

Fixed Point of Expansive Type Mapping in *N*-Cone Metric Space

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Abstract

In this paper, we define expansive mappings in the setting of N-cone metric spaces analogous to expansive mappings in metric spaces. We also obtain some results for two mappings to the setting of N-cone metric space. These results extend main results of Wang et al [14] into the structure of N-cone metric space.
Keywords: Fixed Point, N-cone metric space, Expansive mapping.

1. Introduction and Preliminaries

Over the past two decades a considerable amount of research works for the improvement of fixed point theory have executed by several authors. There has been a number of generalizations of the usual notion of metric spaces such as Gahler [3,4] (called 2-metric spaces). But different authors proved that there is no relation between these two functions, for instance, Ha et al. in [5] show that 2-metric need not be continuous function, further there is no easy
relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D thesis introduce a new class of generalized metric space called D-metric space [1,2]. However, Mustafa and Sims in [10] have pointed out that most of the results claimed by Dhage and others in D-metric spaces are invalid. To overcome these fundamental flaws, they introduced a new concept of generalized metric space called G-metric space[11] and obtained several interesting fixed point results in this structure. Recently in 2012 Sedghi, Shobe and Aliouche have introduced the notion of an S-metric space and proved that this notion is a generalization of a G-metric space. Also, they have proved some properties of S-metric spaces and some fixed point theorems for a self-map on an S-metric space. Another such generalization initiated by Huang and Zhang[6], replacing the set of real numbers by an ordered Banach space, called cone metric space and gave some fixed point theorems for contractive type mappings in a normal cone metric space. In [12], Rezapour and Hamlbarani omitted the assumption of normality in cone metric space. Very recently in 2013 Neeraj Malviya and Brain Fisher[9] introduced an appropriate concept of N-cone metric space which is a generalization of S-metric space and cone metric space. They also proved some properties of these spaces and established some fixed point results for asymptotic regular maps in the setting of N-cone metric space.

On the other hand, the research about fixed points of expansive mapping was initiated by Machuca (see [8]). Later Jungck discussed fixed points for other forms of expansive mapping (see [7]). In 1982, Wang et al. (see [14]) presented some interesting work on expansive mappings in metric spaces which correspond to some contractive mapping in [13]. Also, Zhang has done considerable work in this field. In order to generalize the results about fixed
point theory, Zhang (See [15]) published his work Fixed Point Theory and Its Applications, in which the fixed point problem for expansive mapping is systematically presented in a chapter. As applications, he also investigated the existence of solutions of equations for locally condensing mapping and locally accretive mapping. In present paper we define expansive map and investigate fixed points for these maps in N-cone metric space.

Definition 1.4[9]: Let X be a nonempty set. An N-cone metric on X is a function $N: X^3 \to E$, that satisfies the following conditions for all $x, y, z, a \in X$.

1. $N(x,y,z) \geq 0$
2. $N(x,y,z) = 0$ if and only if $x = y = z$
3. $N(x,y,z) \leq N(x,x,a) + N(y,y,a) + N(z,z,a)$

Then the function N is called an N-cone metric and the pair (X, N) is called an N-cone metric space.

Remark 1.5[9]: It is easy to see that every generalized D^*-metric space is an N-cone metric space but in general, the converse is not true, see the following example.

Example 1.6[9]: Let $E = R^3, P = \{(x, y, z) \in E, x, y, z \geq 0\}, X = R$ and $N: X \times X \times X \to E$ is defined by

$$N(x, y, z) = \big(\alpha(|y + z - 2x| + |y - z|), \beta(|y + z - 2x| + |y - z|), \gamma(|y + z - 2x| + |y - z|)\big)$$

where α, β, γ are positive constants. Then (X, N) is an N-cone metric space but it is not a generalized D^*-metric space because N is not symmetric.

Proposition 1.7[9]: If (X, N) is an N-cone metric space, then for all $x, y, z \in X$, we have $N(x, x, y) = N(y, y, x)$.

Definition 1.8[9]: Let (X, N) be an N-cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N(x_n, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to x and x is the limit of $\{x_n\}$. We denote this by $x_n \to x$ as $(n \to \infty)$.

Lemma 1.9[9]: Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ also converges to y then $x = y$. That is the limit of $\{x_n\}$, if exists is unique.

Definition 1.10[9]: Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$ there is N such that for all $m, n > N, N(x_n, x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition 1.11[9]: Let (X, N) be an N-cone metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete N-cone metric space.

Lemma 1.12[9]: Let (X, N) be an N-cone metric space and $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Definition 1.13[9]: Let (X, N) and (X', N') be N-cone metric spaces. Then a function $f: X \to X'$ is said to be continuous at a point $x \in X$ if and only if it is sequentially continuous at x, that is whenever $\{x_n\}$ is convergent to x we have $\{fx_n\}$ is convergent to $f(x)$.

Lemma 1.14[9]: Let (X, N) be an N-cone metric space and P be a normal cone with normal constant k. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and suppose that $x_n \to x, y_n \to y$ as $n \to \infty$. Then $N(x_n, x_n, y_n) \to N(x, x, y)$ as $n \to \infty$.

Remark 1.15[9]: If $x_n \to x$ in an N-cone metric space X then every subsequence of $\{x_n\}$ converges to x in X.

Proposition 1.16[9]: Let (X, N) be an N-cone metric space and P be a cone in a real Banach space E. If $u \leq v$, $v \ll w$ then $u \ll w$.

Lemma 1.17[9]: Let (X, N) be an N-cone metric space, P be an N-cone in a real Banach space E and $k_1, k_2, k_3, k_4, k > 0$. If $x_n \to x, y_n \to y, z_n \to z$ and $p_n \to p$ in X and

$$ka \leq k_1 N(x_n, x_n, x) + k_2 N(y_n, y_n, y) + k_3 N(z_n, z_n, z) + k_4 N(p_n, p_n, p) \tag{1.4}$$

then $a = 0$.

Expansive map: We define expansive map in N-cone metric space as follows

Definition 1.18: Let (X, N) be an N-cone metric space. A map $f: X \to X$ is said to be an expansive mapping if there exists a constant $L > 1$ such that $N(fx, fx, fy) \geq LN(x, x, y)$ for all $x, y \in X$.

Example 1.8: Let (X, N) be an N-cone metric space as defined in example 1.6. Define a self map $f: X \to X$ by $fx = \beta x$ where $\beta > 1$, for all $x \in X$. Clearly f is an expansive map in X.

2. Main Results

The first main result is

Theorem 2.1. Let (X, N) be a complete N-cone metric space with respect to a cone P contained in a real Banach space E. Let $f$and g be two surjective self map of X satisfying.

$$N(fx, fx, gy) + k[N(x, x, gy) + N(y, y, fx)] \geq aN(x, x, fx) + bN(y, y, gy) + cN(x, x, y) \qquad (2.1.1)$$

for every $x, y \in X, x \neq y$ where $a, b, c, k \geq 0$, and $a < 2k + 1, b < 2k + 1$ and $3k + 1 < c$. Then f and g have a unique common fixed point in X.

Proof: We define a sequence $\{x_n\}$ as follows for $n = 0,1,2,3, \ldots$

$$x_{2n} = fx_{2n+1}, x_{2n+1} = gx_{2n+2} \qquad (2.1.2)$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n then we see that x_{2n} is a fixed point of f and g. Therefore, we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal.

Now we put $x = x_{2n+1}$ and $y = x_{2n+2}$ in (2.1.1) we get

$$N(fx_{n+1}, fx_{2n+1}, gx_{2n+2}) + k[N(x_{2n+1}, x_{2n+1}, gx_{2n+2}) + N(x_{2n+2}, x_{2n+2}, fx_{2n+1})]$$
$$\geq aN(x_{2n+1}, x_{2n+1}, fx_{2n+1})$$
$$+bN(x_{2n+2}, x_{2n+2}, gx_{2n+2}) + cN(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$N(x_{2n}, x_{2n}, x_{2n+1}) + k[N(x_{2n+1}, x_{2n+1}, x_{2n+1}) + N(x_{2n+2}, x_{2n+2}, x_{2n})] \geq aN(x_{2n+1}, x_{2n+1}, x_{2n})$$
$$+bN(x_{2n+2}, x_{2n+2}, x_{2n+1}) + cN(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow N(x_{2n}, x_{2n}, x_{2n+1}) + k[2N(x_{2n+2}, x_{2n+2}, x_{2n+1}) + N(x_{2n}, x_{2n}, x_{2n+1})] \geq aN(x_{2n+1}, x_{2n+1}, x_{2n})$$
$$+bN(x_{2n+2}, x_{2n+2}, x_{2n+1}) + cN(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow (1 + k - a)N(x_{2n}, x_{2n}, x_{2n+1}) \geq (b + c - 2k)N(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

$$\Rightarrow N(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \frac{(1+k-a)}{(b+c-2k)}N(x_{2n}, x_{2n}, x_{2n+1})$$

$$\Rightarrow N(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq K_1 N(x_{2n}, x_{2n}, x_{2n+1})$$

where $K_1 = \frac{(1+k-a)}{(b+c-2k)} < 1$ $(As \;\; a + b + c > 1 + 3k)$
Similarly we can calculate

$$\Rightarrow N(x_{2n+2}, x_{2n+2}, x_{2n+3}) \leq K_2 N(x_{2n+1}, x_{2n+1}, x_{2n+2})$$

where $K_2 = \frac{(1+k-a)}{(b+c-2k)} < 1$ $(As \;\; a + b + c > 1 + 3k)$
and so on

So, in general

$$\Rightarrow N(x_n, x_n, x_{n+1}) \leq KN(x_{n-1}, x_{n-1}, x_n) \text{ for } n = 1,2,3, \ldots$$

where $K = max\{K_1, K_2\}$ then $K < 1$

$$\Rightarrow N(x_n, x_n, x_{n+1}) \leq K^n N(x_0, x_0, x_1)$$

Now we shall prove that $\{x_n\}$ is a cauchy sequence. For this for every positive integer p, we have

$$\begin{aligned} N(x_n, x_n, x_{n+p}) &\le 2N(x_n, x_n, x_{n+1}) + 2N(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + 2N(x_{n+p-2}, x_{n+p-2}, x_{n+p-1}) \\ &+ N(x_{n+p-1}, x_{n+p-1}, x_{n+p}) \\ &\le 2N(x_n, x_n, x_{n+1}) + 2N(x_{n+1}, x_{n+1}, x_{n+2}) + \cdots + 2N(x_{n+p-2}, x_{n+p-2}, x_{n+p-1}) \\ &+ 2N(x_{n+p-1}, x_{n+p-1}, x_{n+p}) \\ &\le [2K^n + 2K^{n+1} + \cdots + 2K^{n+p-2} + 2K^{n+p-1}]N(x_0, x_0, x_1) \\ &= 2K^n[1 + K + K^2 + \cdots + K^{p-1}]N(x_0, x_0, x_1) \\ &< \frac{2K^n}{(1-K)} N(x_0, x_0, x_1) \\ &\Rightarrow \|N(x_n, x_n, x_{n+p})\| < \frac{2K^n}{(1-K)} \mathbf{K}\, \|N(x_0, x_0, x_1)\| \end{aligned}$$

Which implies that $\|N(x_n, x_n, x_{n+p})\| \to 0$ as $n \to \infty$.
Since $\frac{2K^n}{(1-K)} \mathbf{K}\, \|N(x_0, x_0, x_1)\| \to 0$ as $n \to \infty$.
Therefore $\{x_n\}$ is a cauchy sequence in X, which is complete space, so $\{x_n\} \to x \in X$.

Existence of fixed point: Since f and g are surjective maps and hence there exist two points y and y' in X such that

$$x = fy \text{ and } x = gy' \tag{2.1.3}$$

Consider,

$$\begin{aligned} N(x_{2n}, x_{2n}, x) &= N(fx_{2n+1}, fx_{2n+1}, gy') \\ &\ge -k[N(x_{2n+1}, x_{2n+1}, gy') + N(y', y', fx_{2n+1})] + aN(x_{2n+1}, x_{2n+1}, fx_{2n+1}) + bN(y', y', gy') \\ &+cN(x_{2n+1}, x_{2n+1}, y') \\ &= -k[N(x_{2n+1}, x_{2n+1}, gy') + N(y', y'x_{2n})] + aN(x_{2n+1}, x_{2n+1}, x_{2n}) + bN(y', y', gy') \\ &+cN(x_{2n+1}, x_{2n+1}, y') \end{aligned}$$

As $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are subsequences of $\{x_n\}$ as $n \to \infty$, $\{x_{2n}\} \to x$, $\{x_{2n+1}\} \to x$.

Therefore

$$N(x, x, x) \ge -k[N(x, x, x) + N(y', y', x)] + aN(x, x, x) + bN(y', y', x) + cN(x, x, y') \quad [by\ Lemma\ 1.14\ and\ 2.1.3]$$

$$0 \ge (b + c - k)N(x, x, y') \quad [by\ Prob\ 1.7]$$

$$\Rightarrow N(x, x, y') = 0 \quad (As\ \ 1 + 3k < a + b + c < 2k + 1 + b + c \Rightarrow k < b + c)$$

$$\Rightarrow x = y' \tag{2.1.4}$$

In an exactly similar way ($Using\ b < 2k + 1$) we can prove that,

$$x = y \tag{2.1.5}$$

The fact (2.1.3) along with (2.1.4) and (2.1.5) shows that x is a common fixed point of f and g.

Uniqueness: Let z be another common fixed point of f and g, that is

$$fz = z \text{ and } gz = z \tag{2.1.6}$$

$$\begin{aligned} N(x, x, z) &= N(fx, fx, gz) \\ &\ge -k[N(x, x, gz) + N(z, z, fx)] + aN(x, x, fx) + bN(z, z, gz) + cN(x, x, z) \end{aligned}$$

$$\begin{aligned} &\Rightarrow N(x, x, z) \ge -k[N(x, x, z) + N(z, z, x)] + aN(x, x, x) + bN(z, z, z) \\ &\qquad + cN(x, x, z) \quad [by\ 2.1.6] \\ &\Rightarrow (1 - c + 2k)N(x, x, z) \ge 0 \\ &\Rightarrow N(x, x, z) = 0 \quad [As\ \ c > 3k + 1 > 2k + 1] \\ &\Rightarrow x = z \end{aligned}$$

This completes the proof of the theorem 2.1.

Corollary 2.2. Let (X, N) be a complete N-cone metric space with respect to a cone P contained in a real Banach space E. Let f and g be two surjective self maps of X satisfying.

$$N(fx, fx, gy) \geq cN(x, x, y) \tag{2.2.1}$$

where $c > 1$. Then f and g have a unique common fixed point in X.

Proof: If we put $k, a, b = 0$ in theorem 2.1 then we get above corollary 2.2

Corollary 2.3. Let (X, N) be a complete N-cone metric space with respect to a cone P contained in a real Banach space E. Let f be a surjective self map of X satisfying.

$$N(fx, fx, fy) \geq cN(x, x, y) \tag{2.3.1}$$

where $c > 1$. Then f has a unique fixed point in X.

Proof: If we put $f = g$ in corollary 2.2 then we get above corollary 2.3 which is an extension of theorem 1 of Wang et al. [14] in N-cone metric space.

Corollary 2.4. Let (X, N) be an N-cone metric space and $f: X \to X$ be a surjection. Suppose that there exist a positive integer n and a real number $C > 1$ such that $N(f^n x, f^n x, f^n y) \geq CN(x, x, y)$ for all $x, y \in X$. Then f has a unique fixed point in X.

Proof: From corollary 2.3 f^n has a unique fixed point z. But $f^n(fz) = f(f^n z) = fz$, so fz is also a fixed point of f^n. Hence $fz = z$, z is a fixed point of f. Since the fixed point of f is also fixed point of f^n, the fixed point of f is unique.

Corollary 2.5. Let (X, N) be a complete N-cone metric space with respect to a cone P contained in a real Banach space E. Let f and g be two surjective self maps of X satisfying.

$$N(fx, fx, gy) \geq aN(x, x, fx) + bN(y, y, gy) + cN(x, x, y) \tag{2.5.1}$$

for every $x, y \in X, x \neq y$ where $a, b, c \geq 0$ and $c > 1$. Then f and g have a unique common fixed point in X.
Proof: The proof is similar to proof of the theorem 2.1.

Corollary 2.6. Let (X, N) be a complete N-cone metric space with respect to a cone P contained in a real Banach space E. Let f be surjective self map of X satisfying.

$$N(fx, fx, fy) \geq aN(x, x, fx) + bN(y, y, fy) + cN(x, x, y) \tag{2.6.1}$$

for every $x, y \in X, x \neq y$ where $a, b, c \geq 0$ and $c > 1$. Then f has a unique fixed point in X.

Proof: If we put $f = g$ in corollary 2.5 then we get above corollary 2.6 which is an extension of theorem 2 of Wang et al. [14] in N-cone metric space.

The following example demonstrates corollary 2.3.

Example 2.7. Let $E = R^3, P = \{(x, y, z) \in E, x, y, z \geq 0\}$ and $X = R$ and $N: X \times X \times X \to E$ is defined by

$$N(x, y, z) = \big(\alpha(|x - z| + |y - z|), \beta(|x - z| + |y - z|), \gamma(|x - z| + |y - z|)\big)$$

where α, β, γ are positive constants. Then (X, N) is an N-cone metric space. Define a self map f on X as follows $fx = 2x$ for all $x \in X$. Clearly f is an expansive mapping. If we take $c = 2$ then condition (2.3.1) holds trivially good and 0 is the unique fixed point of the map f.

Remark 1: If mappings are continuous in theorem 2.1 then existence of fixed point follows very easily. As shown below.

$$x = \lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} fx_{2n+1} = f\lim_{n\to\infty} x_{2n+1} = fx \qquad (as\ \ n \to \infty \{x_{2n+1}\} \to x)$$

Similarly

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} gx_{2n+2} = g\lim_{n\to\infty} x_{2n+2} = gx \qquad (as\ \ n \to \infty \{x_{2n+2}\} \to x)$$

Remark 2: In corollary 2.6 we proved the fixed point is unique by using only $c > 1$ and there is no need of $a < 1$ and $b < 1$, so it extend and unify the theorem 2 of Wang et al.[14] in N-cone metric space.

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