frac{\alpha}{s}\right)^3\beta^2 + 2\left(\frac{\alpha}{s}\right)^3\beta$ ……*(3)*

$\frac{1}{n}\sum_{i=1}^{n} {x_i}^4 = \left(\frac{\alpha\beta+\epsilon}{s}\right)^4 + 6\left(\frac{\epsilon}{s}\right)^2\left(\frac{\alpha}{s}\right)^2\beta + 12\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^3\beta^2 + 8\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)^3\beta + 11\left(\frac{\alpha}{s}\right)^4\beta^2 + 6\left(\frac{\alpha}{s}\right)^4\beta$ …………………*(4)*

By substitution (1) in (2) to have the equation (5)

$\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 = \left(\frac{\alpha}{s}\right)^2\beta$ …………*(5)*

Also substitution (1) and (5) in (3) and (4) we have :

$\frac{1}{n}\sum_{i=1}^{n} {x_i}^3 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^3 + 3\left(\frac{\epsilon}{s}\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) + 3\left(\frac{\alpha}{s}\beta\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) + 2\frac{\alpha}{s}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right)$…………*(6)*

$\frac{1}{n}\sum_{i=1}^{n} {x_i}^4 = \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^4 + 6\left(\frac{\epsilon}{s}\right)^2\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) + 12\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\beta\right)\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2\right) +$

$8\left(\frac{\epsilon}{s}\right)\left(\frac{\alpha}{s}\right)\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)+11\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)^2+6\left(\frac{\alpha}{s}\right)^2\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)\ldots(7)$

We can have β from (1) and (5)respectively as follows :

$\hat{\alpha}=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\ldots(8)$

Substitution in (2) we have

$\beta=\frac{s\bar{x}-\epsilon}{\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2}\quad\ldots(9)$

Which can be substitution (9) in (6) and (7)

$\hat{s}=\frac{2\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]^2}{\frac{1}{n}\sum_{i=1}^{n}x_i^3-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^3-3\bar{x}\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)}\quad\ldots(10)$

Which can be substituted (10),(8),(1)and(5) in (4)to have

$\epsilon=\frac{12\hat{s}\bar{x}+8\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}{12}\mp\frac{\sqrt{\left(12\hat{s}\bar{x}+8\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]\right)^2-24(\hat{s})^2\,s(x)+6\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}}{12}\ldots(11)$

Such that $\hat{s}$ is the above equation number(10)

Finally we get ϵby using the following formula after substitute (8),(10) and (11)

$\hat{\beta}=\frac{s\bar{x}-\epsilon}{s}\ldots(12)$

Finally that is

$\hat{\alpha}=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\quad\ldots(13)$

$\hat{s}=\frac{2\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]^2}{\frac{1}{n}\sum_{i=1}^{n}x_i^3-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^3-3\bar{x}\left(\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right)}\ldots(14)$

$\epsilon=\frac{12\hat{s}\bar{x}+8\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}{12}\mp\frac{\sqrt{\left(12\hat{s}\bar{x}+8\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]\right)^2-24(\hat{s})^2\,s(x)+6\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}}{12}\ldots(15)$

such that

$$s(x)=\frac{\frac{1}{n}\sum_{i=1}^{n}x_i^4-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^4-11\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}{\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}$$

$$=\frac{\frac{1}{n}\sum_{i=1}^{n}x_i^4-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^4}{\left[\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2\right]}-11$$

$\hat{\beta}=\frac{s\bar{x}-\epsilon}{s}\ldots(16)$ ■

b. Estimate the parameter by using method of maximum likelihood estimation .

Let $x_1,x_2,\ldots,x_n\sim EGGD(.)$then the likelihood function is

$L(x)=\prod_{i=1}^{n}f(x_i)=\prod\frac{s}{\alpha\Gamma\beta}\left(\frac{sx-\epsilon}{\alpha}\right)^{\beta-1}e^{-\left(\frac{sx-\epsilon}{\alpha}\right)}=\frac{s^n}{\alpha^n(\Gamma(\beta))^n}\left(\frac{s\sum x_i-n\,\epsilon}{\alpha^n}\right)^{\beta-1}e^{-\left(\frac{s\sum x_i-n\epsilon}{\alpha}\right)}$

$$\ln L(x)=n\ln s-n\ln\alpha-n\ln\Gamma(\beta)+(\beta-1)\ln\left(s\sum x_i-n\epsilon\right)-(\beta-1)n\ln\alpha-\frac{s\sum x_i-n\epsilon}{\alpha}$$

$$=n\ln s-n\ln\alpha-n\ln\Gamma\beta+\beta\ln\left(s\sum x_i-n\,\epsilon\right)-\ln(s\sum x_i-n\epsilon)-\beta n\ln\alpha+n\ln\alpha-\frac{s\sum x_i-n\epsilon}{\alpha}$$

$$=n\ln s-n\ln\Gamma\beta+\beta\ln(s\sum x_i-n\epsilon)-\ln(s\sum x_i-n\epsilon)-\beta n\ln\alpha-\frac{s\sum x_i-n\epsilon}{\alpha}$$

So that we have :

$$\frac{\partial\ln l(x)}{\partial\alpha}=\frac{-n\beta}{\alpha}+\frac{s\sum x-n\epsilon}{\beta^2}$$

$$\frac{\partial\ln l(x)}{\partial\beta}=-\frac{n}{\Gamma\beta}\Gamma'(\beta)+\ln(s\sum x_i-n\epsilon)-n\ln\alpha$$

$$\frac{\partial\ln l(x)}{\partial s}=\frac{n}{s}+\frac{\beta\sum x_i}{s\sum x_i-n\epsilon}-\frac{\sum x_i}{s\sum x_i-n\epsilon}-\frac{\sum x_i}{\alpha}$$

$$\frac{\partial \ln l(x)}{\partial \epsilon} = \frac{-n\beta}{s\sum x_i - n\epsilon} + \frac{n}{s\sum x_i - n\epsilon} + \frac{n}{\alpha}$$

So that to . solve the above system of equation we have

$$\frac{\partial \ln L(x)}{\partial \alpha} = 0 \quad , \frac{\partial \ln L(x)}{\partial \beta} = 0 \quad \frac{\partial \ln L(x)}{\partial \epsilon} = 0 \quad , \quad \frac{\partial \ln L(x)}{\partial s} = 0$$

So that for which we have that

$$\hat{\alpha} = s\left(\bar{x} - \frac{1}{n}\right) \ldots (1)$$

$$\hat{\beta} = \left(\frac{1}{1-n}\right) \quad \ldots (2)$$

$$\hat{s} = \frac{\frac{1}{1-n} - 1}{\left(\sum x_i - \left(\bar{x} - \frac{1}{n}\right)n\right)e^{n\frac{\Gamma'\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n}\right)}}} \ldots (3)$$

$$\hat{\epsilon} = (1-n)e^{n\frac{\Gamma'\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n}\right)}} + \frac{\frac{1}{n}\left(\frac{1}{1-n} - 1\right)}{\sum x_i - \left(\bar{x} - \frac{1}{n}\right)e^{n\frac{\Gamma'\left(\frac{1}{1-n}\right)}{\Gamma\left(\frac{1}{1-n}\right)}}} \quad \ldots .(4)$$

Conclusion

We can construct new distribution based dependent on the Pearson type three distribution (gamma distribution).

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The plotter:

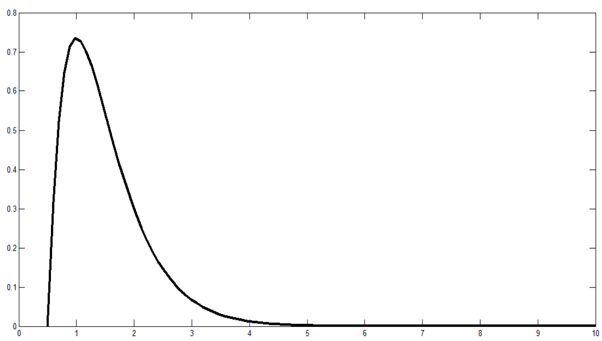

Figure 1: plot of p.d.f. of the **EGGD** with the above parameter $A = \alpha = 1$, $B = \beta = 2$, $ap = \epsilon = 1$,s=2.

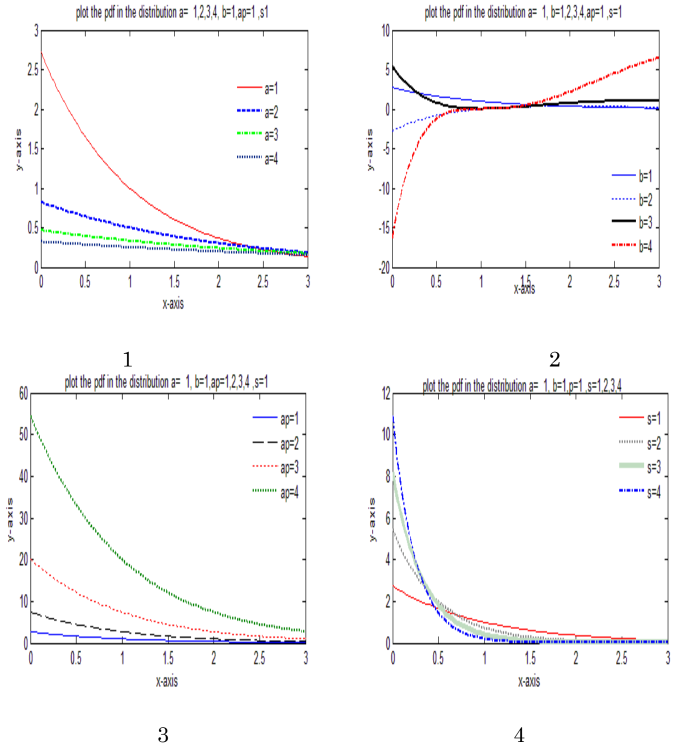

Figure 2: Plot of p.d.f. of the function with the all parameter changed with the parameter EGGD $a = \alpha$, $b = \beta, ap = \epsilon, s = s$

Figure 2 shows us that the p.d.f. of **EGGD** monotonically decreases at fixed β, s, ϵ while α increases **1** in part , and fixed α, β, s while ϵ increases in **3**.

And it takes different shapes in 2 based on change of β at fixed α, ϵ, s such that it monotonically decreases at $\beta = 1$, monotonically increases and then decreases to constantly decreases at $\beta = 2$, while it decreases and then increases at $\beta = 3$.Finally at $\beta = 4$, it monotonically increases , and then constantly increases on the interval (0.5,1.5) ,while it increases after that . It monotically decreases based on changes at fixed α, β, ϵ.

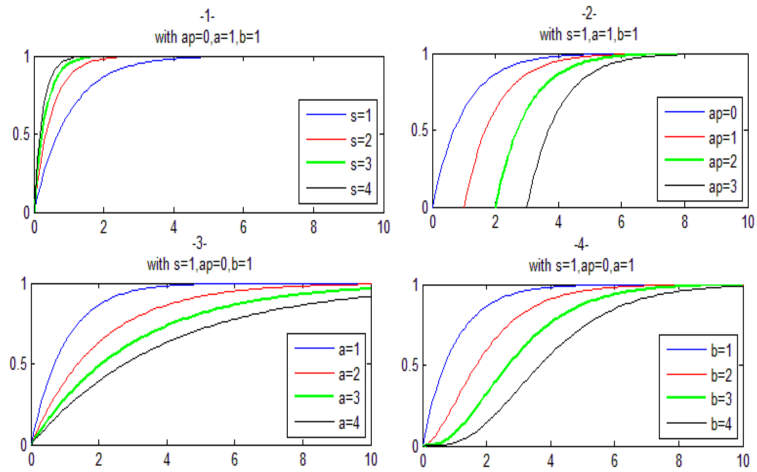

Figure 3 : plot of c.d.f. with the parameter change parameter.

Figure 3 shows us that the c.d.f is increasing with increasing x and fixed parameters $s, a = \alpha, b = \beta, ap = \epsilon$ in each case.

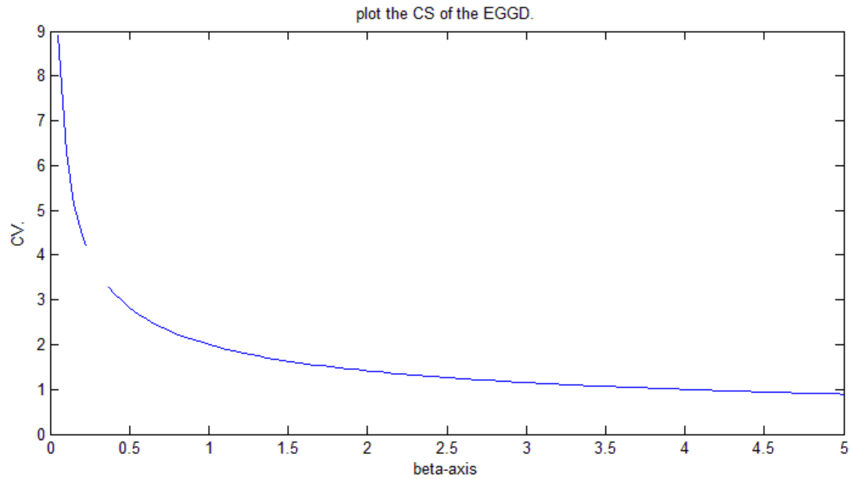

Figure 4 : plot of CS with parameters explained.

Figure 4 shows us the plot of the CS, that is just dependent on the parameter beta.

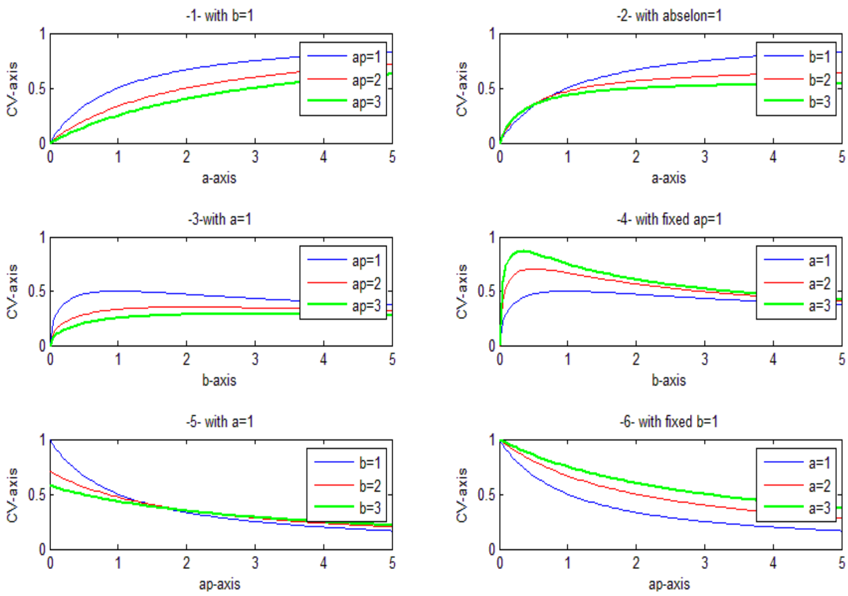

Figure 5 : plot of CV with the change parameter .

In figure 5 show that the CV is increasing in subfigure 1and 2 when the fixed parameter beta and abselon respectively but in sub figure 3 and 4the CV is increasing and decreasing finally in subfigure 5 and 6 the CV is decreasing .

The CK is given in **Figure 6**.

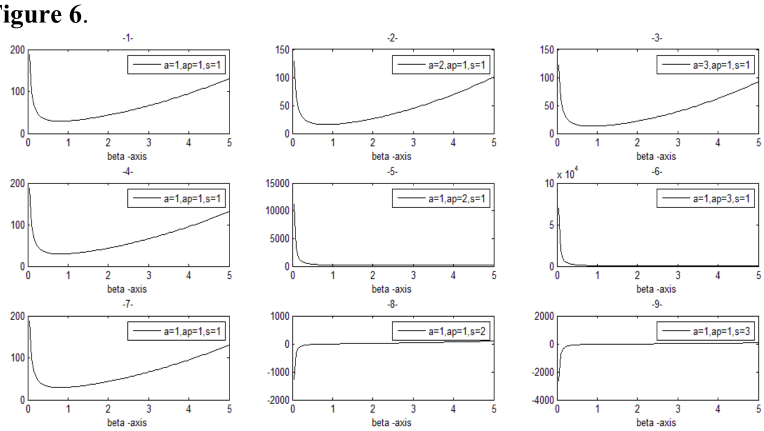

Figure 6: the coeffient of kurtosis with different parameter.

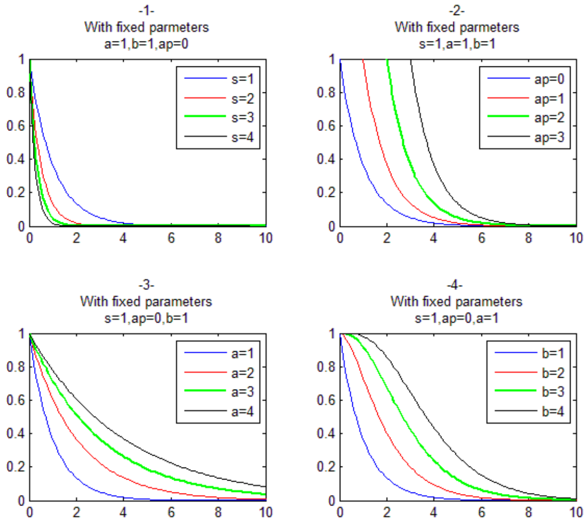

Figure7: plot of reliability function with change parameter.

Figure 7 show that the plots of the reliability function with increasing parameter of x. In **subfigure -1-** show that when the parameter s is increasing the curve in more coming down but in **subfigure -2-** the parameter ϵ is change just of the location of this function, In **subfigure -3-** that is show 2 when the parameter α increasing the $R(x)$ is more increasing finally in **subfigure -4-** the $R(x)$ takee the curve of the bundle with begging in the point (1,0) and end in the point (0.10) with fixed parameter $s = 1, \epsilon = 0$ and $\alpha = 1$.

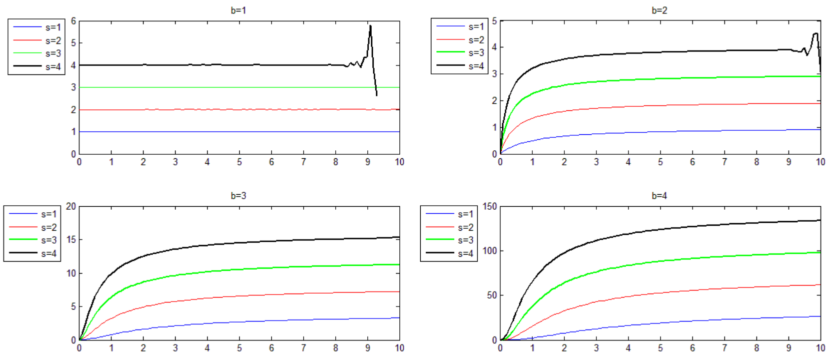

Figure 8: plot of hazard function with the parameter$\alpha = 1$ and $\epsilon = 0$.

In **Figure 8** show that the $h(x)$ with fixed parameter $\alpha = 1$ and $\epsilon = 0$ in the sub figure when the parameter beta =1 the curve take that the line with increasing parameters $'s'$ but in the parameter $s = 4$ that is taken the shape of line and increasing to the point 9.1 approximation and decreasing , when beta $= 2$ that is increasing with all parameters just in the parameter $s = 4$ that is increasing and more increasing, in the figures with the parameter $b = 3$ and 4 that is increasing with increasing parameter s.