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The Dynamics of the 2-D Piecewise Tinkerbell Map

Dr.Ifitchar M.T.Al-Shara'a, Dunia mohomod kreem Al-Ftlawy
Mathematic Department, , University of Babylon, Hilla, Babylon, Iraq
Email:Iftichar_talb@yahoo.com and Email:Duniamohomod@yahoo.com

Abstract
We introduce a new map we call it a Piecewise Tinkerbell , we will prove some general properties of it, also we study the type of fixed points with respect of the parameter spaces. We prove it has positive Lypaunov exponents. We search the chaotic of Piecewise Tinkerbell by use the Wiggins's definition ,we show that the Piecewise Tinkerbell map has sensitive depends on initial condition by using the software (Matlab). Finally we investigate transitivity of it by varying the parameter of map.
Keywords: The Piecewise Tinkerbell Map, Sensitive Depends on Initial Condition, Topological Transitive Map, Lyapunov exponents .

1.Introduction

The Tinkerbell is an iterated discrete – time dynamical system that exhibits chaotic behavior in two – dimension. Viktor Avrutin ,Bernd Eckstein and Michael Schanz presented the attractor of the Tinkerbell map at a=0.9, b=-0.5169, c=2,d=0.5 (Viktor A ,B . and M. , 2007). Vicente Aboites ,Mario Wilson introduced The total transformation matrix [a,b,c,d] for a complete round trip of the above phase conjugating ring resonator(Vicente,M.Wilson ,2009).Yuan,S.,Jiang,T.& Jing,Z discaused Bifurcation and chaos in the Tinkerbell map(Yuan,S.,Jiang,T.& Jing,Z ,2011). Zhonggang Zeng given as an iteration and plot the point set $\{(x_k, y_k)| k= 0,\cdots,n\}$(Zhonggang Zeng ,2011). V. Aboites, Y. Barmenkov, A. Kir'yanov and M. Wilson They studied of the stability and chaos dynamic of the Tinkerbell(V. Aboites, Y. B., A.and M. Wilson,2012).In this work,we investigate the dynamical properties of the Piecewise Tinkerbell map which exhibit transition to chaos.The Piecewise Tinkerbell map is a simplification of the Tinkerbell map. The quadratic term in (x_n^2) is replaced by $|x_n|$ and (y_n^2) is replaced by $|y_n|$Then, we analyze the fixed points of the Piecewise Tinkerbell map and present algorithm to obtain Piecewise Tinkerbell attractor . Piecewise Tinkerbell's attractor is an attractor with a non – integer dimension(so called fractal dimension). The discreet mathematical models are gotten directly via scientific experiences, or by the use of the Poincar´e section for the study of a continuous model. One of these models is the Tinkerbell map. Many papers have described chaotic systems, one of the most famous being a two-dimensional discrete map which models the original Tinkerbell map . Moreover, it is possible to change the form of the Tinkerbell map for obtaining others chaotic attractors, this type of applications is used in secure communications using the notions of chaos. The Piecewise Tinkerbell map is 2-D noninvertible iterated map as follow by the following difference equations :

$$X_{n+1}=|x_n| -|y_n| +ax_n+by_n$$
…………(1.1)
$$Y_{n+1}=2x_ny_n+cx_n+dy_n$$

Where a,b,c and d are real parameters.

2. Definitions and Notations

In this section we introduce many fundamental definitions which we use in this work:

Let F:$R^2 \to R^2$ such that $F\binom{x}{y}=\binom{f(x,y)}{g(x,y)}$ be a map. Any pair$\binom{p}{q}$for which $f\binom{p}{q}$=p , $g\binom{p}{q}$=q is called a fixed point of the two dimensional dynamical system.F is C^1,if all of its first partial derivatives exist and are continuous.F is C^∞, if it's mixed k^{th} partial derivatives exist and are continuous for all K∈ Z^+,F is called a diffeomorphism provided it is F is one-to-one, F is onto, F is C^∞,it's inverse
F^{-1}: $R^2 \to R^2$ is C^∞ too. Let V be a subset of R^2, and v_o be any element in R^2. Consider F:V→R^2 be a map. Furthermore assume that the first partials of the coordinate maps f and g of exist at v_o .The differential of F at v_o is the linear map DF(v_o) defined on R^2 by $DF(v_o)=\begin{bmatrix}\frac{\partial f(v_o)}{\partial x} & \frac{\partial f(v_o)}{\partial y}\\ \frac{\partial g(v_o)}{\partial x} & \frac{\partial g(v_o)}{\partial y}\end{bmatrix}$, for all v in R^2.The determinant of DF(v_o) is called the Jacobian of F at v_o and is denoted by J=|det DF(v_o) |. If |det DF(v_o)|<0 then F is called area-contracting at v_o and if |det DF(v_o)|>0then F is called area -expanding at v_o . A point x ∈ X is a periodic point of period n > 0 if f^n (x) =x ,and f^r(x)≠ x for all r< n .

Definition(2.1) (Kolyada S., Snoha L. ,1997)

The f:X→X is said to be sensitive dependence on initial conditions if there exists ε > 0 such that for any x_o ∈ X and any open set U⊂X containing xo there exists yo ∈U and n∈ Z^+ such that d(f^n (x_o), f^n (y_o)) > ε That is ∃ ε> 0,∀x, ∀δ > 0, ∃y∈ B δ(x), ∃n:d(f^n (x_o), f^n (y_o))≥ ε.

Remark(2.2) (Gulick D. ,1992)

If p is period –n point of f such that $|(f^{(n)})'(p)| < 1$ then f cannot has sensitive dependence on initial conditions at p. The iterates of a fixed point not wander at all; they remain the same point .At the other end of the spectrum are points whose iterates wander all over the domain of the function. with such points are called transitive .

Definition(2.3) (Kolyada S., Snoha L. ,1997)

Every nonempty open subset U of X visits every nonempty open subset V of X in the following sense :f n(U) ∩V≠ ∅ for some n∈ N. If the system (X,f)has this property then it is called topologically transitive, simply called transitive.

Definition (2.4) (Fotion A.,2005)

Let f:X→X be a continuous map and X be a metric space. Then the map f is said to be chaotic according to Wiggins or W-chaotic if :

1-f is topologically transitive.

2-f is exhibits sensitive dependent on initial condition.

Definition(2.5)(Zhang W. ,2006)

An attractor is said to be strange if it contains a dense orbit with positive Lyapunov exponent .

Definition(2.6) (Sturman R.Ottino J.M.and Wiggins's., 2006)

The map f : R^n→R^n will have n Lyapunov exponents, say L_1 (x,v) , L_2 (x,v) , …, L_n (x,v) for a system of n variable. Then the Lyapunov exponent is the maximum n Lyapunov exponents that is L(x,v)=max{ L_1 (x,v) , L_2 (x,v) ,…, L_n (x,v) }. Where v=(v_1, v_2,…, v_n).

3. General Properties of the Piecewise Tinkerbell Map

The goal of this section is to study the type of fixed points in different parameter spaces . It is clear that $\binom{0}{0}$is one of the fixed points of $T^p_{a,b,c,d}$ also we can find the other fixed point by four cases, we show that in the next table.

Table: A fixed points of four cases of $T^p_{a,b,c,d}$:

	$x \geq 0, y \geq$	$x<0, y<0$	$x<0, y>0$	$x>0, y<0$
x	$\frac{a-c-ad+bc}{2a}$	$\frac{2d-2+c+a-ad+bc}{2(a-2)}$	$\frac{2d-c-ad+bc}{2(a-2)}$	$\frac{bc+c-ad}{2a}$
y	$\frac{ac-c^2-acd+bc^2}{2(ac-abc)}$	$\frac{2cd-2c+c^2+ac-acd+bc^2}{4c+4bc-2ac-2abc}$	$\frac{2cd-c^2-acd+bc^2}{4c+4bc-2ac-2abc}$	$\frac{c^2+bc^2-acd}{-2ac-2abc}$

Proposition(3.1)

Let $T^p_{a,b,c,d}$:R^2→R^2 be thePiecewiseTinkerbell map and a,b,c,d be any real constants$\binom{x}{y}$thefixed points, if x≥0 , y≥0 then :

(1)$DT^p_{a,b,c,d}\binom{0}{0}=\left|\det T^p_{a,b,c,d}\binom{0}{0}\right|=|d(1+a)-c(-1+b)|$

(2)$DT^p_{a,b,c,d}\begin{pmatrix}\frac{a-c-ad+bc}{2a}\\ \frac{ac-c^2-acd+bc^2}{2(ac-abc)}\end{pmatrix}=\left|\det T^p_{a,b,c,d}\begin{pmatrix}\frac{a-c-ad+bc}{2a}\\ \frac{ac-c^2-acd+bc^2}{2(ac-abc)}\end{pmatrix}\right| = |a^3+a^2-ac+abc-a^2c+a^2bc+a^2c^2-ac^3-$ a2c2d+3abc3+a2c3−3a2bc3+2a2bc2d−2a2bc2−2ab2c3+3a2b2c3+a2b2c2−a2b2c2d−ab2c3+ab3c3− a2b3c3.

(3) if $(1+a+d)^2-4(c-cb+d+ad)\geq 0$ then the eigen values of $DT^p_{a,b,c,d}\binom{0}{0}$ are the real numbers $\lambda_{1,2}=\frac{1+a+d\pm\sqrt{(1+a+d)^2-4(c-cb+d+ad)}}{2}$

(4)if $(2a^2+a^3-ac+abc)^2-4a^2(M_1)\geq 0$ then the eigen values of $DT^p_{a,b,c,d}\begin{pmatrix}\frac{a-c-ad+bc}{2a}\\ \frac{ac-c^2-acd+bc^2}{2(ac-abc)}\end{pmatrix}$ are the real number

$\lambda_{1,2}=\frac{2a^2+a^3-ac+abc\pm\sqrt{(2a^2+a^3-ac+abc)^2-4a^2(M_1)}}{2a^2}$

(5)$T^p_{a,b,c,d}$is not one-to-one map.

(6)$T^p_{a,b,c,d}$is C^∞.

(7)$T^{p}_{a,b,c,d}$ is not diffeomorphism.

Proof :(1)
By hypothesis x≥0 , y≥0 then $T^{p}_{a,b,c,d}$ is
$T^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x(1+a)-y(1-b)\\2xy+cx+dy\end{pmatrix}$,thus
$f\begin{pmatrix}x\\y\end{pmatrix}$= x(1+a)-y(1-b) , $g\begin{pmatrix}x\\y\end{pmatrix}$=2xy+cx+dy
$\frac{\partial f}{\partial x}$=1+a , $\frac{\partial f}{\partial y}$= -1+b ,$\frac{\partial g}{\partial x}$= 2y+c , $\frac{\partial g}{\partial y}$= 2x+d
$detT^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+a & -1+b\\2y+c & 2x+d\end{pmatrix}$, $detT^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}1+a & -1+b\\c & d\end{pmatrix}$
$J\,T^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}=\left|detT^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}\right|=|d(1+a)-c(-1+b)|\ldots$ ……...(3.1) .
(2):By part (1)$DT^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+a & -1+b\\2y+c & 2x+d\end{pmatrix}$
Then $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{a-c-ad+bc}{2a}\\ \frac{ac-c^2-acd+bc^2}{2(ac-abc)}\end{pmatrix}=\begin{pmatrix}1+a & -1+b\\ \frac{ac-c^2-acd+bc^2+ac^2-abc^2}{ac-abc} & \frac{a-c+bc}{a}\end{pmatrix}$

$J=|(1+a)(a)(a-c+bc)-(-1+b)(ac-abc)(ac-c^2-acd+bc^2+ac^2-abc^2)|$

$= |a^3+a^2-ac+abc-a^2c+a^2bc+a^2c^2-ac^3-a^2c^2d+3abc^3+a^2c^3-3a^2bc^3+2a^2bc^2d-2a^2bc^2-$ 2ab2c3+3a2b2c3+a2b2c2−a2b2c2d−ab2c3+ab3c3−a2b3c3… . ...(3.2)

(3):
To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}$then must be satisfied the characteristic equation λ^2-λ(1+a+d)+(c+ad+d-bc)=0 and the solutions of this equation are $\lambda_{1,2}$ where
$\lambda_{1,2}=\frac{1+a+d\pm\sqrt{(1+a+d)^2-4(c-cb+d+ad)}}{2}$ ....(3.3)
By hypothesis $(1+a+d)^2-4(c-cb+d+ad)\geq 0$,so $\lambda_{1,2}$ are real number. Thus, the proposition is satisfy .

(4): To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{a-c-ad+bc}{2a}\\ \frac{ac-c^2-acd+bc^2}{2(ac-abc)}\end{pmatrix}$ then must be satisfied the characteristic equation $a^2\lambda^2-\lambda(2a^2+a^3-ac+abc)+ a^2-ac+abc+a^3-a^2c+a^2bc+a^2c^2-ac^3-a^2c^2d+3abc^3+a^2c^3-4a^2bc^3-3ab^2c^3+3a^2b^2c^3+a^2bc^2d+a^2b^2c^2-a^2b^2c^2d+ab^3c^3-a^2b^3c^3=0$ and the solutions of this equation are $\lambda_{1,2}$ where $\lambda_{1,2}=\frac{2a^2+a^3-ac+abc\pm\sqrt{(2a^2+a^3-ac+abc)^2-4a^2(M_1)}}{2a^2}$ ……(3.4)
(5):
$x-y+ax+by=0$ and $2xy+cx+dy=0$,so $x,y\neq 0$, then the kernel of ($T^{p}_{a,b,c,d}$) is not singleton set which hence the identity of R^2 so $T^{p}_{a,b,c,d}$ is not one-to-one .
(6):
Since F and G are polynomial of degree two, so all first partial derivatives exist and continuous, we get that all its mixed k^{th} partial derivatives exist from section two $T^{p}_{a,b,c,d}$is C^{∞}.
(7):
By part(5) of this proposition $T^{p}_{a,b,c,d}$is not diffeomorphism from section two.

■

Assume $M_2=(8a-3ac-3abc-5a^2-4+2c+2bc+a^2c+a^3+a^2bc-4c^2d+4c^2-6c^3-4ac^2+4ac^2d-18bc^3+5ac^3+12abc^3-8bc^2d+8bc^2-6abc^2+6abc^2d-18b^2c^3+12ab^2c^3-8b^2c^2d+4b^2c^2-4ab^2c^2+4ab^2c^2d-2b^3c^3+5ab^3c^3-4b^3c^2d+a^2c^2-a^2c^2d-a^2c^3-2a^2bc^3+a^2bc^2-a^2bc^2d-2a^2b^2c^3+a^2b^2c^2-a^2b^2c^2d-a^2b^3c^3+2a\,b^3c^2d)$.
Assume C_1=($ac+a^3-4a^2+4a+abc-2c-2bc$).

Proposition(3.2)

Let $T^{p}_{a,b,c,d}: R^2 \to R^2$ be the Piecewise Tinkerbell map and a,b,c,d be any real constants and $\binom{0}{0}$ is a fixed point, If x<0, y<0 then :

(1)$J\,T^{p}_{a,b,c,d}\binom{0}{0} = \left|\det T^{p}_{a,b,c,d}\binom{0}{0}\right| = |d(-1+a) - c(1+b)|$

(2)$JT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-2+c+a-ad+bc}{2(a-2)}\\ \frac{2cd-2c+ac-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix} = \left|\det T^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-2+c+a-ad+bc}{2(a-2)}\\ \frac{2cd-2c+ac-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix}\right| = |8a - 3ac - 3abc - 5a^2 + a^3 - 4 -$ 16bc3−6c3−6b3c3+5ac3+5abc3+12ab2c3+5ab3c3+2c+2bc+a2bc−14b2c3+4c2+4c2d−4ac2+4ac2d +10abc3−8bc2d+8bc2+a2c−6abc2+6abc2d−8b2c2d+4b2c2−4ab2c2−4b3c2d+4ab2c2d−2a2bc3−2a 2b2c3−a2b2c2d−a2b3c3+2ab3c2d−a2c3+a2c2−a2c2d+a2bc2−a2bc2d

(3)If $(a-1+d)^2 - 4(ad-d-c-cb) \geq 0$ then the eigen values of det $T^{p}_{a,b,c,d}\binom{0}{0}$ are the real numbers

$\lambda_{1,2} = \frac{(a-1+d) \pm \sqrt{(a-1+d)^2 - 4(ad-d-c-cb)}}{2}$

(4) If $C_1{}^2 - 4(4+a^2 4a)(M_2) \geq 0$ then the eigen values of det $T^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-2+c+a-ad+bc}{2(a-2)}\\ \frac{2cd-2c+ac-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix}$ are the real numbers

$\lambda_{1,2} = \frac{C_1 \pm \sqrt{C_1{}^2 - 4(4+a^2-4a)(M_2)}}{2(4+a^2-4a)}$

(5)$T^{p}_{a,b,c,d}$ is not one-to-one map .

(6)$T^{p}_{a,b,c,d}$ is C^{∞}.

(7)$T^{p}_{a,b,c,d}$ is not diffeomorphism.

Proof(1):

By hypothesis x<0, y<0 then $T^{p}_{a,b,c,d}$ is

$T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}x(a-1)+y(1+b)\\ 2xy+cx+dy\end{pmatrix}$, thus

$\det T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}-1+a & 1+b\\ 2y+c & 2x+d\end{pmatrix}$,hence

$\det T^{p}_{a,b,c,d}\binom{0}{0} = \begin{pmatrix}-1+a & 1+b\\ c & d\end{pmatrix}$

$J\,T^{p}_{a,b,c,d}\binom{0}{0} = \left|\det T^{p}_{a,b,c,d}\binom{0}{0}\right| = |d(-1+a) - c(1+b)|$ ....(3.5)

(2):By part (1)$\det T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}-1+a & 1+b\\ 2y+c & 2x+d\end{pmatrix}$,hence

Then

$DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-2+c+a-ad+bc}{2(a-2)}\\ \frac{2cd-2c+ac-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix} =$

$\begin{pmatrix}-1+a & 1+b\\ \frac{-2c+3c^2-acd+3bc^2-ac^2-abc^2+2cd+ac}{2c+2bc-ac-abc} & \frac{a-2+c+bc}{a-2}\end{pmatrix} = |8a - 3ac - 3abc - 5a^2 + a^3 - 4 - 16bc^3 - 6c^3 - 6b^3c^3 +$ 5ac3+5abc3+12ab2c3+5ab3c3+2c+2bc+a2bc−14b2c3+4c2+4c2d−4ac2+4ac2d+10abc3−8bc2d+8bc 2+a2c−6abc2+6abc2d−8b2c2d+4b2c2−4ab2c2−4b3c2d+4ab2c2d−2a2bc3−2a2b2c3−a2b2c2d−a2b3 c3+2ab3c2d−a2c3+a2c2−a2c2d+a2bc2−a2bc2d ....(3.6)

(3):

To find the eigenvalues of $DT^{p}_{a,b,c,d}\binom{0}{0}$ then must be satisfied the characteristic equation $\lambda^2 - \lambda(a-1+d) + (ad-d-c-bc) = 0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2} = \frac{(a-1+d) \pm \sqrt{(a-1+d)^2 - 4(ad-d-c-cb)}}{2}$.. ....(3.7)

By hypothesis $(a-1+d)^2 - 4(ad-d-c-cb) \geq 0$,so $\lambda_{1,2}$ are real number.

(4):

To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-2+c+a-ad+bc}{2(a-2)}\\ \frac{2cd-2c+ac-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix}$ then must be satisfied the characteristic equation $\lambda^2(4+a^2-4a)$-$\lambda C_1 + M_2 = 0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2} = \frac{C_1 \pm \sqrt{C_1{}^2-4(4+a^2-4a)M_2}}{2(4+a^2-4a)}$. ....(3.8)

(5):

$-x+y+ax+by=0$ and $2xy+cx+dy=0$, so $x,y \neq 0$,then the kernel of ($T^{p}_{a,b,c,d}$) is not singleton set which hence the identity of R^2 so$T^{p}_{a,b,c,d}$ is not one-to-one.

(6): and (7): has the same proof of the proposition(3.1)(6,7). ■

Assume $M_3 = 3ac - 3abc$ +2bc-2c+ $a^2bc - a^2c + 4c^2d + 2c^3 - 2ac^2d + 6bc^3 - 7abc^3 - b^2c^3 + 3ab^2c^3 - 2ac^2d - ac^3 + a^2c^2 + a^2bc^3 - a^2b^2c^3 - 4b^2c^2d + 4ab^2c^2d - 6b^3c^3 + 5ab^3c^3 - a^2b^2c^2d - a^2b^3c^3$.

Assume $C_2 = 12a - 4 - 5a^2 + a^3 + abc - ac - 2bc + 2c$

Proposition(3.3)

Let $T^{p}_{a,b,c,d}: R^2 \to R^2$ be the Piecewise Tinkerbell map and a,b,c,d be any real constants and $\binom{x}{y}$the fixed points, If x<0, y> 0then

(1) $J T^{p}_{a,b,c,d}\binom{0}{0} = \left|\det T^{p}_{a,b,c,d}\binom{0}{0}\right| = |d(-1+a) - c(-1+b)|$

(2) $J\, T^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd-c^2+bc^2}{4c+4bc-2ac-2abc}\end{pmatrix} = \left|\det T^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd-c^2+bc^2}{4c+4bc-2ac-2abc}\end{pmatrix}\right| = |3ac - 3abc + 2bc - a^2c - 2c + a^2bc +$ 4c2d+2c3−4ac2d+6bc3−7abc3−ac3+a2c2d+a2c3−2b2c3+3ab2c3+a2bc3+4ab2c2d−4b2c2d−6b3c3 +5ab3c3−a2b2c2d−a2b2c3−a2b3c3.

(3) If $(a-1+d)^2 - 4(ad-d+c-cb) \geq 0$ then the eigen values of $\det T^{p}_{a,b,c,d}\binom{0}{0}$ are the real numbers

$\lambda_{1,2} = \frac{(a-1+d) \pm \sqrt{(a-1+d)^2-4(ad-d+c-cb)}}{2}$

(4)If$C_2{}^2 - 4(a^2-4a+4)M_2 \geq 0$ then the eigen values of $\det T^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd-c^2+bc^2}{4c+4bc-2ac-2abc}\end{pmatrix}$ are the real numbers

$\lambda_{1,2} = \frac{C_2 \pm \sqrt{C_2{}^2-4(a^2-4a+4)M_2}}{2(a^2-4a+4)}$

(5)$T^{p}_{a,b,c,d}$is not one-to-one map.

(6)$T^{p}_{a,b,c,d}$is C^∞.

(7)$T^{p}_{a,b,c,d}$is not diffeomorphism .

Proof(1):

By hypothesis x<0, y> 0 then

$T^{p}_{a,b,c,d}$ is$T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}x(a-1) - y(1-b)\\ 2xy + cx + dy\end{pmatrix}$, thus

$\det T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}-1+a & -1+b\\ 2y+c & 2x+d\end{pmatrix}$,hence

$\det T^{p}_{a,b,c,d}\binom{0}{0} = \begin{pmatrix}-1+a & -1+b\\ c & d\end{pmatrix}$

$JT^{p}_{a,b,c,d}\binom{0}{0} = \left|\det T^{p}_{a,b,c,d}\binom{0}{0}\right| = |d(-1+a) - c(-b)|$ ....(3.9)

(2): By part (1)$\det T^{p}_{a,b,c,d}\binom{x}{y} = \begin{pmatrix}-1+a & -1+b\\ 2y+c & 2x+d\end{pmatrix}$,hence

Then

$DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix} = \begin{pmatrix}-1+a & -1+b\\ \frac{-acd+3bc^2-ac^2-abc^2+2cd+ac^2}{2c+2bc-ac-abc} & \frac{bc-c}{a-2}\end{pmatrix} = JT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix} = |3ac -$
3abc+2bc−a2c−2c+a2bc+4c2d+2c3−4ac2d+6bc3−7abc3−ac3+a2c2d+a2c3−2b2c3+3ab2c3+a2bc3 +4ab2c2d−4b2c2d−6b3c3+5ab3c3−a2b2c2d−a2b2c3−a2b3c3.
...................................(3.10)

(3):
To find the eigenvalues of $DT^{p}_{a,b,c,d}\binom{0}{0}$ then must be satisfied the characteristic equation $\lambda^2 - \lambda(a-1+d) + (c-bc+ad-d) = 0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2} = \frac{(a-1+d)\pm\sqrt{(a-1+d)^2-4(ad-d+c-cb)}}{2}$. ....(3.11)

By hypothesis $(a-1+d)^2 - 4(ad-d+c-cb) \geq 0$ so $\lambda_{1,2}$ are real number.

(4):
To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{2d-c-ad+bc}{2(a-2)}\\ \frac{2cd-acd+bc^2+c^2}{4c+4bc-2ac-2abc}\end{pmatrix}$ then must be satisfied the characteristic equation $\lambda^2(a^2-4a+4)$- $\lambda C_2 + M_2 = 0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2} = \frac{C_2\pm\sqrt{C_2{}^2-4(a^2-4a+4)M_2}}{2(a^2-4a+4)}$. ....(3.12)

By hypothesis $C_2{}^2 - 4(a^2-4a+4)M_2 \geq 0$ so $\lambda_{1,2}$ are real number.

(5):
$-x-y+ax+by=0$ and $2xy+cx+dy=0$
$x, y \neq 0$, Then $T^{p}_{a,b,c,d}$ is not one-to-one map.

(6): and (7) has the same proof of the proposition (2.2.1)(6,7). ∎

Assume $M_3 = (-ac^3 + a^2c^2d - a^2c^3 + a^2bc^3 - abc^3 + ab^2c^3 + ab^3c^3 - a^2b^2c^2d + a^2b^2c^3 - a^2b^3c^3)$

Proposition(3.4)
Let $T^{p}_{a,b,c,d}: R^2 \rightarrow R^2$ be the Piecewise Tinkerbell map and a,b,c,d be any real constants at $\binom{x}{y}$ the fixed points, If x>0, y< 0 then:

(1) $JT^{p}_{a,b,c,d}\binom{0}{0} = \left|\det T^{p}_{a,b,c,d}\binom{0}{0}\right| = |d(1+a) - c(1+b)|$.

(2) $JT^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix} = \left|\det T^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix}\right| = \left|abc + ac + a^2bc + a^2c - ac^3 - abc^3 + a^2c^2d - a^2c^3 + a^2bc^3 + ab^2c^3 + ab^3c^3 - a^2b^2c^2d + a^2b^2c^3 - a^2b^3c^3\right|$.

(3) If $(a+1+d)^2 - 4(ad+d-c-cb) \geq 0$ then the eigen values of $\det T^{p}_{a,b,c,d}\binom{0}{0}$ are the real numbers $\lambda_{1,2} = \frac{(a+1+d)\pm\sqrt{(a+1+d)^2-4(ad+d-c-cb)}}{2}$

(4) If $(ac+abc+a^2+a^3)^2 - 4a^2M_3 \geq 0$ then the eigen values of $\det T^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix}$ are the real numbers . $\lambda_{1,2} = \frac{ac+abc+a^2+a^3\pm\sqrt{(ac+abc+a^2+a^3)^2-4a^2M_3}}{2a^2}$

(5) $T^{p}_{a,b,c,d}$ is not one-to-one map.
(6) $T^{p}_{a,b,c,d}$ is C^{∞}.
(7) $T^{p}_{a,b,c,d}$ is not diffeomorphism.

Proof(1):
By hypothesis x>0, y< 0then $T^{p}_{a,b,c,d}$ is

$T^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x(1+a)+y(1+b)\\2xy+cx+dy\end{pmatrix}$,thus

$DetT^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+a & 1+b\\2y+c & 2x+d\end{pmatrix}$,hence $detT^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}=\begin{pmatrix}1+a & 1+b\\c & d\end{pmatrix}$

$J\,T^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}=\left|\det T^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}\right|=|d(-1+a)-c(-b)|.$ ....(3.13)

(2): By part (1)$detT^{p}_{a,b,c,d}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1+a & 1+b\\2y+c & 2x+d\end{pmatrix}$,hence

Then $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix}=\begin{pmatrix}1+a & 1+b\\ \frac{-acd+ac^2-abc^2+bc^2+ac^2}{ac-abc} & \frac{bc+c}{a}\end{pmatrix}$

$=JT^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix}$

$\left|\begin{matrix}abc+ac+a^2bc+a^2c-ac^3-abc^3+a^2c^2d-a^2c^3+a^2bc^3+ab^2c^3+a\\ b^3c^3-a^2b^2c^2d+a^2b^2c^3-a^2b^3c^3\end{matrix}\right|$.........(3.14)

(3):

To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}0\\0\end{pmatrix}$then must be satisfied the characteristic equation $\lambda^2-\lambda(a-1+d)+(c-bc+ad-d)=0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2}=\frac{(a-1+d)\pm\sqrt{(a-1+d)^2-4(ad-d+c-cb)}}{2}$ ....(3.15)

By hypothesis$(a-1+d)^2-4(ad-d+c-cb)\geq 0$ so $\lambda_{1,2}$are real number.

(4):

To find the eigenvalues of $DT^{p}_{a,b,c,d}\begin{pmatrix}\frac{bc+c-ad}{2a}\\ \frac{c^2+bc^2-acd}{-2ac-2abc}\end{pmatrix}$then must be satisfied the characteristic equation $\lambda^2a^2-\lambda(a^2+a^3+abc+ac)+M_3=0$ and the solutions of this equation are $\lambda_{1,2}$ where

$\lambda_{1,2}=\frac{a^2+a^3+abc+ac\pm\sqrt{(a^2+a^3+abc+ac)^2-4a^2M_3}}{2a^2}.$ ....(3.16)

By hypothesis$(a^2+a^3+abc+ac)^2-4a^2M_3\geq 0$, so $\lambda_{1,2}$ are real number.

(5):

$x+y+ax+by=0$ and $2xy+cx+dy=0$

$x,y\neq 0$,Then$T^{p}_{a,b,c,d}$ is not one-to-one map.

(6): and (7) has the same proof of the proposition(3.1)(6,7).

■

Remark(3.5)

The Tinkerbell and the Piecewise Tinkerbell map has the same properties in Proposition(3.4)(5,6,7), but is the different of Jacobian values and the eigenvalues in all cases.

4. Lyapunov exponents of The Piecewise Tinkerbell Map

For a map on R^m, each orbit has m Lyapunov numbers, which measure the rates of separation from the current orbit point along m orthogonal directions. These directions are determined by the dynamics of the map. The first will be the direction along which the separation between nearby points is the greatest (or which is least contracting, if the map is contracting in all directions). The second will be the direction of greatest separation, chosen from all directions perpendicular to the first. The third will have the most stretching of all directions perpendicular to the first two directions, and so on.

We can find the Lyapunov exponents of the Piecewise Tinkerbell map

Assume $k_1=\frac{\sqrt{(1+2x+a+d)^2-4(2x+d+2ax+ad+2y-2by+c-bc)}}{2}$

Lemma(4.1)

Let $T^{p}_{a,b,c,d}: R^2\to R^2$be the Piecewise Tinkerbell map ,x≥0 , y≥ 0, if $\frac{(1+2x+a+d)}{2}>0$,k_1^2>0,then the map has positive Lyapunov exponents .

Proof: : Let X= $\binom{x}{y} \in R^2$ the Lyapunov exponents of $T^p_{a,b,c,d}$ is given by the formula by $L_1\left(\binom{x}{y}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\|$, since $T^p_{a,b,c,d}$ have two eigenvalue $|\lambda_1| = \frac{1}{|\lambda_2|}$ .If $|\lambda_1|$ <1 then $L_1\left(\binom{x}{y}, v_1\right)=\lim_{n\to\infty} \ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\| > \ln\left(\frac{(1+2x+a+d)}{2} + k_1\right)$ by hypothesis L_1>0 ,such that if $|\lambda_1|$ >1 then $L_2\left(\binom{x}{y}, v_2\right)=\lim_{n\to\infty} \ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_2\right\| < \ln\left(\frac{(1+2x+a+d)}{2} - k_1\right)$, thus the Lyapunov exponents L(x,v)=max{L_1 (x_1 ,v), L_2 (x_2 ,v)}, hence the Lyapunov exponents of the Piecewise Tinkerbell map is positive.

■

Assume $k_2 = \frac{\sqrt{(-1+2x+a+d)^2-4(-2x-d+2ax+ad-2y-2by-c-bc)}}{2}$

Lemma(4.2)

Let $T^p_{a,b,c,d}$: $R^2 \to R^2$ be the Piecewise Tinkerbell map ,x<0 , y< 0, if $\frac{(-1+2x+a+d)}{2} > 0, K_2^2 > 0$,then the map has positive Lyapunov exponents .

Proof: : Let X= $\binom{x}{y} \in R^2$ the Lyapunov exponents of $T^p_{a,b,c,d}$ is given by the formula by $L_1\left(\binom{x}{y}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}\ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\|$, since $T^p_{a,b,c,d}$ have two eigenvalue $|\lambda_1| = \frac{1}{|\lambda_2|}$ .If $|\lambda_1|$ <1 then $L_1\left(\binom{x}{y}, v_1\right)=\lim_{n\to\infty} \ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\| > \ln\left(\frac{-1+2x+a+d}{2} + k_2\right)$ by hypothesis L_1>0 ,such that if $|\lambda_1|$ >1 then $L_2\left(\binom{x}{y}, v_2\right)=\lim_{n\to\infty} \ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_2\right\| < \ln\left(\frac{(-1+2x+a+d)}{2} - k_2\right)$, thus the Lyapunov exponents L(x,v)=max{L_1 (x_1 ,v), L_2 (x_2 ,v)}, hence the Lyapunov exponents of the Piecewise Tinkerbell map is positive.

■

Assume $k_3 = \frac{\sqrt{(1+2x+a+d)^2-4(2x+d+2ax+ad-2y-2by-c-bc)}}{2}$

Lemma(4.3)

Let $T^p_{a,b,c,d}$: $R^2 \to R^2$ be the Piecewise Tinkerbell map ,x>0 , y< 0, if $\frac{(1+2x+a+d)}{2} > 0, k_3^2 > 0$,then the map has positive Lyapunov exponents .

Proof: : Let X= $\binom{x}{y} \in R^2$ the Lyapunov exponents of $T^p_{a,b,c,d}$ is given by the formula by $L_1\left(\binom{x}{y}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\|$, since $T^p_{a,b,c,d}$ have two eigenvalue $|\lambda_1| = \frac{1}{|\lambda_2|}$ .If $|\lambda_1|$ <1 then $L_1\left(\binom{x}{y}, v_1\right)=\lim_{n\to\infty} ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\| > ln\left(\frac{1+2x+a+d}{2} + k_3\right)$ by hypothesis L_1 >0 ,such that if $|\lambda_1|$ >1 then $L_2\left(\binom{x}{y}, v_2\right)=\lim_{n\to\infty} ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_2\right\| < \ln\left(\frac{(1+2x+a+d)}{2} - k_3\right)$, thus the Lyapunov exponents L(x,v)=max{L_1 (x_1 ,v), L_2 (x_2 ,v)}, hence the Lyapunov exponents of the Piecewise Tinkerbell map is positive.

■

Assume $k_4 = \frac{\sqrt{(-1+2x+a+d)^2-4(-2x-d+2ax+ad+2y-2by+c-bc)}}{2}$

Lemma(4.4)

Let $T^p_{a,b,c,d}$: $R^2 \to R^2$ be the Piecewise Tinkerbell map ,x<0 , y> 0, if $\frac{(-1+2x+a+d)}{2} > 0, k_4^2 > 0$,then the map has positive Lyapunov exponents .

Proof: : Let X= $\binom{x}{y} \in R^2$ the Lyapunov exponents of $T^p_{a,b,c,d}$ is given by the formula by $L_1\left(\binom{x}{y}, v_1\right) = \lim_{n\to\infty}\frac{1}{n}ln\left\|\left(DT^p_{a,b,c,d}{}^n\right)\binom{x}{y}v_1\right\|$, since $T^p_{a,b,c,d}$ have two eigenvalue $|\lambda_1| = \frac{1}{|\lambda_2|}$ .If $|\lambda_1|$ <1 then$L_1\left(\binom{x}{y}, v_1\right)=\lim_{n\to\infty} ln\left\|(DT^p_{a,b,c,d})^n\binom{x}{y}v_1\right\| > ln\left(\frac{-1+2x+a+d}{2} + k_4\right)$ by hypothesisL_1>0 ,such that if

$|\lambda_1| > 1$ then $L_2\left(\begin{pmatrix} x \\ y \end{pmatrix}, v_2\right) = \lim_{n\to\infty} \; ln \left\| (DT^{p}_{a,b,c,d})^{n} \begin{pmatrix} x \\ y \end{pmatrix} v_2 \right\| < ln(\frac{(-1+2x+a+d)}{2} - k_4)$, thus the Lyapunov exponents L(x,v)=max{L_1 (x_1 ,v), L_2 (x_2 ,v)}, hence the Lyapunov exponents of the Piecewise Tinkerbell map is positive.

■

Assume $k = k_1, k_2, k_3, k_4$

Theorem(4.5)

Let $T^{p}_{a,b,c,d}$: $R^2 \to R^2$ be the Piecewise Tinkerbell map , if $k^2>0$,then the map has positive Lyapunov exponents .

Proof: By lemma(4.1),(4.2),(4.3),(4.4).

As you can see, the values of the eigenvalues λ_1 and λ_2 of F determine the behavior of the iterates of F .We summarize some of them (Gulick D,1992):

(i)Suppose that λ_1 and λ_2 are real .If $|\lambda_1| < 1$ and $|\lambda_2| < 1$, we have $F^{[n]}$(v)→0 for all v in R^2, so that 0 is an attracting fixed point of F. If $|\lambda_1| > 1$ and $|\lambda_2| \geq 1$,

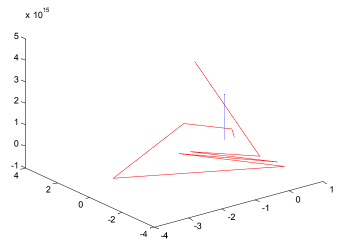
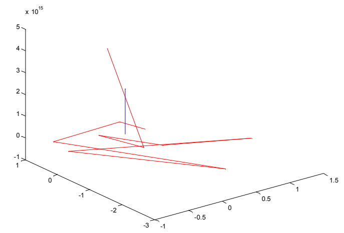
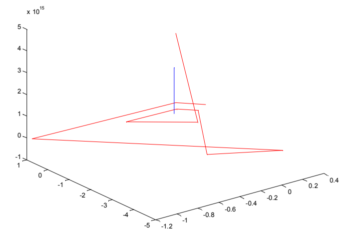
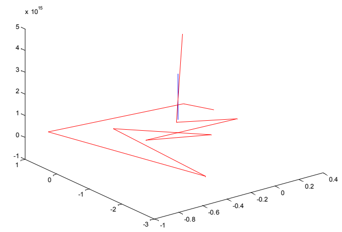
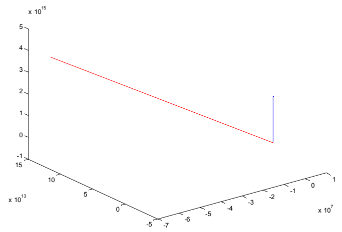
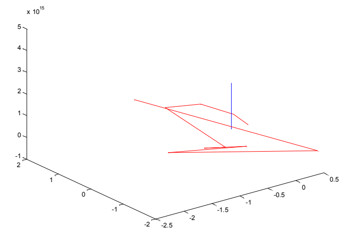
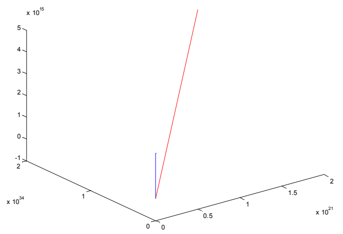
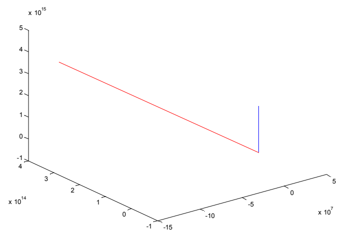
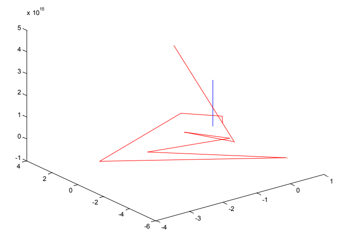
then $\|F^{[n]}$ (v)$\| \to \infty$ for all nonzero v in R^2, so that 0 is a repelling fixed point of F

(ii) Suppose that λ_1 and λ_2 are real .If $|\lambda_1| > 1$ and $|\lambda_2| < 1$ then 0 is a saddle point .

(iii)If the eigenvalues λ_1 and λ_2 are complex, then F has a rotation component .

5. Sensitive Dependent on Initial Condition of The Piecewise Tinkerbell Map

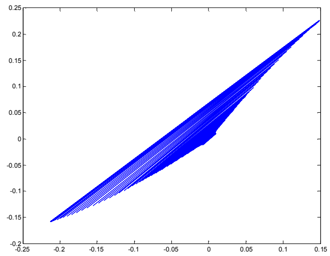
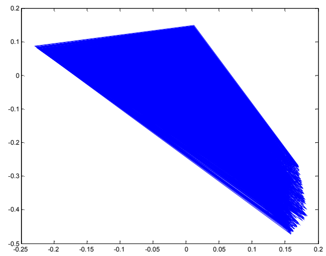
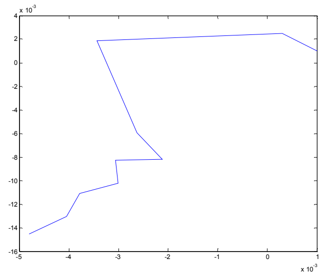
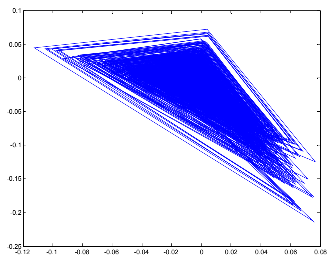
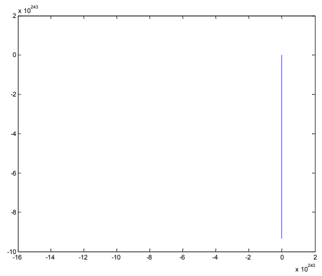
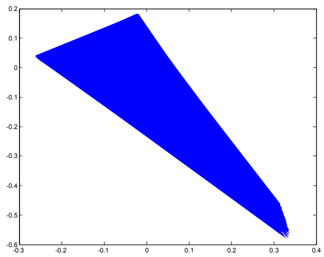
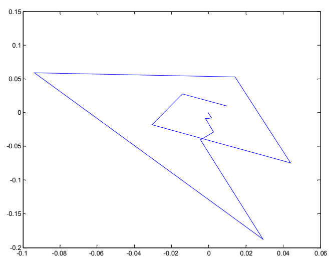
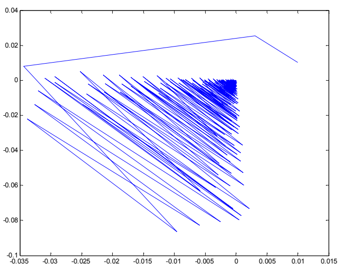
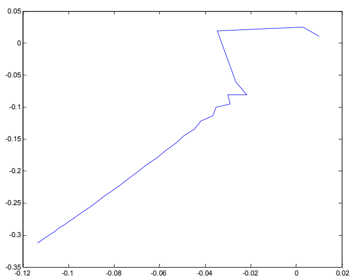
In this section we study the chaotic behavior of The Piecewise Tinkerbell map is depend on definition (2.1)which is refered to in section two. Now , we study the sensitivity to initial condition of map) by varying the control parameters (a,b,c,d) by using (Matlab) to analysis of view for sensitivity dependent on initial condition .this work show as in figure(1) Now consider the map we get sensitivity to initial condition on the initial point (x_i, y_i) as follows (i=1,2) .

 a=0.91,b=0.6,c=2.5,d=0 *with initial points (0, 0)and(0.3,0)*	 a=-0.3,b=-0.6,c=2,d=0.27 *with initial points (0,0)and (0.3,0)*	 a=-0.51,b=-0.9, c=2.2,d=0.5*with initial points (0, 0)and(0.3,0)*
 a=-0.3,b=-0.6,c=2,d=0.5 *with initial points(0, 0)and(0.3,0)*	 a=0.9,b=-0.6,c=2, d=0.5 *with initial points(0, 0)and(0.3,0)*	 a=0.3,b=0.6,c=2.5,d=0.27 *with initial points(0, 0)and(0.3,0*
 a=3.6,b=3.2,c=0.04,d=0.4 *with initial points(0,0)and (0.3,0)*	 a=0.9,b=-0.55,c=2,d=0.5with initial points(0,0) and(0.3,0*)*	 a=0.91,b=-0.55,c=2.5,d=0 with *initial points(0, 0)and(0.3,0)*

Figure(1)

6. Topological Transitive of the Piecewise Tinkerbell Map

In this section we study the chaotic behavior of The Piecewise Tinkerbell Map is depend on definition (2.3) which is refered to in section two.Now , we study topological transitive ofa map by varying the control parameters (a,b,c,d) by using (Matlab) to analysis of view for topological transitive. this work show as in figure(2) Now consider we get topological transitive point $(\boldsymbol{x_i}, \boldsymbol{y_i})$ as follows (i=1,2)

 *a=0 ,b=-0.6, c=0,d=-1 with initial points (0.01, 0.01)*	 *a=-0. 3,b=-0.6, c=2.04,d=0. with initial points (0.01, 0.01)*	 *a=0.9,b=-0.6, c=2,d=0.5with initial points (0.01, 0.01)*
 *a=-0.3,b=-0.6, c=2,d=0.5with initial points (0.01, 0.01)*	 *a=0.3,b=0.6,c=2,d=0.27with initial points (0.01, 0.01)*	 *a=-0.5,b=-0.6,c=2.2,d=0.5with initial points (0.01, 0.01)*
 *a=0.51,b=0.9,c=2.2,d=0.5with initial points (0.01, 0.01*	 *a=0.91,b=-0.6,c=2.5,d=0 with initial points (0.01, 0.01)*	 *a=3.6,b=3.2,c=0.04,d=-0.4 with initial points (0.01, 0.01*

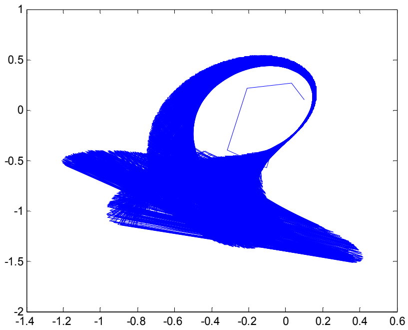
Figure(2)

7. The Lyapunov Dimension and Piecewise Tinkerbell's attractor

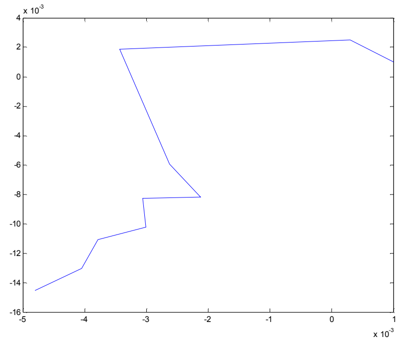
Let V be a subset of X, and let $T^{p}_{a,b,c,d}$:V→X be map with continuous partial derivatives . Also assume that T has an attractor A^{p}_{T}, and that v_o is in A^{p}_{T}. assume that $\lambda_1(v_o)>1>\lambda_2(v_o)$. Then the Lyapunov dimension of A^{p}_{T} at v_o , denoted $dim_L A^{p}_{T}(v_o)$ [13](Gulick D,1992), is given by

$$dim_L\, A^{p}_{T}(v_o) = 1-\frac{\ln\lambda_1(v_o)}{\ln\lambda_2(v_o)} > 0 \qquad \ldots\ldots(7.1)$$

Which has been studied by James York and named the Tinkerbell attractor. By computer he found that $dim_L\,AT(v_o) \approx 1.40$, i.e the Lyapunov dimension of the Piecewise Tinkerbell map is fractal number .By section(5) the Piecewise Tinkerbell map has transitive points and by theorem(4.5) the Piecewise Tinkerbell map has positive Lyapunov exponents then by definition(2.5) the map has strange attractor. Attractors are the pinnacle and origin of chaos theory. An attractor is a 'set', 'curve', or 'space' that a system irreversibly evolves to if left undisturbed. It is other-wise known as a 'limit set'. There are four different types of attractors The point attractors, limit cycle attractors, torus attractors and strange attractors. A very simple iterated mapping that showed a chaotic attractor, now called Piecewise Tinkerbell's attractor. It allowed him to make a direct connection between deterministic chaos and fractals. It consists of two X and Y equations that produce a fractal made up of (Devaney R.L,1989)The standard (typical) parameter values of the Tinkerbell map $T_{a,b,c,d}$ has a=0.9 ,b=-0.6, c=2,d=0.5. This Tinkerbell map has a chaotic strange attractor. The result of computation is shown in Figure (3) and The standard (typical) parameter values of the Piecewise Tinkerbell map $T^{p}_{a,b,c,d}$ has a=0.9 ,b=-0.6, c=2,d=0.5. This Piecewise Tinkerbell map has a chaotic strange attractor. The result of computation is shown in Figure (4)created by Matlab program .



Figure(3) TheTinkerbell attractor



Figure(4)The Piecewise Tinkerbell attractor

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