Iterative Procedure for Uniform Continuous Mapping.

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Abstract

Let K be a closed convex nonempty subset of a normed linear space E and let $\{T_i\}_{i=1}^N$ *be a finite family of self maps on K such that T1 is a uniformly continuous uniformly hemicontractive map and* $T_1(K)$ *is a bounded set with* $F = (\bigcap_{i=1}^N F(T_i)) \neq \phi$, *sufficient conditions for the strong convergence of an N-step iteration process to a fixed common point of the family are proved*

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1. Introduction

$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\} \forall x \in E$ Where E^* denotes the dual space of E and $\langle .,. \rangle$ denotes the generalized duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j. A mapping $T : E \to E^*$ is called strongly pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a constant $K \in (0,1)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq (1-k)\|x-y\|^2 .$$

T is called strongly $\phi-pseudocontractractive$ if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,1) \to [0,1)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|.$$

It is called generalized strongly $\psi-pseudocontractractive$ or uniformly pseudocontractive if for all $x, y \in E$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\psi : [0,1) \to [0,1)$ with $\psi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leq \|x-y\|^2 - \psi(\|x-y\|).$$

Every strongly $\phi-pseudocontractractive$ operator is a uniformly $\psi-pseudocontractractive$ operator with $\psi : [0,1) \to [0,1)$ defined by $\psi(s) = \phi(s)s$, but not conversely (see [13]).

These classes of operators have been studied by several authors (see, for example [3], [4], [7], [13], [19], [23], [24] and references therein).

If I denotes the identity operator, then T is strongly pseudocontractive, strongly $\phi-pseudocontractractive$, generalized strongly $\psi-pseudocontractractive$ if and only if $(I-T)$ is strongly accretive, strongly $\phi-accretive$, generalized strongly $\phi-accretive$ operators respectively. The interest in pseudocontractive mappings is mainly due to their connection with the important class of nonlinear accretive operators. In recent years, many authors have given much attention to approximate the fixed points of non-linear operators in Banach space using the Ishikawa and Mann iterative schemes (see, for example [8], [10], [11] and references therein).

Noor [14] introduced the three-step iteration process for solving nonlinear operator equations in real Banach as follows;

Let E be a real Banach space, K a nonempty convex subset of E and $T : K \to K$, a mapping. For an arbitrary $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty} \subset K$ defined by

$$x_{n+1} = (1 - \propto_n)x_n + \propto_n Ty_{n,}$$
$$y_n = (1 - \beta_n)x_n + \beta_n Tz_{n,}$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n Tx_{n,} \quad (1)$$

Where $\{\propto_n\}, \{\beta_n\}$and $\{\gamma_n\}$are three sequences in [0,1] is called the three-step iteration (or the Noor iteration). When $\gamma_n = 0$, then the three-step iteration reduces to the Ishikawa iterative sequence if $\beta_n = \gamma_n = 0$, then the three-step iteration reduces to the Mann iteration.

Rafiq [21], recently introduced a new type of iteration-the modified three-step iteration process which is defined as follows;

Let $T_1\, T_2\, T_3 : K \to K$ be three mappings for any given $x_0 \in K$, the modified three-step iteration is defined by

$$x_{n+1} = (1 - \propto_n)x_n + \propto_n T_1 y_{n,}$$
$$y_n = (1 - \beta_n)x_n + \beta_n T_2 z_n$$
$$z_n = (1 - \gamma_n)x_n + \gamma_n T_3 x_n \quad (2)$$

Where $\{\propto_n\}, \{\beta_n\}$and $\{\gamma_n\}$ are three real sequences staisfying some conditions. It is clear that the iteration scheme (2) includes Noor as special case.

Glowinski and Le Tallec [5] used the three-step iteration schemes to solve elastoviscoplasiticty, liquid crystal and eigen-value problems. They have shown that the three-step approximation scheme performs better than the two-step and one-step iteration methods. Haubruge et al. [6] have studied the convergence analysis of three-step iteration schemes and applied these three-step iteration to obtain new splitting type algorithms for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that three-step iterations also lead to highly parallelized algorithms under certain conditions. Thus, it is clear that three-step schemes play an important part in solving various problems, which arise in pure and applied sciences.

Recently, Xue and Fan [23] used the iteration procedure define by (2) in their theorem as stated below.

Theorem 1.1 Let X be a real Banach space and K be a nonempty closed convex subset of X. Let T_1, T_2 and T_3 be strongly pseudocontractive self maps of K with $T_1(K)$ bounded and T_1, T_2 and T_3 uniformly continuous. Let $\{x_n\}$ be defined by (2),where $\{a_n\}$, $\{b_n\}$and $\{c_n\}$ are three real sequences in [0,1] such that (i) $a_n, b_n \to 0 \; as \; n \to \infty$. (ii) $\sum_{n\geq 0}^{\infty} a_n = \infty$, and $F(T_1) \cap F(T_2) \cap F(T_3) \neq 0$ then the sequence$\{x_n\}$ converges strongly to the common fixed point of T_1, T_2 and T_3

Olaleru and Mogbademu [17] established the strong convergence of a modified Noor iterative process when applied to three generalized strongly $\phi - pseudocontractractive$, operators or generalized strongly $\phi - accretive$ operators in Banach space. Thus, generalizing the recent results of Fan and Xue (2009). In fact the stated and proved the following result.

Theorem 1.2 let E be a real Banach space, K a nonempty closed convex subset of E, $\phi - pseudocontractractive$ mappings such that $T_1(K)\; is$ bounded. Let $\{x_n\}$ be a sequence defined by (2) where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in [0,1] satisfiying the following conditions:

$\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \gamma_n = 0$

$\sum_{n\geq 0}^{\infty} \alpha_n = \infty$ if $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}$ converges to the unique common fixed points T_1, T_2 and T_3 .

Remark 1.1 In theorem 1.2, it is required that;

All the 3 maps be generalized strongly $\phi - pseudocontractractive$ with the same function ϕ (which is rather strong function).

All the 3 maps are required to be uniformly continuous (and thus bounded).

Our purpose in this paper is to extend is to extend and generalized the result of Olaleru and Mogbademu (17) in the following ways:

We introduce m-step iteration scheme

We extend the result to any finite family of m-maps.

The conditions of our theorems are less restrictive and more general than the one used in (17), (23). For instance, the demand that the three maps must be uniformly continuous is weakened by allowing some of the maps to be free.

1.1. *The M-Step Iteration Process*

Let K be a non-empty convex subset of a normed linear space E and let T: K→K be a map. For any given $x_0 \in K$. The m-step iteration process is defined by

$$y_{n,0} = x_n$$
$$y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i} Ty_{n,i-1}\,; i = 1,2,\dots,m$$
$$y_{n,m} = x_{n+1} = y_{n+1,0} \quad where\ n+1 = i\ mod\ m; n \geq 0 \qquad (3)$$

For a finite family $\{T_i\}_{i=1}^{m}$ of m-maps, the m-step iterative process becomes

$$y_{n,0} = x_n$$
$$y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i} T_{m+1-i}y_{n,i-1}\,; i = 1,2,\dots,m$$
$$y_{n,m} = x_{n+1} = y_{n+1,0} \qquad (4)$$
$$where\ n+1 = i\ mod\ m \left(or\ i(n) = Res\left[\frac{n+1}{m}\right] = m\left(\frac{n+1}{m}\right) - \left(\frac{n+1}{m}\right)\right), n \geq 0$$

In the case where at least one of the maps in the finite family has some asymptotic behaviour (satisfies an asymptotic condition) then the iterative process becomes:

$$y_{n,0} = x_n$$
$$y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i} T^{r}_{m+1-i} Ty_{n,i-1}\,; i = 1,2,\dots,m$$
$$y_{n,m} = x_{n+1} = y_{n+1,0} \qquad (5)$$

With n and m as in equation (4) and $r = 1 - \left(\frac{n}{m}\right)$

We need the following lemma in this work:

Lemma 1.1 [13] Let $\{\mu_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be sequence of nonnegative numbers satisfying the conditions: $\sum_{n\geq 0}^{\infty} \beta_n = \infty$, $\beta_n \to 0$ as n→ ∞ and $\gamma_n = 0\{\beta_n\}$. Suppose that

$$\mu_{n+1}^2 \leq \mu_n^2 - \beta_n\psi(\mu_{n+1}) + \gamma_n;\ n = 1,2,\dots\dots$$

Where ψ:[0,1)→ [0,1) is a strictly increasing function with $\psi(0) = 0$. Then $\mu_n \to 0$ as $n \to \infty$.

2. Main Result

2.1 Theorem 2.1 Let E be a normed linear space and K a nonempty closed convex subset of E, let $\{T_i\}_{i=1}^{m}$ be a finite family of self maps on the K such that :

- $T_1(K)$, $T_2(K)$ are bounded
- T_1 is a uniformly continuous uniformly hemicontractive map on K
- $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ where $F(T_i)$ is the set of fixed points of T_i in K

Starting with an arbitrary $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by

$$y_{n,0} = x_n$$
$$y_{n,i} = (1-\alpha_{n,i})x_n + \alpha_{n,i} T_{m+1-i}y_{n,i-1};\ i = 1,2,\dots,m-1$$
$$y_{n,m} = x_{n+1} \qquad (6)$$

Where $\{\alpha_{n,i}\} \subset (0,1)$ is a finite family of real sequence such that $\lim_{n\to\infty} \alpha_{n,i} = 0; \forall_i \in \{1,2,\dots m\}$ and $\sum_{n\geq 0}^{\infty} \alpha_{n,m} = \infty$. Then, $\{x_n\}$ converges strongly to a common fixed point of the finite family.

Proof Let $x^* \epsilon F$. It suffices to prove that:

- $\{x_n\}$ is bounded.
- Let $\delta_n = \|T_1 y_{n,m-1} - T_1 x_{n+1}\|$ Then $\delta_n \to 0$ as n→ ∞
- x_n converges to x^*.

Now, since $T_1(K)$ is bounded, let $D_1 = \|x_n - x^*\| + sup_{n\geq 0}\|T_1 y_{n,m-1} - x^*\| < \infty$. We establish by induction that $\|x_n - x^*\| \leq D_1 \forall_n \geq 0$. The case n = 0 is trivial, so assume it is true for n = v +1

$$\|x_{v+1} - x^*\| \leq (1-\alpha_{v,m})\|x_v - x^*\| + \alpha_{v,m}\|T_1 y_{v,m-1} - x^*\| \leq D1$$

Thus, $\|x_n - x^*\| \leq D_1 \forall_n \geq 0$ which gives $\{x_n\}$ are bounded.

More so, since $T_1(K)$, $T_2(K)$ and $\{x_n\}$ are bounded sets, let

$D_2 = sup_{n\geq 0}\{\|x_n - x^*\| + \|T_1 y_{n,m-1} - x^*\| + \|T_2 y_{n,m-2} - x^*\| < \infty$

Then,

$\|y_{n,m-1} - x_{n+1}\| \leq \alpha_{n,m-1}\|x_n - T_2 y_{n,m-2}\| + \alpha_{n,m}\|x_n - T_1 y_{n,m-1}\|$
$\leq \alpha_{n,m-1}(\|x_n - x^*\| + \|T_2 y_{n,m-2} - x^*\| + \alpha_{n,m}\|x_n - x^*\| + \|T_1 y_{n,m-1} - x^*\|)$
$\leq (\alpha_{n,m-1} + \alpha_{n,m})(\|x_n - x^*\| + \max\{\|T_1 y_{n,m-1} - x^*\|, \|T_2 y_{n,m-2} - x^*\|\})$
$\leq (\alpha_{n,m-1} + \alpha_{n,m}) D_2$

Thus, $\|y_{n,m-1} - x_{n+1}\| \to 0$ as $n \to \infty$.

Then by uniform continuity of T_1, $\delta_n \to 0$ as $n \to \infty$. Thus proving (ii).

Also,

$$\|x_{n+1} - x^*\|^2 = (x_{n+1} - x^*, j(x_{n+1} - x^*))$$
$$= (1-\alpha_{n,m})(x_{n+1} - x^*, j(x_{n+1} - x^*)) + \alpha_{n,m}(T_1 y_{n,m-1} - x^*, j(x_{n+1} - x^*))$$
$$\leq (1-\alpha_{n,m})\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_{n,m}\|T_1 y_{n,m-1} - T_1 x_{n+1}\|\|x_{n+1} - x^*\| + \alpha_{n,m}(T_1 x_{n+1} - x^*, j(x_{n+1} - x^*))$$
$$\leq (1-\alpha_{n,m})\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_{n,m}\|T_1 y_{n,m-1} - T_1 x_{n+1}\|\|x_{n+1} - x^*\| + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)$$
$$\leq \frac{1}{2}[(1-\alpha_{n,m})^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] + \frac{1}{2}\alpha_{n,m}\delta_n(1 + \|x_{n+1} - x^*\|^2) + \alpha_{n,m}\|x_{n+1} - x^*\|^2 - \psi_1(\|x_{n+1} - x^*\|)$$

So that

$$2\|x_{n+1} - x^*\|^2 \leq (1-\alpha_{n,m})^2\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 + \frac{1}{2}\alpha_{n,m}\delta_n + \alpha_{n,m}\delta_n\|x_{n+1} - x^*\|^2 + 2\alpha_{n,m}\|x_{n+1} - x^*\|^2 - \alpha_{n,m}\psi_1(\|x_{n+1} - x^*\|)$$

Hence,

$$(1 - 2\alpha_{n,m} - \alpha_{n,m}\delta_n)\|x_n - x^*\|^2 \leq (1-\alpha_{n,m})^2\|x_{n+1} - x^*\|^2 + \alpha_{n,m}\delta_n - 2\alpha_{n,m}\psi_1\|x_{n+1} - x^*\|^2$$

(7)

Since

$alpha_{n,i} \to 0$ as $n \to \infty \forall_i$, there exist $n_0 \in N$ such that $\forall_n \geq n_0$ $1/2 < 1 - \alpha_{n,m}(2+\delta_n) < 1$

$$\|x_{n+1} - x^*\|^2 \leq \frac{(1-\alpha_{n,m})^2}{1-\alpha_{n,m}(2+\delta_n)}\|x_n - x^*\|^2 + \frac{\alpha_{n,m}\delta_n}{1-\alpha_{n,m}(2+\delta_n)} - \frac{2\alpha_{n,m}}{1-\alpha_{n,m}(2+\delta_n)}\psi_1(\|x_{n+1} - x^*\|^2)$$
$$\leq \|x_{n+1} - x^*\|^2 + 2\alpha_{n,m}(\alpha_{n,m} + \delta_n)D + 2\alpha_{n,m}\delta_n - 2\alpha_{n,m}\psi_1(\|x_{n+1} - x^*\|^2)$$

Let $\mu_n = \|x_n - x^*\|$; $\beta_n = 2\alpha_{n,m}$, $\upsilon = \delta_n + (\alpha_{n,m} + \delta_n)D^2$. Then, we have that

$$\mu_{n+1}^2 \leq \mu_n^2 + \beta_n\upsilon_n - \beta_n\psi_1(\mu_{n+1}) \quad (8)$$

By Lemma, $\mu_n \to 0$ as $n \to \infty$. Hence, the theorem.

*2.2 **Theorem 2.2***

Let E be a normal linear space, K a non-empty closed convex subset of E, and let $\{T_i\}_{i=1}^m$ be a finite family of self maps on K such that:

- $T_1(K)$ is bounded
- $T_1$is a uniformly continuous uniformly hemicontractive map on K
- $T_2, \ldots, T_m$ are bounded maps
- $F = \bigcap_{i=1}^m F(T_i) \neq \phi$ where $F(T_i)$ is the set of fixed points of T_i in K

Starting with an arbitrary $x_0 \in K$, let $\{x_n\}$ be the iterative sequence defined by Equation 6. Then, $\{x_n\}$

converges strongly to a common fixed point of the finite family.

Proof Let $x^* \in F$. Then

$$\|x_{n+1}-x^*\|^2 = (x_{n+1}-x^*, j(x_{n+1}-x^*))$$
$$= (1-\alpha_{n,m})(x_{n+1}-x^*, j(x_{n+1}-x^*)) + \alpha_{n,m}(T_1 y_{n,m-1}-x^*, j(x_{n+1}-x^*))$$
$$\le (1-\alpha_{n,m})\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_{n,m}\|T_1 y_{n,m-1}-T_1 x_{n+1}\|\|x_{n+1}-x^*\| + \alpha_{n,m}(T_1 x_{n+1}-x^*, j(x_{n+1}-x^*))$$
$$\le (1-\alpha_{n,m})\|x_n-x^*\|\|x_{n+1}-x^*\| + \alpha_{n,m}\|T_1 y_{n,m-1}-T_1 x_{n+1}\|\|x_{n+1}-x^*\| + \alpha_{n,m}\|x_{n+1}-x^*\|^2 - \psi_1(\|x_{n+1}-x^*\|)$$
$$\le \frac{1}{2}[(1-\alpha_{n,m})^2\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2] + \frac{1}{2}\alpha_{n,m}\delta_n(1+\|x_{n+1}-x^*\|^2) + \alpha_{n,m}\|x_{n+1}-x^*\|^2 - \psi_1(\|x_{n+1}-x^*\|)$$

Where $\delta_n = \|T_1 y_{n,m-1} - T_1 x_{n+1}\|$; So that

$$2\|x_{n+1}-x^*\|^2 \le (1-\alpha_{n,m})^2\|x_n-x^*\|^2 + \|x_{n+1}-x^*\|^2 + \frac{1}{2}\alpha_{n,m}\delta_n + \alpha_{n,m}\delta_n\|x_{n+1}-x^*\|^2 + 2\alpha_{n,m}\|x_{n+1}-x^*\|^2$$
$$-\alpha_{n,m}\psi_1(\|x_{n+1}-x^*\|)$$

So that

$$(1-2\alpha_{n,m}-\alpha_{n,m}\delta_n)\|x_n-x^*\|^2 \le (1-\alpha_{n,m})^2\|x_{n+1}-x^*\|^2 + \alpha_{n,m}\delta_n - 2\alpha_{n,m}\psi_1\|x_{n+1}-x^*\|^2 \quad \textbf{(9)}$$

Now

$\|y_{n,m-1} - x_{n+1}\| \le \alpha_{n,m-1}\|x_n - T_2 y_{n,m-2}\| + \alpha_{n,m}\|x_n - T_1 y_{n,m-1}\|$
$\le \alpha_{n,m-1}(\|x_n - x^*\| + \|T_2 y_{n,m-2} - x^*\| + \alpha_{n,m}\|x_n - x^*\| + \|T_1 y_{n,m-1} - x^*\|)$
$\le (\alpha_{n,m-1} + \alpha_{n,m})(\|x_n - x^*\| + \max\{\|T_2 y_{n,m-2} - x^*\|, \|T_1 y_{n,m-1} - x^*\|\})$

Since $T_1(K)$ is bounded by the same argument in the prove of part (i) of theorem 2.1 we establish that $\{x_n\}$ is bounded. Since $T_2,\dots,T_n$ are bounded maps, we have that $\{T_n\ x_n\}$ is bounded. Let $D_2 = \max\{D_1, sup_{n\ge 0}\|T_m x_n - x^*\| < \infty$

$\|y_{n,1} - x^*\| \le (1-\alpha_{n,1})\|x_n - x^*\| + \alpha_{n,1}\|T_m x_n - x^*\| \le D_2\ \forall_n \ge 0$ (10)

Thus, $\{T_{m-1}\ y_{n,1}\}$ is bounded. Let $D_3 = = \max\{D_1, sup_{n\ge 0}\|T_{m-1} y_{n,1} - x^*\| < \infty$

$\|y_{n,2} - x^*\| \le (1-\alpha_{n,2})\|x_n - x^*\| + \alpha_{n,2}\|T_{m-1} y_{n,1} - x^*\| \le D_3\ \forall_n \ge 0$ (11)

So, $\{T_{m-2}\ y_{n,2}\}$ is bounded. Let $D_4 = \max\{D_1, sup_{n\ge 0}\|T_{m-2} y_{n,2} - x^*\| < \infty$. Proceeding thus, we obtain that if $\{y_{n,i}\}$ is bounded then $\{T_{m-i}\ y_{n,i}\}$ is bounded, and $D_{i+2} = \max\{D_1, sup_{n\ge 0}\|T_{m-i} y_{n,i} - x^*\| < \infty$ so that

$\|y_{n,i+1} - x^*\| \le (1-\alpha_{n,i+1})\|x_n - x^*\| + \alpha_{n,i+1}\|T_{m-i} y_{n,i} - x^*\| \le D_{i+2}\ \forall_n \ge 0$ (12)

Hence, $\{y_{n,i+1}\}$ and $\{T_{m-1-i} y_{n,i+1}\}$ are bounded. We have thus established that there exists a constant $D_o>0$ such that

$D_0 \ge \max\{\|x_n - x^*\|, \|y_{n,i} - x^*\|, \|T_{m-i} y_{n,i} - x^*\|\} \forall_i \in \{1,2,\dots,m\}$. Thus, let $D=2D_0$

$$\|y_{n,m-i} - x_{n+1}\| \le (\alpha_{n,m-1} + \alpha_{n,m})D \to 0 \text{ as } n \to \infty \quad (13)$$

So that $\delta_n \to 0$ as $n \to 0$. There exists $n_0 \in N$ such that $\forall_n \ge n_0$

$$\|x_{n+1}-x^*\|^2 \le \frac{(1-\alpha_{n,m})^2}{1-\alpha_{n,m}(2+\delta_n)}\|x_n-x^*\|^2 + \frac{\alpha_{n,m}\delta_n}{1-\alpha_{n,m}(2+\delta_n)} - \frac{2\alpha_{n,m}}{1-\alpha_{n,m}(2+\delta_n)}\psi(\|x_{n+1}-x^*\|^2)$$
$$\le \|x_{n+1}-x^*\|^2 + 2\alpha_{n,m}(\alpha_{n,m}+\delta_n)D^2 + 2\alpha_{n,m}\delta_n - 2\alpha_{n,m}\psi_1(\|x_{n+1}-x^*\|^2)$$

Let $\mu_n = \|x_n - x^*\|$; $\beta_n = 2\alpha_{n,m}$, $\upsilon = \delta_n + (\alpha_{n,m}+\delta_n)D^2$. Then, we have that

$$\mu_{n+1}^2 \le \mu_n^2 + \beta_n \upsilon_n - \beta_n \psi_1(\mu_{n+1}) \tag{14}$$

By Lemma 1.1, $\mu_n \to 0$ as $n \to \infty$. Hence, the theorem.

A map $A : E \to E$ is said to be accretive if $\forall_{x,y} \in E$ and $\forall_\alpha > 0$

$$\|x - y + \alpha(Ax - Ay)\| \ge \|x - y\|. \tag{15}$$

If the above holds, $\forall_{x,y} \in E$ and $\forall_y \in Z(A) := \{w \in E \mid Aw = 0\}$ (the zero set of A), then A is said to be quasi-accretive. It is easy to see that T is hemicontractive if and if A=I-T is quasi-accretive. We have the following theorem as an easy corollary to Theorem 2.2 and Theorem 2.1

2.3 ***Theorem 2.3***

Let E be a normed liner space and let $A_i : E \to E; i \in \{1,2,\ldots,m\}$ be a finite family of maps such that:

- The simultaneous nonlinear equations $A_i x = 0; i \in \{1,2,\ldots,m\}$ have a common solution $x^* \in E$
- $R(I-A_i)$ is bounded.
- $(I-A_2), \ldots, (I-A_m)$ are bounded maps.

Starting with an arbitrary $x_0 \in E$, define the iterative sequence $\{x_n\}$ by

$$\begin{aligned} y_{n,0} &= x_n \\ y_{n,i} &= (1-\alpha_{n,i})x_n + \alpha_{n,i}(1 - A_{m+1-i})y_{n,i-1}; i = 1,\ldots,m \\ &= (1-\alpha_{n,i})x_n + \alpha_{n,i}T_{m+1+i}y_{n,i-1}; i = 1,\ldots,m \\ y_{n,m} &= x_{n+1} \quad \text{where } n+1 \equiv i \text{ mod m}; \end{aligned} \tag{16}$$

Where $\{\alpha_{n,i}\} \subset (0,1)$ is a family of real sequences such that $\lim_{n\to\infty} \alpha_{n,i} = 0; \forall_i \in \{1,2,\ldots,m\}$ and $\sum_{n\ge 0}^{\infty} \alpha_{n,m} = \infty$. Then, $\{x_n\}$ converges strongly to a solution of the simultaneous nonlinear equations.

Proof: Let $T_i = I-A_i$. Then T_i is a uniform continuous uniformly hemicontraction. Further,

$$\begin{aligned} y_{n,0} &= x_n \\ y_{n,i} &= (1-\alpha_{n,i})x_n + \alpha_{n,i}(1 - A_{m+1-i})y_{n,i-1}; i = 1,\ldots,m \\ &= (1-\alpha_{n,i})x_n + \alpha_{n,i}T_{m+1+i}y_{n,i-1}; i = 1,\ldots,m \\ y_{n,m} &= x_{n+1} \quad \text{where } n+1 \equiv i \text{ mod m}; \end{aligned}$$

Thus, Theorem 2.2 applies and we have the stated results. Similarly

Theorem 2.4 Let E be a normed linear space and let $A_i : E \to E; i \in \{1,2,\ldots,m\}$ be a finite family of maps such that:

- The simultaneous nonlinear equations $A_i x = 0$ $\{1,2,\ldots,m\}$ have a common solution $x^* \in E$ that is $\bigcap_{i=1}^m Z(A_i) \ne \phi$.
- $R(I-A_1)$, $R(I-A_2)$ are bounded
- A_i is a uniformly continuous quasi-accretive map

Starting with an arbitrary $x_0 \in E$, define the iterative sequence $\{x_n\}$ by equation (16). Then $\{x_n\}$ converges strongly to a solution of the simultaneous nonlinear equations.

3. Conclusion\Remark

A Theorem 2.1 extends theorem 1.2 in the following ways; in Equation (6) of Theorem 2.1, let m = 3, $\alpha_{n,3} = \alpha_n, \alpha_{n,2} \equiv \beta_n, \alpha_{n,1} \equiv \gamma_n, y_{n,2} \equiv y_n, y_{n-1} \equiv z_n$, then we have Equation (2) of Theorem 1.2. Theorem 2.1 is proved for any finite family of maps, so if they are just three in the family, we have Theorem 1.2. More so, the conditions of Theorem 2.1. For instance, whereas the three maps in Theorem 1.2 are required to be uniformly continuous and uniformly pseudocontractive with the same function ψ, Theorem 2.1 requires that only one map in the family satisfies such conditions. Also, Theorem 2.2 extends Theorem 1.2 in a similar manner. Hence, Theorem 2.1, Theorem 2.2 and their corollaries are rather interesting.

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