A Weaker Version of Continuity and a Common Fixed Point Theorem

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Abstract
A generalization of results of the second author [4] and the authors [5] have been proved through the notions of property E.A. and weak compatibility and restricting the orbital completeness of the space.
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1. Introduction

Let (X, d) be a metric space. Given $x_0 \in X$ and self-maps A, S and T on X, if there exist points x_1, x_2, …, x_n, … in X such that

$$Sx_{2n-2} = Ax_{2n-1},\ Tx_{2n-1} = Ax_{2n} \text{ for } n = 1, 2, 3, \ldots, \qquad \ldots \qquad (1)$$

Then the sequence $\langle Ax_n \rangle_{n=1}^{\infty}$ is an (S,T)-orbit at x_0 with respect to A.

Definition 1.1. The space X is (S,T)- orbitally complete with respect to A at x_0 [4] if every Cauchy sequence in some orbit of the form from equation (1) converges in X.

Definition 1.2. The pair (S,T) is Asymptotically regular at x_0 with respect to A [4] if the orbit (1) satisfies the condition that $d(Ax_n + Ax_{n+1}) \to 0$ as $n \to \infty$.

Definition 1.3. Self-map A on X is orbitally continuous at x_0 if it is continuous at every point of some orbit at x_0.

Obviously every continuous self-map on X is orbitally continuous at each $x_0 \in X$. However the converse is not true as seen from [4].

Definition 1.4. Self-maps A and S are compatible [2] if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$ whenever $\lim_{n\to\infty} d(Sx_n, Ax_n) = 0$.

Definition 1.5. Self-maps A and S are said to be weakly compatible [3] if they commute at their coincidence points.

With these notions the following theorem was proved in [4]:

Theorem A: Let A, S and T be self-maps on X satisfying the inequality.

$$d(Sx, Ty) \le c \max\{d(Ax, Ay), d(Ax, Sx), d(Ay, Ty), d(Ax, Ty), d(Ay, Sx)\}$$
$$\text{for all } x, y \in X, \ldots \qquad (2)$$

where $0 \le c < 1$. Suppose that at $x_0 \in X$,

(a) The pair (S,T) is asymptotically regular with respect to A.
(b) The space X is orbitally complete
(c) A is orbitally continuous
(d) Either (A,S) or (A,T) is compatible pair.

Then A, S and T will have a unique common fixed point.

In this paper, we prove a generalization of Theorem A by using the property EA (cf. Section 2), relaxing the condition (b), removing the condition (c) and weakening the condition (d).

2. Main result

Definition 2.1. Self-maps A and S satisfy property E.A. [1] if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$.

Theorem B: Let A, S and T be self-maps on X satisfying the inequality (2).
Suppose that

(e) either (A, S) or (A, T) satisfies property E.A.

(f) A(X) is orbitally complete subspace of X.

and

(g) (A, S) or (A, T) is weakly compatible.

Then all the three self-maps will have a unique common fixed point.

Proof. By the property EA for the pair (A, S) we have

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z \text{ for some } z \in X. \quad \ldots \quad \textbf{(3)}$$

Let $\lim_{n\to\infty} Tx_n = p$. Now we prove that p = z.

In fact from the inequality (2) with $x = x_n$ and $y = x_n$ we get

$$d(Sx_n, Tx_n) \le c \max\{d(Ax_n, Ax_n), d(Ax_n, Sx_n), d(Ax_n, Tx_n), d(Ax_n, Tx_n), d(Ax_n, Sx_n)\}$$

applying the limit as $n \to \infty$ and then using (3), this gives

$$d(z, p) \le c \max\{0, 0, d(z, p), d(z, p), 0\} \text{ or } d(z, p) \le c\, d(z, p) \text{ so that } p = z.$$

Thus $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z.$ … **(4)**

Equation (4) can also be obtained in similar lines whenever (A, T) satisfies property E.A.

From the orbital completeness (f) we see that $z \in A(X)$ so that $z = Au$ for some $u \in X$..

Now, taking $x = u$ and $y = x_n$ in (2),

$$d(Su, Tx_n) \le c \max\{d(Au, Ax_n), d(Au, Su), d(Ax_n, Tx_n), d(Au, Tx_n), d(Ax_n, Su)\}$$

Applying the limit as $n \to \infty$, using (4) and $Au = z$ in this, we get

$$d(Su, Au) \le c \max\{0, d(Au, Su), 0, 0 d(Au, Su)\} \Rightarrow d(Su, Au) \le c\, d(Au, Su) \text{ or } Au = Su = z.$$

Then from the weak compatibility of (A, S), we see that $ASu = SAu$ or $Az = Sz$.

Again writing $x = y = z$ in the inequality (2) and using Az = Sz, it follows that

$$d(Sz, Tz) \le c \max\{0, 0, d(Sz, Tz), d(Sz, Tz), 0\} = c d(Sz, Tz) \text{ so that } Sz = Tz.$$

That is, $Az = Sz = Tz$. … **(5)**

Taking $x = x_n, y = z$ in (2), we have

$$d(Sx_n, Tz) \le c \max\{d(Ax_n, Az), d(Ax_n, Sx_n), d(Az, Tz), d(Ax_n, Tz), d(Az, Sx_n)\}$$

As limit $n \to \infty$, this along with (4) and (5) implies that

$$d(z, Tz) \le c \max\{d(z, Tz), d(z, z), d(Az, Az), d(z, Tz), d(Tz, z)\} = c d(z, Tz) \text{ or } z = Tz.$$

Thus z is a common fixed point of self-maps A, S and T.

On the other hand, with minor changes in the above proof we can prove that Au = Tu = z.

Suppose that the pair (A, T) is weakly compatible. Then it follows that $ATu = TAu$ or Az = Tz.

Proceeding as in the previous steps, we get that $Az = Tz = Sz = z$.

Uniqueness: Let z, z′ be two common fixed points of A, S, and T. Then from (2) with x = z and y = z′, we get

$$d(Sz, Tz') \le c \max\{d(Az, Az'), d(Az, Sz), d(Az', Tz'), d(Az, Tz'), d(Az', Sz)\}$$

$$\Rightarrow \quad d(z, z') \le c \max\{d(z, z'), d(z, z), d(z', z'), d(z, z'), d(z', z)\} = c d(z, z') \text{ or } z = z'.$$

Hence the fixed point is unique.

Remark 2.1. Suppose at some $x_0 \in X$, (a) holds good then from the proof of Theorem A, the sequnce $\langle Ax_n \rangle_{n=1}^{\infty}$ defined by (1) is Cauchy and hence by the orbital completeness, we can find some $w \in X$ such that

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Ax_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = w,$$

which immediately implies that the pairs (A, T) and (S, T) satisfy the property E.A.

Also every compatible pair is weakly compatible. Therefore a unique common fixed point follows from Theorem B, under the restricted orbital completeness of the space X. It is interesting to note that the orbital continuity is not needed to obtain a common fixed point.

Corollary(Theorem C, [5]): Let A, S and T be self-maps on X satisfying (2). Given $x_0 \in X$, suppose that there is an orbit with choice (1) and the conditions (a) and (b) of Theorem A hold good at x_0. If A is onto and the condition (g) of Theorem B hold good, then A, S and T will have unique common fixed point.

Proof: Since A is onto we see that $A(X) = X$. In view of Remark 2.1, a unique common fixed point follows from Theorem B.

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