Iterative Procedure for Uniform Continuous Mapping.

Chika Moore$^1$ Naubia Agatha C$^1*$ and. Mogbademu Adesanmi$^2$

1. Department of Mathematics, Nnamdi Azikiwe University P.M.B 5025 Awka, Nigeria.
2. Department of Mathematics, University of Lagos, Lagos Nigeria.

* E-mail of the corresponding author: obijiaakuagatha@ymail.com

Abstract

Let K be a closed convex nonempty subset of a normed linear space E and let $\{T_i\}_{i=1}^N$ be a finite family of self maps on K such that $T_1$ is a uniformly continuous uniformly hemicontractive map and $T(K)$ is a bounded set with $F(\bigcap_{i=1}^N F(T_i)) \neq \emptyset$, sufficient conditions for the strong convergence of an N-step iteration process to a fixed common point of the family are proved.

Keywords: key words, uniformly continuous, uniformly hemicontractive, finite family, common fixed point, Noor iteration, strong convergence.

1. Introduction

Let $E^*$ denote the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and $E^*$. The single-valued normalized duality mapping is denoted by $j$. A mapping $T : E \to E^*$ is called strongly pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a constant $K \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq (1 - k) \|x - y\|^2,$$

T is called strongly $\psi - pseudocontractive$ if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\psi : [0,1) \to [0,1)$ with $\psi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - \psi(\|x - y\|) \|x - y\|.$$  

It is called generalized strongly $\psi - pseudocontractive$ or uniformly pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\psi : [0,1) \to [0,1)$ with $\psi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x - y\|^2 - \psi(\|x - y\|).$$

Every strongly $\phi - pseudocontractive$ operator is a uniformly $\psi - pseudocontractive$ operator with $\psi : [0,1) \to [0,1)\] \psi(s) = \phi(s)s$, but not conversely (see [13]).

These classes of operators have been studied by several authors (see, for example [3], [4], [7], [13], [19], [23], [24] and references therein).

If I denotes the identity operator, then T is strongly pseudocontractive, strongly $\phi - pseudocontractive$ , generalized strongly $\psi - pseudocontractive$ and only if $(I - T)$ is strongly accretive, strongly $\phi - accretive$ , generalized strongly $\phi - accretive$ operators respectively. The interest in pseudocontractive mappings is mainly due to their connection with the important class of nonlinear accretive operators. In recent years, many authors have given much attention to approximate the fixed points of non-linear operators in Banach space using the Ishikawa and Mann iterative schemes (see, for example [8], [10], [11] and references therein).

Noor [14] introduced the three-step iteration process for solving nonlinear operator equations in real Banach as follows;

Let E be a real Banach space, K a nonempty convex subset of E and $T : K \to K$, a mapping. For an arbitrary $x_0 \in K$, the sequence $\{x_n\}_{n=0}^\infty \subset K$ defined by...
\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \]
\[ y_n = (1 - \beta_n)x_n + \beta_n Tz_n, \]
\[ z_n = (1 - \gamma_n)x_n + \gamma_n Tx_n. \]  

(1)

Where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0,1]\) is called the three-step iteration (or the Noor iteration). When \( \gamma_n = 0 \), then the three-step iteration reduces to the Ishikawa iterative sequence if \( \beta_n = \gamma_n = 0 \), then the three-step iteration reduces to the Mann iteration.

Rafiq [21], recently introduced a new type of iteration-the modified three-step iteration process which is defined as follows;

Let \( T_1, T_2, T_3 : K \to K \) be three mappings for any given \( x_0 \in K \), the modified three-step iteration is defined by

\[ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1y_n, \]
\[ y_n = (1 - \beta_n)x_n + \beta_n T_2z_n, \]
\[ z_n = (1 - \gamma_n)x_n + \gamma_n T_3x_n \]  

(2)

Where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three real sequences satisfying some conditions. It is clear that the iteration scheme (2) includes Noor as special case.

Glowinski and Le Tallec [5] used the three-step iteration schemes to solve elastoviscoplasticity, liquid crystal and eigen-value problems. They have shown that the three-step approximation scheme performs better than the two-step and one-step iteration methods. Haubrige et al. [6] have studied the convergence analysis of three-step iteration schemes and applied these three-step iteration to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations also lead to highly parallelized algorithms under certain conditions. Thus, it is clear that three-step schemes play an important part in solving various problems, which arise in pure and applied sciences.

Recently, Xue and Fan [23] used the iteration procedure define by (2) in their theorem as stated below.

**Theorem 1.1** Let \( X \) be a real Banach space and \( K \) be a nonempty closed convex subset of \( X \). Let \( T_1, T_2 \) and \( T_3 \) be strongly pseudocontractive self maps of \( K \) with \( T_1(K) \) bounded and \( T_1, T_2 \) and \( T_3 \) uniformly continuous. Let \( \{x_n\} \) be defined by (2), where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three real sequences in \([0,1]\) such that (i) \( \alpha_n, \beta_n \to 0 \) as \( n \to \infty \), (ii) \( \sum_{n=0}^{\infty} \alpha_n = \infty \), and \( F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset \) then the sequence \( \{x_n\} \) converges strongly to the common fixed point of \( T_1, T_2 \) and \( T_3 \).

Olaleru and Mogbademu [17] established the strong convergence of a modified Noor iterative process when applied to three generalized strongly \( \phi - \text{pseudocontractive} \) operators or generalized strongly \( \phi - \text{accretive} \) operators in Banach space. Thus, generalizing the recent results of Fan and Xue (2009). In fact the stated and proved the following result.

**Theorem 1.2** Let \( E \) be a real Banach space, \( K \) a nonempty closed convex subset of \( E \), \( \phi - \text{pseudocontractive} \) mappings such that \( T_i(K) \) is bounded. Let \( \{x_n\} \) be a sequence defined by (2) where \( \{\alpha_n\}, \{\beta_n\} \) and \( \{\gamma_n\} \) are three sequences in \([0,1]\) satisfying the following conditions:

\[ \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n = \lim_{n \to \infty} \gamma_n = 0. \]

\[ \sum_{n=0}^{\infty} \alpha_n = \infty \text{ if } F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset, \text{ then the sequence } \{x_n\} \text{ converges to the unique common fixed points } T_1, T_2 \text{ and } T_3. \]

**Remark 1.1** In theorem 1.2, it is required that;

All the 3 maps be generalized strongly \( \phi - \text{pseudocontractive} \) with the same function \( \phi \) (which is rather strong function).

All the 3 maps are required to be uniformly continuous (and thus bounded).

Our purpose in this paper is to extend is to extend and generalized the result of Olaleru and Mogbademu (17) in the following ways:

We introduce m-step iteration scheme.

We extend the result to any finite family of m-maps.

The conditions of our theorems are less restrictive and more general than the one used in (17), (23). For instance, the demand that the three maps must be uniformly continuous is weakened by allowing some of the maps to be free.
1.1. The M-Step Iteration Process

Let \( K \) be a non-empty convex subset of a normed linear space \( E \) and let \( T: K \rightarrow K \) be a map. For any given \( x_0 \in K \). The m-step iteration process is defined by
\[
y_{n,0} = x_n \\
y_{n,i} = (1 - \alpha_{n,i})x_n + \alpha_{n,i} T y_{n,i-1}; i = 1, 2, ..., m \\
y_{n,m} = x_{n+1} = y_{n+1,0}
\]
where \( n + 1 = i \mod m; n \geq 0 \) (3)

For a finite family \( \{T_i\}_{i=1}^m \) of m-maps, the m-step iterative process becomes
\[
y_{n,0} = x_n \\
y_{n,i} = (1 - \alpha_{n,i})x_n + \alpha_{n,i} T_{m+1-i} y_{n,i-1}; i = 1, 2, ..., m \\
y_{n,m} = x_{n+1} = y_{n+1,0}
\]
where \( n + 1 = i \mod m \) (or \( i(n) = \text{Rem} \left\{ \frac{n + 1}{m} \right\} = \frac{n + 1}{m} - \left( \frac{n + 1}{m} \right) \), \( n \geq 0 \) (4)

In the case where at least one of the maps in the finite family has some asymptotic behaviour (satisfies an asymptotic condition) then the iterative process becomes:
\[
y_{n,0} = x_n \\
y_{n,i} = (1 - \alpha_{n,i})x_n + \alpha_{n,i} T_{m+1-i} y_{n,i-1}; i = 1, 2, ..., m \\
y_{n,m} = x_{n+1} = y_{n+1,0}
\]
(5)

With \( n \) and \( m \) as in equation (4) and \( r = 1 - \left( \frac{n}{m} \right) \)

We need the following lemma in this work:

Lemma 1.1 [13] Let \( \{\mu_n\}, \{\beta_n\}, \{\gamma_n\} \) be sequence of nonnegative numbers satisfying the conditions:
\[
\sum_{n=0}^{\infty} \beta_n = \infty, \beta_n \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \gamma_n = 0(\beta_n). \text{ Suppose that } \\
\mu_{n+1}^2 \leq \mu_n^2 - \beta_n \psi(\mu_{n+1}) + \gamma_n; n = 1, 2, \ldots
\]
Where \( \psi:[0,1) \rightarrow [0,1) \) is a strictly increasing function with \( \psi(0) = 0 \). Then \( \mu_n \rightarrow 0 \) as \( n \rightarrow \infty \).

2. Main Result

2.1 Theorem 2.1 Let \( E \) be a normed linear space and \( K \) a nonempty closed convex subset of \( E \), let \( \{T_i\}_{i=1}^m \) be a finite family of self maps on the \( K \) such that:
- \( T_i(K), T_2(K) \) are bounded
- \( T_i \) is a uniformly continuously uniformly hemicontractive map on \( K \)
- \( F = \bigcap_{i=1}^m F(T_i) \neq \emptyset \) where \( F(T_i) \) is the set of fixed points of \( T_i \) in \( K \)

Starting with an arbitrary \( x_0 \in K \), let \( \{x_n\} \) be the iterative sequence defined by
\[
y_{n,0} = x_n \\
y_{n,i} = (1 - \alpha_{n,i})x_n + \alpha_{n,i} T_{m+1-i} y_{n,i-1}; i = 1, 2, ..., m - 1 \\
y_{n,m} = x_{n+1}
\]
(6)

Where \( \{\alpha_{n,i}\} \subset (0,1) \) is a finite family of real sequence such that \( \lim_{n \rightarrow \infty} \alpha_{n,i} = 0; \forall i \in \{1, 2, \ldots m\} \) and
\[
\sum_{n \geq 0} \alpha_{n,i} = \infty. \text{ Then, } \{x_n\} \text{ converges strongly to a common fixed point of the finite family.}
\]

Proof Let \( x^* \in F \). It suffices to prove that:
- \( \{x_n\} \) is bounded.
- Let \( \delta_n = \|T_1 y_{n,m-1} - T_2 x_{n+1}\| \) Then \( \delta_n \to 0 \) as \( n \to \infty \)
- \( x_n \) converges to \( x^* \).

Now, since \( T_1(K) \) is bounded, let \( D_1 = \|x_n - x^*\| + \sup_{n \geq 0} \|T_1 y_{n,m-1} - x^*\| < \infty \). We establish by induction that \( \|x_n - x^*\| \leq D_1 \forall n \geq 0 \). The case \( n = 0 \) is trivial, so assume it is true for \( n = v +1 \)
\[
\|x_{v+1} - x^*\| \leq (1 - \alpha_{v,m})\|x_v - x^*\| + \alpha_{v,m} \|T_1 y_{v,m-1} - x^*\| \leq D_1 \]
Thus, \( \|x_n - x^*\| \leq D_1 \forall n \geq 0 \) which gives \( \{x_n\} \) are bounded.

More so, since \( T_1(K), T_2(K) \) and \( \{x_n\} \) are bounded sets, let
D_2 = \sup_{n \in \mathbb{N}} \{ \| x_n - x^* \| + \| T_1 y_{n,m-1} - x^* \| + \| T_2 y_{n,m-2} - x^* \| < \infty \}

Then,
\| y_{n,m-1} - x_{n+1} \| \leq \alpha_{m,n-1} \| x_n - T_2 y_{n,m-2} \| + \alpha_{m,n} \| x_n - T_1 y_{n,m-1} \|
\leq \alpha_{m,n-1} (\| x_n - x^* \| + \| T_2 y_{n,m-2} - x^* \|) + \alpha_{m,n} \| x_n - x^* \| + \| T_1 y_{n,m-1} - x^* \|
\leq (\alpha_{m,n-1} + \alpha_{m,n}) (\| x_n - x^* \| + \max (\| T_1 y_{n,m-1} - x^* \|, \| T_2 y_{n,m-2} - x^* \|))
\leq (\alpha_{m,n-1} + \alpha_{m,n}) D_2

Thus, \| y_{n,m-1} - x_{n+1} \| \to 0 \quad \text{as} \quad n \to \infty.

Then by uniform continuity of T_1, \delta_n \to 0 \quad \text{as} \quad n \to \infty. \quad \text{Thus proving (ii).}

Also,
\| x_{n+1} - x^* \|^2 = (x_{n+1} - x^*, j(x_{n+1} - x^*))
= (1 - \alpha_{m,n}) (x_{n+1} - x^*, j(x_{n+1} - x^*)) + \alpha_{m,n} (T_1 y_{n,m-1} - x^*, j(x_{n+1} - x^*))
\leq (1 - \alpha_{m,n}) \| x_{n+1} - x^* \|^2 + \alpha_{m,n} \| T_1 y_{n,m-1} - T_1 x_{n+1} \| \| x_{n+1} - x^* \| + \alpha_{m,n} \| T_1 x_{n+1} - x^* \| \| x_{n+1} - x^* \|^2 - \psi_1 (\| x_{n+1} - x^* \|)
\leq (1 - \alpha_{m,n}) \| x_{n+1} - x^* \|^2 + \alpha_{m,n} \| T_1 y_{n,m-1} - T_1 x_{n+1} \| \| x_{n+1} - x^* \|^2 - \psi_1 (\| x_{n+1} - x^* \|)
\leq \frac{1}{2} (1 - \alpha_{m,n}) \| x_{n+1} - x^* \|^2 + \alpha_{m,n} \| T_1 y_{n,m-1} - T_1 x_{n+1} \|^2 - \psi_1 (\| x_{n+1} - x^* \|)
\leq \frac{1}{2} (1 - \alpha_{m,n}) \| x_{n+1} - x^* \|^2 + \frac{1}{2} \alpha_{m,n} \delta_n (1 + \| x_{n+1} - x^* \|^2) + \alpha_{m,n} \| x_{n+1} - x^* \|^2 - \psi_1 (\| x_{n+1} - x^* \|)

So that
2 \| x_{n+1} - x^* \|^2 \leq (1 - \alpha_{m,n}) \| x_{n+1} - x^* \|^2 + \alpha_{m,n} \| T_1 y_{n,m-1} - T_1 x_{n+1} \|^2 + \alpha_{m,n} \| x_{n+1} - x^* \|^2 - \psi_1 (\| x_{n+1} - x^* \|)
\leq \alpha_{m,n} \psi_1 (\| x_{n+1} - x^* \|)

Hence,
\| x_{n+1} - x^* \|^2 \leq \alpha_{m,n} \psi_1 (\| x_{n+1} - x^* \|)
(7)

Since \alpha_{n,i} \to 0 \quad \text{as} \quad n \to \infty \quad \forall i, \exists n_0 \in \mathbb{N} \quad \text{such that} \quad \forall n \geq n_0 \frac{1}{2} < 1 - \alpha_{m,n} (2 + \delta_n) < 1
\| x_{n+1} - x^* \|^2 \leq \frac{(1 - \alpha_{m,n})^2}{1 - \alpha_{m,n} (2 + \delta_n)} \| x_n - x^* \|^2 + \frac{\alpha_{m,n} \delta_n}{1 - \alpha_{m,n} (2 + \delta_n)} - \frac{2 \alpha_{m,n} \psi_1 (\| x_{n+1} - x^* \|)}{(1 - \alpha_{m,n} (2 + \delta_n))}
\leq \| x_{n+1} - x^* \|^2 + 2 \alpha_{m,n} (\alpha_{m,n} + \delta_n) D + 2 \alpha_{m,n} \delta_n - 2 \alpha_{m,n} \psi_1 (\| x_{n+1} - x^* \|)

Let \mu_n = \| x_n - x^* \|; \beta_n = 2 \alpha_{m,n}; \nu = \delta_n; \theta = \alpha_{m,n} + \delta_n; D = 2 \alpha_{m,n} \delta_n - 2 \alpha_{m,n} \psi_1 (\| x_{n+1} - x^* \|)
\mu_{n+1} \leq \mu_n + \beta_n \nu_n - \theta \psi_1 (\mu_{n+1})
(8)

By Lemma, \mu_n \to 0 \quad \text{as} \quad n \to \infty. \quad \text{Hence, the theorem.}

2.2 Theorem 2.2
Let E be a normal linear space, K a non-empty closed convex subset of E, and let \{ T_i \}_{i=1}^m be a finite family of self maps on K such that:
- T(K) is bounded
- T_i is a uniformly continuous uniformly hemicontractive map on K
- T_2, \ldots, T_m are bounded maps
- F = \bigcap_{i=1}^m F(T_i) \neq \emptyset \quad \text{where} \quad F(T_i) \quad \text{is the set of fixed points of} \ T_i \quad \text{in K}

Starting with an arbitrary \ x_0 \in K \ , \ let \{ x_i \} \ be the iterative sequence defined by Equation 6. Then, \{ x_n \}
converges strongly to a common fixed point of the finite family.

**Proof** Let \( x^* \in F \). Then
\[
\|x_{n+1} - x^*\|^2 = (x_{n+1} - x^*, j(x_{n+1} - x^*))
\]
\[
= (1 - \alpha_{n,m})(x_{n+1} - x^*, j(x_{n+1} - x^*)) + \alpha_{n,m}(T_{1n+1,m-1} - x^*, j(x_{n+1} - x^*))
\]
\[
\leq (1 - \alpha_{n,m})\|x_{n+1} - x^*\|\|x_{n+1} - x^*\| + \alpha_{n,m}\|T_{1n+1,m-1} - T_{1n+1,m-1}\|\|x_{n+1} - x^*\|^2 + \alpha_{n,m}(T_{1n+1,m-1} - x^*, j(x_{n+1} - x^*))
\]
\[
\leq (1 - \alpha_{n,m})\|x_{n+1} - x^*\|\|x_{n+1} - x^*\| + \alpha_{n,m}\|T_{1n+1,m-1} - T_{1n+1,m-1}\|\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\|T_{1n+1,m-1} - x^*\|^2 + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)
\]
\[
\leq \frac{1}{2}(1 - \alpha_{n,m})\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)
\]
\[
2\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_{n,m})\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)
\]
\[
(1 - 2\alpha_{n,m} - \alpha_{n,m}\delta_n)\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_{n,m})\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|) \tag{9}
\]
Now
\[
\|y_{n+1} - x_{n+1}\| \leq \|x_{n+1} - T_{2n+1}y_{n+1}\| + \alpha_{n,m}\|y_{n+1} - T_{2n+1}y_{n+1}\| \leq \|x_{n+1} - T_{2n+1}y_{n+1}\| + \alpha_{n,m}\|y_{n+1} - T_{2n+1}y_{n+1}\| + \alpha_{n,m}\|T_{2n+1}y_{n+1} - x^*\| \leq (\alpha_{n,m} + \alpha_{n,m})\|y_{n+1} - x^*\| + \alpha_{n,m}\|T_{2n+1}y_{n+1} - x^*\|
\]
\[
(\alpha_{n,m} + \alpha_{n,m})\|y_{n+1} - x^*\| \leq (\alpha_{n,m} + \alpha_{n,m})\|y_{n+1} - x^*\| + \alpha_{n,m}\|T_{2n+1}y_{n+1} - x^*\| \leq \|y_{n+1} - x^*\| + \alpha_{n,m}\|T_{2n+1}y_{n+1} - x^*\| \leq D_2 \forall n \geq 0 \tag{10}
\]
Thus, \( \{T_{n+1}y_{n+1}\} \) is bounded. Let \( D_3 = \max\{D_1, \sup_{n \geq 0}\|T_{n+1}y_{n+1} - x^*\| \} \leq \infty \)
\[
\|y_{n+2} - x^*\| \leq (\alpha_{n,m} + \alpha_{n,m})\|y_{n+1} - x^*\| + \alpha_{n,m}\|T_{n+1}y_{n+1} - x^*\| \leq D_3 \forall n \geq 0 \tag{11}
\]
So, \( \{T_{n+2}y_{n+2}\} \) is bounded. Let \( D_4 = \max\{D_1, \sup_{n \geq 0}\|T_{n+2}y_{n+2} - x^*\| \} < \infty \). Proceeding thus, we obtain that if \( \{y_{n+1}\} \) is bounded then \( \{T_{n+1}y_{n+1}\} \) is bounded, and \( D_4 = \max\{D_1, \sup_{n \geq 0}\|T_{n+2}y_{n+2} - x^*\| \} < \infty \) so that
\[
\|y_{n+1} - x^*\| \leq (\alpha_{n,m} + \alpha_{n,m})\|y_{n+1} - x^*\| + \alpha_{n,m}\|T_{n+1}y_{n+1} - x^*\| \leq D_4 \forall n \geq 0 \tag{12}
\]
Hence, \( \{y_{n+1}\} \) and \( \{T_{n+1}y_{n+1}\} \) are bounded. We have thus established that there exists a constant \( D_0 > 0 \) such that
\[
D_0 = \max\{\|x_n - x^*\|, \|y_{n+1} - x^*\|, \|T_{n+1}y_{n+1} - x^*\| \} \forall j \in \{1, 2, \ldots, m\}. \text{ Thus, let } D = 2D_0
\]
\[
\|y_{n+1} - x^*\| \leq (\alpha_{n,m} + \alpha_{n,m})D \rightarrow 0 \text{ as } n \rightarrow \infty \tag{13}
\]
So that \( \delta_n \rightarrow 0 \) as \( n \rightarrow 0 \). There exists \( n_0 \in N \) such that \( \forall n \geq n_0 \)
\[
\|x_{n+1} - x^*\|^2 \leq (1 - \alpha_{n,m})\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)
\]
\[
\|x_{n+1} - x^*\|^2 \leq \|x_{n+1} - x^*\|^2 + 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 + 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|) \tag{14}
\]
Let \( \mu_n = \|x_{n+1} - x^*\|, \beta_n = 2\alpha_{n,m}, \nu = \delta_n + (\alpha_{n,m} + \delta_n)D^2 \). Then, we have that
\[
\mu_{n+1}^2 \leq \mu_n^2 + \beta_n \nu_n - \beta_n \psi_f(\mu_{n+1}) \tag{14}
\]

By Lemma 1.1, \( \mu_n \to 0 \) as \( n \to \infty \). Hence, the theorem.

A map \( A : E \to E \) is said to be accretive if \( \forall x,y \in E \) and \( \forall \alpha > 0 \)
\[
\|x - y + \alpha (Ax - Ay)\| \geq \|x - y\|. \tag{15}
\]

If the above holds, \( \forall x,y \in E \) and \( \exists \alpha \) such that \( x \in A(x) \), then \( A \) is said to be quasi-accretive. It is easy to see that \( T \) is hemicontractive if and if \( A=I-T \) is quasi-accretive. We have the following theorem as an easy corollary to Theorem 2.2 and Theorem 2.1

### 2.3 Theorem 2.3

Let \( E \) be a normed linear space and let \( A_i : E \to E; i \in \{1,2,...,m\} \) be a finite family of maps such that:

- The simultaneous nonlinear equations \( A_i x = 0 \) \( \{1,2,...,m\} \) have a common solution \( x^* \in E \)
- \( R(I-A_1), R(I-A_2) \) are bounded
- \( (I-A_3), ..., (I-A_m) \) are bounded maps.

Starting with an arbitrary \( x_0 \in E \), define the iterative sequence \( \{x_n\} \) by
\[
\begin{align*}
y_{n,0} &= x_n \\
y_{n,i} &= (1-\alpha_{n,i})x_n + \alpha_{n,i}(1-A_{m+1-i})y_{n,i-1}; i = 1,...,m \\
y_{n,m} &= x_{n+1} \text{ where } n+1 \equiv i \mod m; \tag{16}
\end{align*}
\]

Where \( \{\alpha_{n,i}\} \subset (0,1) \) is a family of real sequences such that \( \lim_{n \to \infty} \alpha_{n,i} = 0; \forall i \in \{1,2,...,m\} \) and
\[
\sum_{n \geq 0} \alpha_{n,m} = \infty .
\]

Then, \( \{x_n\} \) converges strongly to a solution of the simultaneous nonlinear equations.

**Proof:** Let \( T_i = I-A_i \). Then \( T_i \) is a uniform continuous uniformly hemicontraction. Further,
\[
\begin{align*}
y_{n,0} &= x_n \\
y_{n,i} &= (1-\alpha_{n,i})x_n + \alpha_{n,i}(1-A_{m+1-i})y_{n,i-1}; i = 1,...,m \\
y_{n,m} &= x_{n+1} \text{ where } n+1 \equiv i \mod m;
\end{align*}
\]

Thus, Theorem 2.2 applies and we have the stated results. Similarly

### Theorem 2.4

Let \( E \) be a normed linear space and let \( A_i : E \to E; i \in \{1,2,...,m\} \) be a finite family of maps such that:

- The simultaneous nonlinear equations \( A_i x = 0 \) \( \{1,2,...,m\} \) have a common solution \( x^* \in E \) that is \( \bigcap_{i=1}^{m} Z(A_i) \neq \emptyset . \)
- \( R(I-A_1), R(I-A_2) \) are bounded
- \( A_i \) is a uniformly continuous quasi-accretive map

Starting with an arbitrary \( x_0 \in E \), define the iterative sequence \( \{x_n\} \) by equation (16). Then \( \{x_n\} \) converges strongly to a solution of the simultaneous nonlinear equations.

### 3. Conclusion/Remark

A Theorem 2.1 extends theorem 1.2 in the following ways; in Equation (6) of Theorem 2.1, let \( m = 3 \), \( \alpha_{n,3} = \alpha_n, \alpha_{n,2} \equiv \beta_n, \alpha_{n,1} \equiv \gamma_n, \gamma_{n,2} \equiv \gamma_n, \gamma_{n,1} \equiv \zeta_n \), then we have Equation (2) of Theorem 1.2. Theorem 2.1 is proved for any finite family of maps, so if they are just three in the family, we have Theorem 1.2. More so, the conditions of Theorem 2.1. For instance, whereas the three maps in Theorem 1.2 are required to be uniformly continuous and uniformly pseudocontractive with the same function \( \psi \), Theorem 2.1 requires that only one map
in the family satisfies such conditions. Also, Theorem 2.2 extends Theorem 1.2 in a similar manner. Hence, Theorem 2.1, Theorem 2.2 and their corollaries are rather interesting.

References
This academic article was published by The International Institute for Science, Technology and Education (IISTE). The IISTE is a pioneer in the Open Access Publishing service based in the U.S. and Europe. The aim of the institute is Accelerating Global Knowledge Sharing.

More information about the publisher can be found in the IISTE’s homepage: http://www.iiste.org

**CALL FOR PAPERS**

The IISTE is currently hosting more than 30 peer-reviewed academic journals and collaborating with academic institutions around the world. There’s no deadline for submission. **Prospective authors of IISTE journals can find the submission instruction on the following page:** http://www.iiste.org/Journals/

The IISTE editorial team promises to the review and publish all the qualified submissions in a fast manner. All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Printed version of the journals is also available upon request of readers and authors.

**IISTE Knowledge Sharing Partners**

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digitial Library, NewJour, Google Scholar