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On the Rational Valued Character of the Group $D_n \times C_3$ when n is an Odd Number

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Abstract

Let D_n be the dihedral group , C_2 be the cyclic group of order 2 and D_{nh} is the direct product group of D_n and C_3 (i.e. $D_n \times C_3$) . Let cf($D_n \times C_3$,Z) be the abelian group of Z-valued class functions of the group $D_n \times C_3$. The intersection cf($D_n \times C_3$,Z) with the group of all generalized characters of $D_n \times C_3$ which is denoted by R ($D_n \times C_3$) , is a normal subgroup of cf($D_n \times C_3$,Z) denoted by

$\overline{R}$ ($D_n \times C_3$) , then factor group cf($D_n \times C_3$,Z)/ $\overline{R}$ ($D_n \times C_3$) is a finite abelian group denoted by K($D_n \times C_3$) .

The problem of determining the cyclic decomposition of the group K($D_n \times C_3$) seem to be untouched .

The aim of this paper is to find the cyclic decomposition of this group.

We find that when n is an odd number such that $n = \prod_{i=1}^{m} p_i^{\alpha_i}$, where all p_i 's are distinct primes , then

$$K(D_n \times C_3) = \bigoplus_{i=1}^{4}\left[\bigoplus\sum_{i=1}^{m}\left(\bigoplus\sum K\left(C_{p_i}^{\alpha_i}\right)\right)\left[\prod_{\substack{j\neq i\\ j=1}}^{m}\left(\alpha_j+1\right)\right] time\right] \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+2} C_4 \bigoplus_{i=1}^{2(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+1} C_2 \oplus C_8.$$

1. Introduction

Let G be a finite group ,two elements of G are said to be ***Γ- conjugate*** if the cyclic subgroups they generate are conjugate in G, this defines an equivalence relation on G .Its classes are called ***Γ- classes*** . The Z - valued class function on the group G , which is constant on the ***Γ- classes*** forms a finitely generated abelian group cf(G,Z) of a rank equal to the number of ***Γ- classes*** .

The intersection of cf(G,Z) with the group of all generalized characters of G ,R(G) is a normal subgroup of cf(G,Z) denoted by $\overline{R}(G)$,then cf(G,Z)/ $\overline{R}(G)$ is a finite abelian factor group which is denoted by K(G).

Each element in $\overline{R}(G)$ can be written as $u_1\theta_1 + u_2\theta_2 + \ldots\ldots + u_l\theta_l$, where l is the number of ***Γ- classes*** , u_1 , $u_2, \ldots\ldots, u_l \in Z$ and $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$, where χ_i is an irreducible character of the group G and σ is any element in Galios group $Gal(Q(\chi_i)/Q)$. Let ≡*(G) denotes the $l \times l$ matrix which corresponds to the θ_i 's and columns correspond to the ***Γ- classes*** of G .The matrix expressing $\overline{R}(G)$ basis in terms of the cf(G,Z) basis is ≡*(G) .

We can use the theory of invariant factors to obtain the direct sum of the cyclic Z-module of orders the distinct invariant factors of ≡*(G) to find the cyclic decomposition of K(G).In1982 M.S.Kirdar[4] studied the K(C_n).In 1994 H.H. Abass[2]studied the K(D_n) and found ≡*(D_n).In 1995 N.R. Mahamood [5] studied the factor group cf(Q_{2m},Z) / $\overline{R}$ (Q_{2m}).In 2005 N.S. Jasim [6] studied the factor group cf(G,Z)/ $\overline{R}(G)$ for the special linear group SL(2,p).

In this paper we study K($D_n \times C_3$) and find ≡*($\boldsymbol{D_n \times C_3}$) when n is an odd number .

2. Preliminaries

In this section we review definitions and some results which will be used in later section.

Definition (1.1):[1]

The set of all $l \times l$ non-singular matrices over the field F which forms a group under the operation of the matrix multiplication is called *the general linear group* of the dimension l over the field F, denoted by GL (l,F).

Definition (1.2): [1]

A matrix representation of a group G is a group homomorphism T of G into GL (l,F) , l is called *the degree of matrix representation T* .

Definition (1.3): [1]

The trace of an $l \times l$ matrix is the sum of the main diagonal elements , denoted by tr(A) .

Definition (1.4): [3]

A matrix representation T: G→GL (l, F) is said to be *reducible* if there exists a non-singular matrix A over F such that:

$$A^{-1}\,T(g)\,A=\begin{bmatrix} T_1(g) & E(g) \\ o & T_2(g) \end{bmatrix}, \text{ for all } g\in G.$$

Where $T_1(g)$, $T_2(g)$ are matrices of representations T_1 and T_2 of a group over F of the dimension r×r, s×s respectively and E(g) is a matrix of the dimension r×s such that $0< r < l$ and $r+s= l$. If no such reducible matrix exists then T is called *an irreducible matrix representation*.

Theorem (1.5):[1]

Let $T_1 : G_1 \rightarrow GL(V_1)$ and $T_2: G_2 \rightarrow GL(V_2)$ be two irreducible representations of the groups G_1 and G_2 respectively, then $T_1 \otimes T_2$ is irreducible representations of the group $G_1 \times G_2$.

Definition (1.6): [3]

Let T be a matrix representation of a group G over the field F, *the character* χ of a matrix representation T is the mapping χ: G→F defined by χ(g)=Tr(T(g)) for all g∈G where Tr(T(g)) refers to the trace of the matrix T(g) and χ(1) is the degree of χ .

Remark (1.7):

(i)A finite group G has a finite number of conjugacy classes and a finite number of distinct irreducible character, the group character of a group representation is constant on a conjugacy class , the values of irreducible characters can be written as a table whose columns are the conjugacy class and rows the value of irreducible characters on each conjugacy class , this table of the group G, denoted by ≡(G).

(ii) If G = C_n =< r > is the cyclic group of order n generated by r .If ω = $e^{2\pi i/n}$ is the primitive n-th root of unity , the

≡(C_n)=

CL_α	1	r	r^2	…	r^{n-1}
$\|CL_\alpha\|$	1	1	1	…	1
$\|C_G(C_\alpha)\|$	n	n	n	…	n
χ_1	1	1	1	…	1
χ_2	1	ω	ω^2	…	ω^{n-1}
χ_3	1	ω^2	ω^4	…	ω^{n-2}
⋮	⋮	⋮	⋮	⋱	⋮
χ_n	1	ω^{n-1}	ω^{n-2}	…	ω

Definition (1.8):[3]

Let χ and ψ as characters of a group G, then :

1. The sum of characters is defined by:
   (χ+ψ)(g) = χ(g)+ψ(g) , for all g∈G
2. The product of characters is defined by : (χ.ψ)(g) = χ(g).ψ(g) , for all g∈G.

Theorem (1.9):[3]

Let $T_1 : G_1 \rightarrow GL(n,K)$ and $T_2 : G_2 \rightarrow GL(m,K)$ are two matrix representations of the groups G_1 and G_2 , χ_1 and χ_2 be two characters of T_1 and T_2 respectively, then the character of $T_1 \otimes T_2$ is $\chi_1 \chi_2$.

Definition (1.10):[1]

A rational valued character θ of G is a character whose values are in the set of integers Z, which is $\theta(g) \in Z$, for all $g \in G$.

Proposition (1.11):[4]

The rational valued characters $\theta_i = \sum_{\sigma \in Gal(Q(\chi_i)/Q)} \sigma(\chi_i)$ form basis for $\overline{R}(G)$, where χ_i are the irreducible characters of G and their numbers are equal to the number of all distinct Γ- classes of G.

2. The factor group K(G)

In this section , we study the factor K(G) and discuss the cyclic decomposition of the factor groups $K(C_n)$ and $K(D_n)$.

Definition (2.1):[4]

Let M be a matrix with entries in a principal ideal domain R , a k-minor of M is the determinate of k×k sub matrix preserving rows and columns order.

Definition (2.2):[4]

A k-th determinant divisor of M is the greatest common divisor (g.c.d) of all the k-minors of M . This is denoted by $D_k(M)$.

Lemma (2.3):[4]

Let M, P and W be matrices with entries in a principal ideal domain R , if P and W are invertible matrices, then $D_k(P\,M\,W) = D_k(M)$ modulo the group of unites of R.

Theorem (2.4):[4]

Let M be an $l \times l$ matrix entries in a principal ideal domain R, then there exists matrices P and W such that:

1- P and W are invertible.
2- P M W = D.
3- D is diagonal matrix.
4-if we denote D_{ii} by d_i then there exists a natural number m ;

$0 \le m \le l$ such that $j > m$ implies $d_j = 0$ and $j \le m$ implies $d_j \ne 0$

and $1 \le j \le m$ implies $d_j \mid d_{j+1}$.

Definition (2.5):[4]

Let M be a matrix with entries in a principal ideal domain R be equivalent to a matrix $D = diag\{d_1, d_2, \ldots, d_m, 0, 0, \ldots, 0\}$ such that $d_j \mid d_{j+1}$ for $1 \le j < m$.

We call D *the invariant factor matrix of* M and $d_1, d_2, \ldots, d_m$ the invariant factors of M.

Theorem (2.6):[4]

Let K be a finitely generated module over a principal ideal domain R, then K is the direct sum of a cyclic submodules with an annihilating ideal $<d_1>, <d_2>, \ldots, <d_m>$, $d_j \mid d_{j+1}$ for $j = 1, 2, \ldots, K-1$.

Proposition(2.7):[4]

Let A and B be two non-singular matrices of the rank n and m respectively, over a principal ideal domain R . Then the invariant factor matrices of $A \otimes B$ equals $D(A) \otimes D(B)$, where D(A)and D(B) are the invariant factor matrices of A and B respectively .

Theorem(2.8):[4]

Let H and L be p_1-group and p_2-group respectively ,where p_1 and p_2 are distinct primes . Then , $\equiv^*(H \times L) = \equiv^*(H) \otimes \equiv^*(L)$.

Remark (2.9):[4]

Suppose cf(G,Z) is of the rank l , the matrix expressing the $\overline{R}$ (G) basis in terms of the cf(G,Z) = Z^l basis is $\equiv^*$(G).

Hence by theorem (2.4),we can find two matrices P and Q with a determinant ± 1 such that P. $\equiv^*$(G).Q =D($\equiv^*$(G))= diag{d_1,d_2,………,d_l },

$$d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G)) .$$

this yields a new basis for $\overline{R}$ (G) and Cf(G,Z),{v_1,v_2,…….., v_l }and {u_1,u_2,…….., u_l } respectively with the property $v_j = d_j u_j$.

Hence by theorem (2.6) the Z-module K(G) is the direct sum of cyclic submodules with annihilating ideals <d_1>, <d_2 >, … ,<d_l>.

Theorem(2.10) :[4]

Let p be a prime number , then :

$$K(G) = \oplus \sum_{i=1}^{s} C_{d_i} \quad \text{such that } d_i = \pm D_i(\equiv^*(G)) / \pm D_{i-1}(\equiv^*(G)) .$$

Theorem (2.11):[4]

$$|K(G)| = \det(\equiv^*(G))$$

Proposition (2.12):[4]

The ***rational valued characters table of the cyclic group*** C $_{p^s}$ of the rank s+1 where p is a prime number which is denoted by ($\equiv^*$(C $_{p^s}$)), is given as follows:

Γ-classes	[1]	$[r^{p^{s-1}}]$	$[r^{p^{s-2}}]$	$[r^{p^{s-3}}]$	…	$[r^{p^2}]$	$[r^p]$	[r]
θ_1	p^{s-1}(p-1)	- p^{s-1}	0	0	…	0	0	0
θ_2	p^{s-2}(p-1)	p^{s-2}(p-1)	- p^{s-2}	0	…	0	0	0
θ_3	p^{s-3}(p-1)	p^{s-3}(p-1)	p^{s-3}(p-1)	- p^{s-3}	…	0	0	0
⋮	⋮	⋮	⋮	⋮	⋱	⋮	⋮	⋮
θ_{s-1}	p(p-1)	p(p-1)	p(p-1)	p(p-1)	…	p(p-1)	-p	0
θ_s	p-1	p-1	p-1	p-1	…	p-1	p-1	-1
θ_{s+1}	1	1	1	1	…	1	1	1

where its rank s+1 represents the number of all distinct Γ-classes.

Example (2.13):

Consider the cyclic group C_{49} by using table (2.3), we can find the rational valued characters table of C_{49} as follows:

$\equiv^*(C_{49}) = \equiv^*(C_{7^2}) =$

Γ-classes	[1]	$[r^7]$	$[r]$
θ_1	42	-7	0
θ_2	6	6	-1
θ_3	1	1	1

Remark (2.14):

In general, for $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \ldots \cdot p_m^{\alpha m}$ where g.c.d $(p_i, p_j) = 1$, if $i \neq j$, p_i's are prime numbers and $\alpha_i \in Z^+$, then we have the following formula :

$$\equiv^*(C_n) = \equiv^*(C\ p_1^{\alpha 1}) \otimes \equiv^*(C\ p_2^{\alpha 2}) \otimes \ldots \otimes \equiv^*(C\ P_m^{\alpha m}).$$

Proposition (2.14):[4]

If p is a prime number , then

$$D(\equiv^*(C_p^{\,s})) = \mathrm{diag}\{p^s, p^{s-1}, \ldots\ldots, p, 1\}.$$

Remark (2.15):[4]

For $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdot \ldots \cdot p_m^{\alpha m}$ where p_i's are distinct primes and $\alpha_i \in Z^+$, then : $D(\equiv^*(C_n)) = D(\equiv^*(C\ p_1^{\alpha 1})) \otimes D(\equiv^*(C\ p_2^{\alpha 2})) \otimes \ldots \otimes D(\equiv^*(C\ P_m^{\alpha m}))$.

Theorem(2.16) :[4]

Let p be a prime number, then :

$$K(C_p^{\,s}) = \oplus \sum_{i=1}^{s} C_{P^i}\ .$$

Example(2.17):-

$$K(C_{25}) = K(C_{5^2}) = C_5 \oplus C_{5^2}$$

Proposition(2.18):[4]

Let $n = \prod_{i=1}^{k} P_i^{a_i}$,where p_i's are distinct primes and $a_i \in Z^+$,then :

$$K(C_n) = \oplus \sum_{i=1}^{k} \left(\oplus \sum K\left(C_{P_i^{a_i}}\right)\right) \left[\prod_{\substack{j \neq i \\ j=1}}^{k} (a_j + 1)\right] \ \textit{time}\ .$$

Example(2.19) :

To find the cyclic decomposition of group $K(C_{15435})$

$$K(C_{15435}) = K(C_{3^2 \cdot 7^3 \cdot 5}) = \underbrace{K(C_{3^2}) \oplus \cdots \cdots \oplus K(C_{3^2})}_{(3+1).(1+1)\ \textit{times}}$$

$$\oplus \underbrace{K(C_{7^3}) \oplus \cdots \cdots \oplus K(C_{7^3})}_{(2+1).(1+1)\ \textit{times}}$$

$$\oplus \underbrace{K(C_5) \oplus \cdots \cdots \oplus K(C_5)}_{(2+1).(3+1)\ \textit{times}}$$

$$= \bigoplus_{i=1}^{8} K(C_{3^2}) \bigoplus_{i=1}^{6} K(C_{7^3}) \bigoplus_{i=1}^{12} K(C_5)$$

$$= \bigoplus_{i=1}^{8} C_{3^2} \bigoplus_{i=1}^{8} C_3 \bigoplus_{i=1}^{6} C_{7^3} \bigoplus_{i=1}^{6} C_{7^2} \bigoplus_{i=1}^{6} C_7 \bigoplus_{i=1}^{12} C_5 .$$

Definition (2.20):[3]

For a fixed positive integer $n \geq 3$,***the dihedral group*** D_n is a certain non- abelian group of the order 2n . In general can write it as:

$$D_n=\{ S^j r^k : 0 \leq k \leq n-1 , 0 \leq j \leq 1\}$$

which has the following properties :

$$r^n = 1 , S^2 = 1 , Sr^k S^{-1} = r^{-k}$$

Definition $D_n \times C_3$ (2.21) :[3]

The group $D_n \times C_3$ is the direct product group $D_n \times C_3$,where C_3 is a cyclic group of the order 3 consisting of elements $\{1, r^* , r^*\}$with $(r^*)^2=1$ .It is order 4n .
So the group $D_n \times C_2$ is the direct product group $D_n \times C_3$, then the order of $D_n \times C_3$ is 6n .

Lemma (2.22):[2]
The rational valued characters table of D_n when n is an odd number is given as follows:

$\equiv^*(D_n)$=

	Γ- classes of C_n	[S]
θ_1	$\equiv^*(C_n)$	0
⋮		⋮
θ_{l-1}	1 1 1 … 1 1	0
θ_l		1
θ_{l+1}	1 1 1 … 1 1	-1

Where l is the number of Γ- classes of C_n .

<u>*Example (2.23):*</u>

To find the rational valued characters table of D_{49}, From example (2.13), we obtain $\equiv^*(C_{49})$ and by using lemma (2.22) , we have.

$\equiv^*(D_{49}) = \equiv^*(D_{7^2}) =$

Γ-classes	[1]	$[r^7]$	$[r]$	[S]
θ_1	$\equiv^*(C_{7^2})$			0
θ_2				0
θ_3	1	1	1	1
θ_4	1	1	1	-1

Table (2.10)

=

Γ-classes	[1]	$[r^7]$	$[r]$	[S]
θ_1	42	-7	0	0
θ_2	6	6	-1	0
θ_3	1	1	1	1
θ_4	1	1	1	-1

Proposition(2.24):[2]

$$D(\equiv^*(D_n))=\left[\begin{array}{c|c} D(\equiv^*(C_n)) & 0 \\ \hline 0 & -2 \end{array}\right]$$ Where $D(\equiv^*(D_n))$ and $D(\equiv^*(C_n))$ are the invariant factors matrices of $\equiv^*(D_n)$ and $\equiv^*(C_n)$ respectively .

Theorem(2.25) :-[7]

If n is an odd number ,then :

$$\equiv^*(D_{nh}) = \equiv^*(D_n) \otimes \equiv^*(C_2) .$$

Theorem(2.26) :- [7]

For a fixed positive odd integer n such that $n = p_1^{\alpha 1} \cdot p_2^{\alpha 2} \cdots p_m^{\alpha m}$ where $p_1, p_2, \ldots, p_m$ are distinct primes and $\alpha_1, \alpha_2, .., \alpha_m$ are positive integers, then ;

$$K(D_{nh})= \bigoplus_{i=1}^{2} K(D_n) \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-2} C_2 \oplus K(C_4) .$$

Example(2.27):-

To find $K(D_{15435h})$

$$K(D_{nh})= \bigoplus_{i=1}^{2} K(D_n) \bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)-2} C_2 \oplus K(C_4) .$$

$$K(D_{15435h}) = K(D_{3^2.7^3.5h}) = \bigoplus_{i=1}^{2} K(D_{3^2.7^3.5}) \bigoplus_{i=1}^{22} C_2 \oplus K(C_4) .$$

$$= \bigoplus_{i=1}^{16} C_{3^2} \bigoplus_{i=1}^{16} C_3 \bigoplus_{i=1}^{12} C_{7^3} \bigoplus_{i=1}^{12} C_{7^2} \bigoplus_{i=1}^{12} C_7 \bigoplus_{i=1}^{24} C_5 \bigoplus_{i=1}^{24} C_2 \oplus C_4 .$$

3. The Main Results

In this section we find the general form of the rational valued characters table of the group $(D_n \times C_3)$ (when n is an odd number) .

Theorem

If n is an odd number then

$$(D_n \times C_3) = \equiv^*(D_n) \times \equiv^*(C_3)$$

We denote by χ_i to the irreducible characters of D_n an θ_i to the rational valued characters of D_n

Now

$\equiv(C_3)=$

cl_α	[1]	$[\chi]$	χ^2
χ_1^*	1	1	1
χ_2^*	1	β	β^2
χ_3^*	1	β^2	β

Where $\beta = e^{2\pi i/3}$

And $\equiv^*(C_3)=$

$\Gamma - class$	[1]	$[\chi]$
θ_1^*	2	-1
θ_2^*	1	1

Every element g_{hk} in the group $D_n \times C_3$ can be written as follows

$g_{hk}=(g_h, g_k^*)$ where $g_h \epsilon D_n$,h=1,2,3,…,2n

And $g_k^* \epsilon C_3$,K=1,2,3

And each irreducible character $\chi_{(i,j)}$ of the group $D_n \times C_3$ can be written as $\chi_{(i,j)}=\chi_i \cdot \chi_j$ where i=1,2,…,$\frac{n-2}{2}+4$

Then

$$\chi_{(i,j)}(g_{hk}) = \begin{cases} \chi_i(g_h) & if\ j = 1\ and\ k = 1,2,3 \\ \chi_i(g_h) & if\ j = 2\ and\ k = 1 \\ \chi_i(g_h)\beta & if\ j = 2\ and\ k = 2 \\ \chi_i(g_h)\beta^2 & if\ j = 2\ and\ k = 3 \\ \chi_i(g_h) & if\ j = 3\ and\ k = 1 \\ \chi_i(g_h)\beta^2 & if\ j = 3\ and\ k = 2 \\ \chi_i(g_h) & if\ j = 3\ and\ k = 3 \end{cases}$$

If we denote by $\theta_{(i,j)}$ to the rational valued characters of the group $D_n \times C_3$,then we have

(i) $\theta_{(i,j)} = \sum_{\sigma \epsilon Gal(Q(\frac{\chi_{(i,2)}}{Q})} \sigma(\chi_{(i,2)}) + \sum_{\sigma \epsilon Gal(Q(\frac{\chi_{(i,3)}}{Q})} \sigma(\chi_{(i,3)})$

Then we have

$\theta_{(i,1)}(g_{hk}) = \sum_{\sigma \epsilon Gal(Q(\frac{\chi_{(i,2)}}{Q})} \sigma(\chi_{(i,2)})(g_{hk}) + \sum_{\sigma \epsilon Gal(Q(\frac{\chi_{(i,3)}}{Q})} \sigma(\chi_{(i,3)})(g_{hk})$

Now we have the following Cases

(a) If k=1 ,then

$\theta_{(i,1)}(g_{hk}) = \sum_{\sigma \epsilon Gal(\frac{Q(\chi_{(i,2)}(g_{hk}))}{Q}} \sigma(\chi_i(g_h)) + \sum_{\sigma \epsilon Gal(\frac{Q(\chi_{(i,3)}(g_{hk}))}{Q}} \sigma(\chi_i(g_h)) = \theta_i(g_h).2$

$= \theta_i(g_h).\theta_1^*(1^*)$

(b) **if k=2**

$\theta_{(i,1)}(g_{hk}) = \sum_{\sigma \epsilon Gal(\frac{Q(\chi_i(g_{hk}))}{Q})} \chi_i(g_h)\beta + \sum_{\sigma \epsilon Gal(\frac{Q(\chi_i(g_h))}{Q}} \chi_i(g_h)\beta^2$

$= \sum_{\sigma \epsilon Gal(\frac{Q(\chi_i(g_h))}{Q})} \chi_i(g_h)(\beta + \beta^2) = \theta_i(g_h)$**-1=**$\theta_i(g_h)$**.**$\theta_1^*(\chi)$

(ii) $\theta_{(i,2)} = \sum_{\sigma \epsilon Gal(\frac{Q(\chi_{(i,1)})}{Q})} \sigma(\chi_{(i,1)})$ **then** $\theta_{(i,2)}(g_{hk}) = \sum_{\sigma \epsilon Gal(Q(\chi_{(i,1)}(g_{hk})/Q} \sigma(\chi_{(i,1)}(g_{hk}))$

Then we have the following cases.

(a) If k=1

$\theta_{(i,2)}(g_{hk}) = \sum_{\sigma \epsilon Gal(\frac{Q(\chi_i(g_h))}{Q})} \sigma(\chi_i(g_h)) = \theta_i(g_h)$

$= \theta_i(g_h).1 = \theta_i.(g_h).\theta_2^*(1^*)$

(b) If k=2

$\theta_{(i,2)}(g_{hk}) = \sum_{\sigma \epsilon Gal(\frac{Q(\chi_i(g_h))}{Q})} \sigma(\chi_i(g_h)) = \theta_i.(g_h).\theta_2(\chi)$**.**

From (t) and (ii) we have

$\theta_{(i,j)} = \theta_i\ .\theta_j^*$ **for all i=1,2…,(n-2)/2 +4**
$and\ j = 1,2,3$ **.**

then we have

$\equiv^*(D_n \times C_3) = \equiv^*(D_n) \otimes \equiv^*(C_3)$**.**

Theorem(3.3) :-

The cyclic decomposition of the group $K(D_{nh} \times C_2)$, when n an odd number such that $n = \prod_{i=1}^{m} p_i^{\alpha_i}$

where $p_1, p_2, \ldots, p_m$ are distinct prime numbers is equal to :

$$K(D_{nh}\times C_2)=\overset{4}{\underset{i=1}{\oplus}} K(C_n) \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+2}{\underset{i=1}{\oplus}} C_4 \overset{2(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+1}{\underset{i=1}{\oplus}} C_2 \oplus C_8$$

$$=\overset{4}{\underset{i=1}{\oplus}}\left[\oplus\sum_{i=1}^{m}\left(\oplus\sum K\left(C_{p_i}^{\alpha_i}\right)\right)\left[\prod_{\substack{j\neq i\\ j=1}}^{m}(\alpha_j+1)\right]time\right] \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+2}{\underset{i=1}{\oplus}} C_4 \overset{2(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+1}{\underset{i=1}{\oplus}} C_2 \oplus C_8$$

Proof:-

By theorem (3.1) and proposition(2.7) we have

$$\equiv^*(D_{nh}\times C_2) = \equiv^*(D_{nh}) \otimes \equiv^*(C_2) = \equiv^*(D_n) \otimes \equiv^*(C_2) \otimes \equiv^*(C_2)$$

then

$$D(\equiv^*(D_{nh}\times C_2)) = D(\equiv^*(D_n)) \otimes D(\equiv^*(C_2)) \otimes D(\equiv^*(C_2))$$

From proposition(2.24) and theorem(2.14) we have

$$D(\equiv^*(D_n))=\begin{bmatrix} D\left(\equiv^*(C_n)\right) & 0 \\ 0 & -2 \end{bmatrix}$$

and $D(\equiv^*(C_2))=\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ then

$$D(\equiv^*(D_{nh}\times C_2)) = \begin{bmatrix} D\left(\equiv^*(C_n)\right) & 0 \\ 0 & -2 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 4D(\equiv^*(C_n)) &&&&&&& \\ & -8 &&&&&& \\ && 2D(\equiv^*(C_n)) &&&&& \\ &&& -4 &&&& \\ &&&& 2D(\equiv^*(C_n)) &&& \\ &&&&& -4 && \\ &&&&&& D(\equiv^*(C_n)) & \\ &&&&&&& -2 \end{bmatrix}$$

$= \text{diag } \{4d_1,4d_2,\ldots,4d_{((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))},-8,2d_1,2d_2,\ldots,$

$2d_{((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))},-4,2d_1,2d_2,\ldots, 2d_{((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))},-4,$

$d_1, d_2,\ldots, d_{((\alpha_1+1)(\alpha_2+1)\cdots(\alpha_m+1))},-2\}.$

where d_i is the invariant factor of $\equiv^*(C_n)$

Then by theorem (2.10) we have

$$K(D_{nh}\times C_2)= \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)}{\underset{i=1}{\oplus}} C_{4d_i} \oplus C_8 \overset{2(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)}{\underset{i=1}{\oplus}} C_{2d_i} \overset{2}{\underset{i=1}{\oplus}} C_4 \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)}{\underset{i=1}{\oplus}} C_{d_i} \oplus C_2$$

$$= \overset{4(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)}{\underset{i=1}{\oplus}} C_{d_i} \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+2}{\underset{i=1}{\oplus}} C_4 \overset{2(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1}{\underset{i=1}{\oplus}} C_2 \oplus C_8$$

$$= \overset{4}{\underset{i=1}{\oplus}} K(C_n) \overset{(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+2}{\underset{i=1}{\oplus}} C_4 \overset{2(\alpha_1+1)\cdot(\alpha_2+1)\cdots(\alpha_m+1)+1}{\underset{i=1}{\oplus}} C_2 \oplus C_8$$

From proposition(2.18) we have

$$K(D_{nh}\times C_2)=\bigoplus_{i=1}^{4}\left[\bigoplus\sum_{i=1}^{m}\left(\bigoplus\sum K\left(C_{p_i}^{\alpha_i}\right)\right)\left[\prod_{\substack{j\neq i\\ j=1}}^{m}(\alpha_j+1)\right] time\right]\bigoplus_{i=1}^{(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+2} C_4 \bigoplus_{i=1}^{2(\alpha_1+1)\cdot(\alpha_2+1)\cdot\ldots\ldots\cdot(\alpha_m+1)+1} C_2 \oplus C_8 \blacksquare$$

Example(3.4):-

To find $K(D_{7h}\times C_2)$.

$$K(D_{7h}\times C_2) = \bigoplus_{i=1}^{4} K(C_7) \bigoplus_{i=1}^{4} C_4 \bigoplus_{i=1}^{5} C_2 \oplus C_8$$

$$= \bigoplus_{i=1}^{4} C_7 \bigoplus_{i=1}^{4} C_4 \bigoplus_{i=1}^{5} C_2 \oplus C_8.$$

And To find $K(D_{63h}\times C_2)$.

$$K(D_{63h}\times C_2) = K(D_{7\cdot 3^2 h}\times C_2)$$

$$= \bigoplus_{i=1}^{4} [K(C_7) \oplus K(C_7) \oplus K(C_7) \oplus K(C_{3^2}) \oplus K(C_{3^2})] \bigoplus_{i=1}^{8} C_4 \bigoplus_{i=1}^{13} C_2 \oplus C_8$$

$$= \bigoplus_{i=1}^{12} C_7 \bigoplus_{i=1}^{8} C_{3^2} \bigoplus_{i=1}^{8} C_3 \bigoplus_{i=1}^{8} C_4 \bigoplus_{i=1}^{13} C_2 \oplus C_8 .$$

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