On the Discretized Algorithm for Optimal Proportional Control Problems Constrained by Delay Differential Equation

Olotu, Olusegun
Department of Mathematical Sciences
Federal University of Technology, Akure
P.M.B. 704, Akure. Ondo-State, Nigeria
Tel: +2348033509040 E-mail: segolotu@yahoo.ca

Dawodu, Kazeem Adebowale (corresponding Author)
Department of Mathematical Sciences
Federal University of Technology, Akure
P.M.B. 704, Akure. Ondo-State, Nigeria
Tel: +2348032541380 E-mail:dawodukazeem@yahoo.com

Abstract

This paper seeks to develop an algorithm for solving directly an optimal control problem whose solution is close to that of analytical solution. An optimal control problem with delay on the state variable was studied with the assumption that the control effort is proportional to the state of the dynamical system with a constant feedback gain, an estimate of the Riccati for large values of the final time. The performance index and delay constraint were discretized to transform the control problem into a large-scale nonlinear programming (NLP) problem using the augmented lagrangian method. The delay terms were consistently discretized over the entire delay interval to allow for its piecewise continuity at each grid point. The real, symmetric and positive-definite properties of the constructed control operator of the formulated unconstrained NLP were analyzed to guarantee its invertibility in the Broydon-Fletcher-Goldberg-Shanno (BFGS) based on Quasi-Newton algorithm. Numerical example was considered, tested and the results responded much more favourably to the analytical solution with linear convergence.

Keywords: Simpson's discretization method, proportional control constant, augmented Lagrangian, Quasi – Newton algorithm, BFGS update formula, delays on state variable, linear convergence.

1.0 Introduction

Differential control systems with delays in state or control variables play important roles in the modelling of real-life phenomena in various fields of applications. The introduction of delay in control theory emanated from the fact that most real life scenarios involve responses with non-zero delays such as models of conveyor belts, urban traffics, transportation, signal transmission, nuclear reactors, heat exchangers and robotics that are synonymous with optimal control models. Falbo **[4 &5]** worked on the complete solutions to certain Functional Differential Equations which seek to address salient approach in developing analytical solutions to delay Differential Equation using either methods of characteristics or Myshkis method of steps **[12]** which were discovered to be very tedious for large space problems. However, many papers have been devoted to delayed (other terminology: time lag, retarded, hereditary) optimal control problems for the derivation of necessary optimality conditions after it was first introduced by Oguztoreli **[13]** in 1966. Most of the adopted methods were for the provision of the analytical maximum principle for the optimal control problems with a constant state delay firstly by Kharatishvili **[10].** Though he later gave similar results for control problems with pure control delays **[11]** while multiple constant delays in state and control variables was by Halanay **[9]** in which the delays are chosen to be equal for both state and control. Banks **[1]** later derived a maximum principle for control systems with a time-dependent delay in the state variable while Guinn **[8]** sketches a simple method for obtaining necessary conditions for control problems with a constant delay in the state variable. The recent work by Gollman et al **[7]** was in the development of the Pontryagin-type minimum (maximum) principle for the optimal control problems with constant delays in state and control variables and mixed control–state inequality constraints with the aim of presenting a discretized nonlinear programming methods that provide the optimal state, control and adjoint functions that allows for an accurate check of the necessary conditions. Colonius and

Hinrichsen **[3]** and Soliman et al **[18]** also provide a unified approach to control problems with delays in the state variable by applying the theory of necessary conditions for optimization problems in function spaces.

However, all these reviewed literature were mainly analytical approach and did not consider any direct method amenable to direct numerical algorithms except for the recent publication by Olotu and Adekunle **[14]** on the algorithm for numerical solution to optimal control problem governed by delay differential equation purely on the state variable with emphasis on vector-matrix coefficients. This research then seeks to address the direct numerical approach to solving this optimal control problem with a pre-shaped function within the delay interval such that the optimal control law has a constant feedback gain as a relationship between its control and state variables. For technical reasons, we used the assumption that the ratio of the time delays in state and control is a rational number based on the analysis of Gollman et al **[7].**

2.0 General formulation of the problem

The optimal control problem is modeled to find the state and control trajectories that optimize minimize the objective function of the statement of problem below.

$$\text{Min } J(x,w) = \tfrac{1}{2}\int_0^T F(t, x(t), w(t))dt \tag{1}$$

subject to;

$$\begin{cases} \dot{x}(t) = g[t, x(t), x(t-r), w(t)] & t \in [0,T] \\ x(t) = h(t) & t \in [-r,0] \\ x(0) = x_0,\ w(t) = m\,x(t), \text{ for } p,q,a,b,r,m \in \mathbb{R} \text{ (real) and } p,q,r > 0 \end{cases} \tag{2}$$

where x and u are the state and control trajectories respectively, describing the system. The numerical solution to the optimal control problem is a direct approximate method requiring the parameterizing of each control history using a set of nodal points which then become the variables in the resulting parameter optimization problem. In the discretization of the continuous-time optimal control problem into a Non-Linear Programming (NLP) problem, we assume the values of the pre-shaped (known historical) function $h(t)$ at each nodal (grid mesh) point within the delay interval $[-r,0]$ to be a constant for each rational value r such that $r = h.s$ where $s \in \mathbb{Z}^+$ and $h = \Delta t_k$ usually expressed in the form $h = u \times 10^{-v}$ for $u, v \in \mathbb{Z}^+$.

3.0 Materials and Method of solution

Consider optimal control problem with time delay of the form

$$\text{Min } J(x,w) = \int_0^T (px^2(t) + qw^2(t))dt \tag{3}$$

subject to:

$$\begin{cases} \dot{x}(t) = ax(t) + bw(t) + cx(t-r), \quad t \in [0,T] & (4a) \\ x(t) = h(t),\ t \in [-r,0] & (4b) \\ x(0) = x_0 \text{ where } p,q,a,b,c,r \in \mathbb{R} \text{ (real) and } p,q,r > 0 \end{cases}$$

We then discretize the performance index of the continuous-time model to generate large sparse discretized matrices using the *composite Simpson's rule* **[2]** of the form

$$\int_0^{t_n} f[x(t)]dt = \frac{h}{3}\left[f[x(t_0)] + f[x(t_n)] + 2\sum_{k=1}^{\frac{n}{2}-1} f[x(t_{2k})] + 4\sum_{k=1}^{\frac{n}{2}} f[x(t_{2k-1})] \right] - \left(\frac{n}{180}\right) h^4 f^4(\xi), \tag{5}$$

Where $x(t_j) = x_j$, $f \in C'[t_0, t_n]$, n is an even positive integer, $h = \frac{t_n - t_0}{n}$ and $x_j = x_0 + jh$ for each $j = 0,1,2,\ldots,n$.

For $E = \frac{(p+qm^2)}{3}$, $w(t) = m\,x(t)$, $h = \frac{T}{n}$ and $\bar{p} = 2h/3$, the discretised performance index becomes;

$$J(x.w) = \tfrac{1}{2}\int_0^T (px^2(t) + qw^2(t))dt = \int_0^T x^2(t)dt \approx E\bar{p}/2 \left[x_0^2/2 + \sum_{k=1}^{\frac{n}{2}-1} (2x_{2k-1}^2 + x_{2k}^2) + x_n^2/2 \right] \quad (6)$$

$$= \tfrac{1}{2}\left(x_1, x_2, ..., x_n\right) \begin{pmatrix} E\bar{p} & 0 & 0 & . & . & 0 & 0 \\ 0 & \frac{E\bar{p}}{2} & 0 & . & . & . & . \\ 0 & 0 & E\bar{p} & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & 0 \\ 0 & 0 & 0 & . & . & 0 & \frac{E\bar{p}}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{pmatrix} + \frac{E\bar{p}x_0^2}{4} = \tfrac{1}{2} Z^T VZ + C \quad (7)$$

where $Z = (x_1, x_2, ...x_n)$ is a *n*-dimensional vector and $V = \left[v_{ij}\right]$ is a n× dimensional coefficient matrix defined below as

$$V = [v_{ij}] = \begin{cases} E\bar{p} & i = j(\text{odd}) \\ \frac{E\bar{p}}{2} & i = j(\text{even}) \\ \frac{E\bar{p}}{4} & i = j = n \\ 0 & \text{elsewhere} \end{cases} \quad (8)$$

$$\text{and} \quad C = E\bar{p}x_o^2/4 \quad (9)$$

The discretization of the constraint of the delayed optimal proportional control problem using the *2-step third order Simpson's rule* gives a discrete constraint equation and initial value profiles from the pre-shaped function over the delay intervals $[-r, 0]$ defined by the following equations;

$$f(x_{k+2}) + 4(x_{k+1}) + (x_k) + O(h^4) \quad (10a)$$

$$x_{-k} = h(-kh), \quad k = 1, 2, ..., s \quad (\text{where } x_{-k} \text{ are known constants}) \quad (10b)$$

By further expansion gives,

$$\approx \bar{T}x_k + \bar{U}x_{k+1} + x_{k+2} = \bar{S}(x_{k-s} + 4x_{k+1-s} + x_{k+2-s}) \qquad (11)$$

Given $0 < m < n-1$ and $A = a + bm < 0$, therefore

$$K = \frac{(Ah+3)}{(Ah-3)}, L = \frac{4Ah}{(Ah-3)} \text{ and } x\left(t_k - r\right) = x\left(t_k - sh\right) = x_{k-s}. \qquad (12)$$

set k = 0, $\bar{U}x_1 + x_2 = -\bar{T}x_0 + \bar{S}(x_{-s} + 4x_{1-s} + x_{2-s})$

set k = 1 $\bar{T}x_1 + \bar{U}x_2 + x_3 = \bar{S}(x_{1-s} + 4x_{2-s} + x_{3-s})$

set k = 2 $\bar{T}x_2 + \bar{U}x_3 + x_4 = \bar{S}(x_{2-s} + 4x_{3-s} + x_{4-s})$

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set k = 2 $\bar{T}x_{s-2} + \bar{U}x_{s-1} + x_s = \bar{S}(x_{-2} + 4x_{-1} + x_0)$

set $k = s-1$ $-\bar{S}x_1 + \bar{T}x_{s-1} + \bar{U}x_s + x_{s+1} = \bar{S}(x_{-1} + 4x_0)$

set $k = s$ $-4\bar{S}x_1 - \bar{S}x_2 + \bar{T}x_s + \bar{U}x_{s+1} + x_{s+2} = \bar{S}x_0$

set $k = s+q$ for $q = 1,2,3........(n-3-s)$

$$-\bar{S}x_s - 4\bar{S}x_{s+1} - \bar{S}x_{s+2} + \bar{T}x_{s+q} + \bar{U}x_{s+q+1} + x_{s+q+2} = 0$$

set k = n − 2 $-\bar{S}X_{n-2-s} - 4\bar{S}X_{n-1-s} - \bar{S}X_{n-s} + \bar{T}X_{n-2} + \bar{U}X_{n-1} + X_n = 0$

The above system becomes

$$\begin{bmatrix} \bar{U} & 1 & 0 & . & . & . & . & . & 0 \\ \bar{T} & \bar{U} & 1 & 0 & . & . & . & . & . \\ 0 & \bar{T} & \bar{U} & 1 & 0 & . & . & . & . & . \\ . & . & . & . \\ 0 & . \\ -\bar{S} & 0 & . & 0 & \bar{T} & \bar{U} & 1 & 0 & . & . & 0 \\ -4\bar{S} & -\bar{S} & 0 & . & 0 & \bar{T} & \bar{U} & 1 & 0 & . & . & . & . \\ -\bar{S} & -4\bar{S} & -\bar{S} & 0 & . & 0 & \bar{T} & \bar{U} & 1 & . & 0 & . & . \\ 0 & -\bar{S} & -4\bar{S} & -\bar{S} & 0 & . & 0 & \bar{T} & \bar{U} & 1 & 0 & . \\ . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & 0 \\ 0 & . & . & 0 & -\bar{S} & -4\bar{S} & -\bar{S} & 0 & . & 0 & \bar{T} & \bar{U} & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ \\ x_{s-2} \\ x_{s-1} \\ x_s \\ x_{s+1} \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} -\bar{T}x_0 + \bar{S}(x_{-s} + 4x_{1-s} + x_{2-s}) \\ \bar{S}(x_{1-s} + 4x_{2-s} + x_{3-s}) \\ . \\ . \\ . \\ \bar{S}(x_{-2} + 4x_{-1} + x_0) \\ \bar{S}(x_{-1} + 4x_0) \\ \bar{S}x_0 \\ 0 \\ 0 \\ . \\ 0 \end{pmatrix} \qquad (13)$$

Therefore,

$$JZ = H$$

(14)

Where J is a $(n-1)$矗 sparse coefficient matrix defined by

$$J=[j_{ij}]=\begin{cases} \bar{U} & 1\le i\le n-1 \qquad j=i \\ \bar{T} & 2\le i\le n-1 \qquad j=i-1 \\ 1 & 1\le i\le n-1 \qquad j=i+1 \\ -\bar{S} & \begin{bmatrix} s-1\le i\le n-1 \qquad j=i+2-s \\ s+1\le i\le n-1 \qquad j=i-s \end{bmatrix} \\ -4\bar{S} & s\le i\le n-1 \qquad j=i-s+1 \\ 0 & \text{elsewhere} \end{cases} \tag{15}$$

H is a $(n-1)?$ column vector defined by

$$H=[h_{i1}]=\begin{cases} -\bar{T}X_0+\bar{S}(X_{-s}+4X_{1-s}+X_{2-s}) & i=1 \\ \bar{S}(X_{i-1-s}+4X_{i-s}+X_{i+1-s}) & 2\le i\le s-1 \\ \bar{S}(X_{-1}+4X_0) & i=s \\ \bar{S}X_0 & i=s+1 \\ 0 \qquad \text{elsewhere} & [\text{i.e } i=s+2,s+3,.........n-1] \end{cases} \tag{16}$$

and $Z=(x_1,x_2............x_n)^T$ is a n-dimensional column vector.

The combination of equations (7 &14) by the parameter optimization gives the constrained discretized non-linear programming (quadratic) problem stated below:

$$\begin{aligned} &\text{Min } F(Z) = \tfrac{1}{2} Z^T VZ + C \\ &\text{Subject } \; JZ = H \end{aligned} \tag{17}$$

Where Z is a column vector of dimension $n?$ for $Z^T=(x_1,x_2,...............,x_n)$, V a sparse tri-diagonal matrix of dimension n靐, J a sparse coefficient matrix of dimension $(n-1)$靐 and H a row vector of dimension $(n-1)?$.

Applying the Augmented Lagrangian Method reviewed by Fiacco et al **[6]** as earlier proposed by Powell **[17]** where the penalty term is added not only to the objective function but also to the lagrangian function to give

$$L_\rho(Z,\lambda,\mu)=\tfrac{1}{2}Z^TVZ+C+\lambda^T|JZ-H|+\frac{\mu}{2}\|JZ-H\|^2 \tag{18}$$

On expanding equation (18) gives the following quadratic programming problem below to be solved by the Quasi-Newton Method (QNM).

$$L_\rho(Z,\lambda,\mu)=\tfrac{1}{2}Z^T\left[V+\left(\mu J^TJ\right)\right]Z+\left(\lambda^TJ-\mu H^TJ\right)Z+\left(\frac{\mu}{2}H^TH-\lambda^TH+C\right) \tag{19}$$

$$L_\rho(Z,\lambda,\mu)=\tfrac{1}{2}Z^TV_\rho Z+J_\rho^TZ+C_\rho \tag{20}$$

where $\mathrm{Dim}(V_\rho)=n$靐, $\mathrm{Dim}(J_\rho^T)=1$靐, and $\mathrm{Dim}(C_\rho)=1?$ represent the dimensions of the various coefficients (discretized matrices stated below) of the lagrangian function.

$$V_\rho=\left[V+\left(\mu J^TJ\right)\right], J_\rho^T=\left(\lambda^TJ-\mu H^TJ\right) \text{ and } C_\rho=\left(\frac{\mu}{2}H^TH-\lambda^TH+C\right)$$

(21)

Lemma 1: The constructed quadratic operator $V_\rho=\left[V+\left(\mu J^TJ\right)\right]$ of the formulated lagrangian function is real, symmetric and positive definite. See proof in **[15].**

The symmetric and positive definite properties of the quadratic operator are to ensure the invertibility of the BFGS in the Quasi-Newton Algorithm (inner loop) and as well enforce the feasibility condition of the augmented Lagrangian Method (outer loop) used in the formulation of the unconstrained NLP problem as expressed by Olotu and Dawodu**[16].**

4.0 The numerical Algorithm for the developed scheme

(1) $Compute\ given\ variables\ V, M, N, m$

(2) $Choose\ Z_{0,0} \in \mathbb{R}^{(n-1)\times n}, B_0 = I,\ T^*(tolerance)\ and\ initialize\ \mu_j > 0,\ \lambda_j > 0\ by\ setting\ j = 0$

(3a) $set\ i = 0\ and\ g_0 = \nabla_z L(Z_{0,0}) = \nabla L_0$

(3b) $compute\ V_i = [V + (\mu_j J^T J)],\ M_i = [\lambda_j J - (\mu_j H^T J)],\ N_i = \left(\frac{\mu_j}{2} H^T H - \lambda_j^T H + C\right)$

(3c) $set\ S_i = -[B_i] g_i$ (search direction) and

(3d) $compute\ \alpha_i^* = \frac{-(M_i S_i + Z_i^T V_i S_i)}{S_i^T V_i S_i}$ (steplength)

(3e) $set\ Z_{j,i+1} = Z_{j,i} + \alpha_i^* S_i$ and

(3f) $compute\ g_{k+1} = 祵 (Z_{j,i+1}, \lambda_j, \mu_j)$

(3g) $if\ \left\| 祵 (Z_{j,i+1}, \lambda_j, \mu_j \right\| £T^*$ go to step 4 else go to (3h)

(inner convergence from Quasi - Newton)

(3h) $set\ q_i = g_{i+1} - g_i\ and\ p_i = z_{i+1} - z_i$

(3i) $compute\ B^u{}_i = [1 + \frac{q_i^T B_i q_i}{p_i^T q_i}][\frac{p_i p_i^T}{p_i^T q_i}] - [\frac{(p_i q_i^T B_i) + (B_i q_i p_i^T)}{p_i^T q_i}]$ (BFGS)

(3j) $set\ B_{i+1} = B_i + B_i^u$ and repeat steps $3(a-f)$ for next $i = i+1$

(4) If $\left\| JZ_{j,i+1} - H \right\| \le T^*$ stop! Choose?$Z^*_{j,i+1}$ and compute $W^*_{j,i+1}$ else go to step 5)

(outer convergence from lagrangian)

(5) Update $\mu_{j+1} = \mu_0 \times 2^{j+1}$ (penalty) and $\lambda_{j+1} = \lambda_j + \mu_j \left(JZ_j - H \right)$ (multiplier)

(6) Go to step (3) for next $j = j+1$

5.0 The analytical optimal proportional control formulation

Consider the re-formulated delay optimal proportional control problem with delay $r \ge 0$ only on the state expressed below as

$$Minimize\ J(x,m) = \frac{x_0^2 (p + qm^2)}{2} \int_0^T x^2(t) dt \tag{22}$$

$$\begin{cases} \dot{x}(t) = (a + bm)x(t) + cx(t-r) = Ax(t) + cx(t-r) \quad t \in [0,T] & (23) \\ x(t) = h(t) \quad t \in [-r, 0] \quad r \ge 0 \\ w(t) = mx(t) \quad a,b,c,p,q,r,m \in \mathbb{R},\ p,q,r > 0\ and\ c < 0 & (24) \end{cases}$$

Theorem 5.1(Myshkis method of steps for first order homogenous linear ordinary delay differential equation)

Given the delay differential equation (23). *let* $h_0 \in \square$, *then* (23) *has a unique solution*
$x_n(t) = e^{At}\left[\int_{(n-1)r}^{t} [ce^{-As}h_{n-1}(s)]ds + x_{n-1}((n-1)r)e^{-(n-1)rA}\right]$ *given on* $[(n-1)r, nr]$ *and satisfies the condition* $x_n(t-r) = h_{n-1}(t-r)$ *and* $x(0) = h_0$ *on* $[-r, 0]$ *for* $n \in \square$ *and* $A, c, h_0 \in \square$.

Proof: see [14]
To further illustrate the analytical solution to the Delay Differential Equation (DDE) with the relevant theorem stated in theorem 5.1 for easy applications, the Method of Steps, though tedious, which is the simplest analytical approach to the DDE compared to methods of Characteristics; least Square or Laplace transforms, can be applied. In the general (linear or nonlinear) first order DDE expressed below, the steps will be illustrated.

$$\dot{y}(t) = f(t, y(t), y(t-r)) \text{ for } t \in [0, r], \ r > 0$$
$$y(t) = p(t) \text{ for } t \in [-r, 0] \qquad (25)$$

Step 1: On the delay interval $[-r, 0]$, function $y(t)$ is the given function $p(t)$; which gives $y_0(t)$.

Step 2: In the interval $[0, r]$ the system in equation (25) above becomes $\dot{y}(t) = f(t, y(t), y_0(t-r))$ on $[0, r]$ subject

to $y(0) = p(0)$ for $y_0(t-r) = p(t-r)$ defined on $[0, r]$ with the solution $y_1(t)$ obtained by "Method of

Integration by parts".

Step 3: On the next interval $[r, 2r]$, the system is defined on $[0, r]$ where $y_1(t-r)$ is reading its values from another

domain $[-r, 0]$ known for all $t \in [-r, 0]$ such that $y_1(t) = p(t)$.

Step 4: This process continues until subsequent solutions of each step of the system on the interval $[(n-1)r, nr]$ for

$T = nr$ is evaluated such that the known function $y_{n-1}(t-r)$ is replaced with $y_n(t-r)$ on the next interval.

Theorem 5.2 *(pontryagin's minimum principle for delay optimal control problem)*

Given the delay proportional optimal control problem witth delay $r > 0$ *such that* $r = T/n \in \square^+$ *and* $x^*(t)$ *is the optimal solution of the state system* (23) *that minimizes the performance index* $J(x)$, *then there exists a costate (adjoint) function* $\lambda \in W^{1/\infty}([0, T]\ \square^n)$ *such that the* (i) *adjoint differential equation and* (ii) *transversality condition below are satisfied for* $t \in [0, T]$.

$$(i) \begin{cases} \dot{\lambda}(t) = -2(p+qm^2)x^*(t) - (a+bm)\lambda(t) - c\lambda(t+r) & 0 \le t \le T-r] \\ \dot{\lambda}(t) = -2(p+qm^2)x^*(t) - (a+bm)\lambda(t) & T-r \le t \le T] \end{cases} \qquad (26)$$

$$(ii) \quad \lambda(T) = 0 \qquad (27)$$

Proof: see [14]

Theorem 5.3

Given the delay differential equation DDE (23) *above with the delay* $r>0$ *and* $A=a+bm<0$ *then there exists a unique real solution of the DDE such that for* $m=\frac{1}{b}\left\{\frac{1}{r}[1+\log_e(-rc)]-a\right\}<0$, *then* $-\frac{1}{re}<c<0.$

Proof:

By the method of characteristics, the solution to the state of equation can be represented with $x(t)=ke^{\theta t}$ and when put into (23) will give a nonlinear characteristic stated in equation (28) below with fixed values of the coefficients a, b, c and r.

$$V(\theta)=(\theta-A)e^{\theta r}-c=0 \tag{28}$$

To ascertain the feedback gain at which the control is at optimum, the gradient of the equation (28) gives

$$\dot{V}(\theta)=(\theta-A)re^{\theta r}+e^{\theta r}=[(\theta-A)r+1]e^{\theta r}=0 \tag{29}$$

$$\theta=A-\frac{1}{r} \text{ to give } V(\theta)=(-\frac{1}{r})e^{Ar-1}-c=0$$

$$A=\frac{1}{r}\left[1+\log_e(-rc)\right]=a+bm<0 \text{ to give}$$

$$m=\frac{1}{b}\left\{\frac{1}{r}[1+\log_e(-rc)]-a\right\}<0 \tag{30}$$

For $\log_e(-rc)$ to exist given that $r>0$, it then implies that $c<0$ and also given that $a+bm<0$,

To therefore guarantee convergence and uniqueness of real solution, the interval below must be satisfied

$$-\frac{1}{re}<c<0 \tag{31}$$

6.0 Numerical examples and presentation of results

Example (6.1): consider a one-dimensional optimal control problem

$$\text{Min } J(x,w)=\frac{1}{2}\int_0^5 (x^2(t)+w^2(t))dt \tag{32}$$

subject to;

$$\begin{cases}\dot{x}(t)=2x(t)+w(t)-0.3x(t-0.5), & 0\le t\le 7\\ x(t)=h(t)=t \text{ and } x(0)=1, & t\in[-0.5,0]\end{cases} \tag{33}$$

Using the method of steps in theorem (5.1) to solve the DOCP analytical with the parameters

$a=2,\ b=1,\ c=-0.3,\ x_0=1,\ r=0.5,\ h_0=1,\ T=7$ and $A=a+bm=-1.7942<0$ with

$m=\frac{1}{b}\left\{\frac{-1}{r}[1+\log_e(-rc)]-a\right\}=-3.7942$ and the constraint of equation (31) becomes

$$\dot{x}(t)=Ax(t)-0.3x(t-0.5)=-1.7942x(t)-0.3x(t-0.5) \quad \text{on } [0,7]$$

$$x(t)=h(t)=t \text{ with } x(0)=h(0)=1 \quad \text{on } [-0.5,0]$$

The solution $x_0(t)=h_0(t)=t$ exist on the interval $-0.5\le t\le 0$ such that for $n=\frac{T}{r}=\frac{7}{0.5}=14$ steps

$$\dot{x}_k(t)-Ax_k(t)=-0.3x_k(t-0.5)=-0.3h_{k-1}(t) \text{ on } [0.5(k-1),\ 0.5k],\ k=1,2,.....,14 \tag{34}$$

and $x_k(t-0.5)=h_{k-1}(t-0.5)$

$$\dot{x}_1(t)+1.7942x_1(t)=-0.3x_1(t-0.5)=-0.3h_0=-0.3 \quad \text{on} \quad [0\ 0.5]$$

$$for \quad x_1(0)=h_0(0)=1 \quad \text{on} \quad [-0,5\ 0]$$

integrating factor (IF) $=e^{\int 1.7942tdt}$ then $x_1(t)=e^{-1.7942t}\left\{-0.3\int_0^t e^{1.7942s}ds+x_1(0)\right\}$ on $[0,0.5]$

$$=e^{-1.7942t}\left\{-0.3\left[\left(\frac{(e^{1.7942t}-1)}{1.7942}\right]+x_1(0)\right\}=(0.1672+1.1672e^{-1.7942t})$$

Therefore,

$$x_1(t)=0.1672+1.1672e^{-1.7942t} \quad \text{on} \quad [0,0.5] \tag{35}$$

Similarly, for next step, the solution exist on the interval $0.5\le t\le 1$ such that $x_2(t-0.5)=h_1(t-0.5)=x_1(t-0.5)=0.1672+1.1672e^{-1.7942(t-0.5)}$ and $x_2(0.5)=h_1(0.5)=0.3087$

$$x_2(t)=e^{-1.7942}\left\{\left[-0.3\int_{0.5}^t[1.1672e^{1.7942(s-0.5)}+0.1672]ds\right]+h_1(0.5)\right\} \quad \text{on} \quad [0.5,1]$$

$$x_2(t)=\left[-0.0796+0.9774e^{-1.7942t}-0.0502te^{-1.7942t}\right] \qquad t\in[0.5,1] \tag{36}$$

.

$$x_k(t)=\sigma_k+\left(\sum_{j=0}^{k-1}\beta_{kj}t^j\right)e^{-1.7942t} \qquad \text{on} \qquad [0.5(k-1),0.5k]$$

for $x_k(t-0.5)=h_{k-1}(t-0.5),\ k=3,4,5..........14;\ j=1,2........14$

Where the coefficients (σ_k *and* β_{kj}) of the rest of the solution for other sub-intervals can be solved with the MATLAB package because of the tedious nature of the analytical computations for various steps to obtain the results in table 3 below as well as the analytical objective value from the proportional DOCP result with the given parameters $p=1,\ q=1,\ a=2,\ b=1,\ c=-0.3,\ x_0=1,\ r=0.5,\ h_0=1$ and $T=7$ to give $J_A=\mathbf{1.92882143}$. The numerical objective value from the *Quasi-Newton based augmented lagrangian method using MATLAB subroutine is* $J_N=\mathbf{1.9296402}$. Here we take $\mu=1000,\ \varepsilon=10^{-5},\ h=0.1$ for large $T=7$ as shown in the selected values of the parameters (**X_N**, **W_N**, **X_A**, **W_A**, **E_X** and **E_W**) representing the state, control and errors for the numerical and analytical results respectively as outlined in the table 3 below:

Table 1: Comparison of analytical and numerical results (DOCP for newly developed scheme)

t	X_N	W_N	X_A	W_A	E_X	E_W
0.0000	1.0000	-3.7942	1.0000	-3.7942	0.0000	0.0000
0.1000	0.8259	-3.1336	0.8031	-3.0471	0.0228	-0.0865
0.2000	0.6441	-2.4438	0.6481	-2.4590	-0.0040	0.0152
0.3000	0.5873	-2.2283	0.5142	-1.9510	0.0731	-0.2773
0.4000	0.4388	-1.6649	0.4023	-1.5264	0.0365	-0.1385
0.5000	0.3653	-1.3860	0.3087	-1.1713	0.0566	-0.2147
0.6000	0.2974	-1.1284	0.2432	-0.9228	0.0542	-0.2056
0.7000	0.2570	-0.9751	0.1888	-0.7163	0.0682	-0.2588
0.8000	0.1980	-0.7513	0.1435	-0.5445	0.0545	-0.2068
0.9000	0.1460	-0.5540	0.1059	-0.4018	0.0401	-0.1521
1.0000	0.1051	-0.3988	0.0746	-0.2830	0.0305	-0.1157
2.0000	-0.0033	0.0125	0.0093	-0.0353	-0.0126	0.0478
3.0000	0.0008	-0.0030	0.0027	-0.0102	-0.0019	0.0072
4.0000	0.0024	-0.0091	0.0012	-0.0046	0.0012	-0.0046
5.0000	0.0040	-0.0152	0.0037	-0.0140	0.0003	-0.0011
6.0000	0.0067	-0.0254	0.0064	-0.0243	0.0003	-0.0011
7.0000	0.0114	-0.0433	0.0112	-0.0425	0.0002	-0.0008

Graphical Representation of numerical result of developed scheme

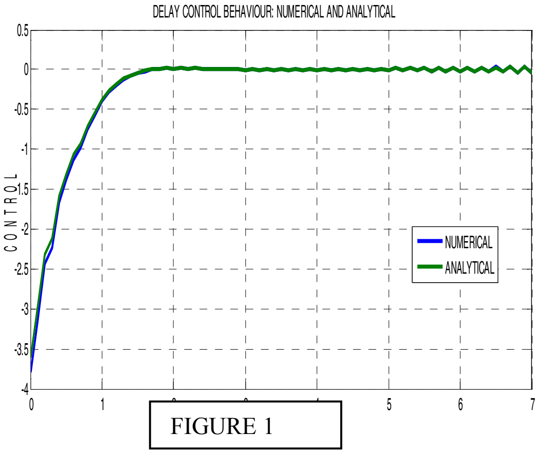
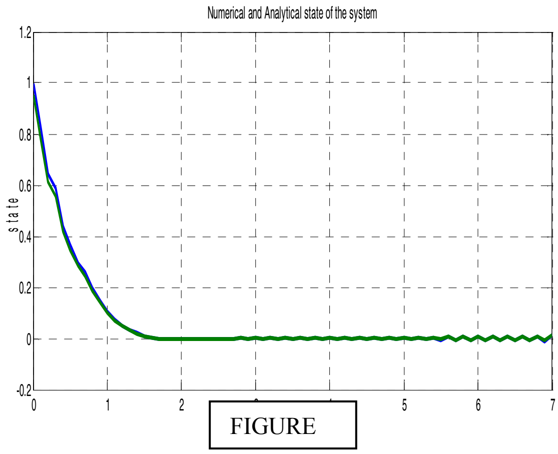


FIGURE 1



FIGURE

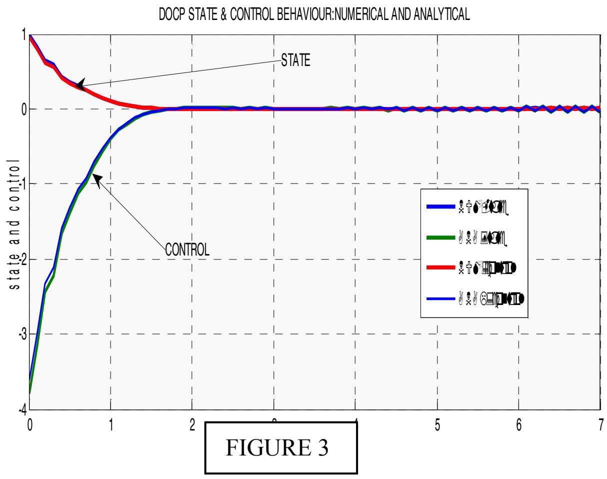
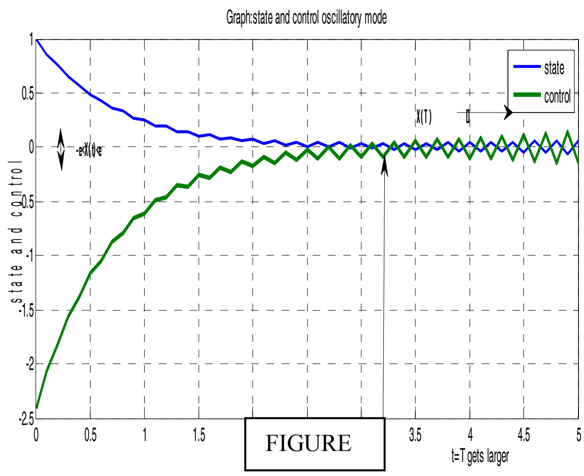


FIGURE 3



FIGURE

Numerically, the stability of the system considering the behaviour of the state (delay differential equation) with respect to the variation of all its relevant parameters was analyzed. It was observed that small values of the coefficient $c < 0$ of the delay term $r > 0$ move the delay differential equation (constraint) towards stable region for increasing values of the final time T with other parameters fixed as presented in figures 1-4. It was also observed that since the nonhomogeneous delay differential equation (DDE) exhibits exponential growth or decay, then the nature of the *pre-shaped function* $h(t)$ within the delay interval $[-r, 0]$ determines to a large extent the *convergence of the solution* of the DDE within the bounded interval $[0, T]$. Therefore the solution of the state of the DOCP depends heavily on the relationship between the values of A, c, r and $h_0(t)$.

6.2 Convergence Analysis

Suppose $\{z_k\} \subset \square^n$ represents the sequence of solution z_k that approaches a limit z^* (say $z_k \to z^*$), then the error $e(z_k) = e_k$ is such that $e(z_k) = e_k = |z_k - z^*| \geq 0$ for $\forall z_k \subset \square^n$ and $e(z^*) \neq 0$.

For purpose of convenience, assuming the convergence ratio is represented with β, then

$$\beta = \lim_{k\to\infty} \frac{e_{k+1}}{e_k} = \lim_{k\to\infty} \frac{\|z_{k+1} - z^*\|}{\|z_k - z^*\|} \quad for \;\; e_k \neq 0 \quad \forall k \;\; \text{where}$$

(37)

$0 < \beta < 1, \beta = 0$ *and* $\beta = 1 \Rightarrow$ *Quadratic*, *super-linear and sub-linear convergence respectively.*

However, the convergence ratio (β) of the earlier example (6.1) expressed in terms of the penalty parameter (μ) used in the newly developed algorithm is shown in table 2 below.

Table 2: Convergence ratio profile

penalty parameter (μ)	Objective value (r)	convergence ratio (β)
1.0×10^2	1.9321041	-
1.0×10^3	1.9296402	0.1335
1.0×10^4	1.9288349	0.1237
1.0×10^5	1.9288228	0.1044

The result on the table shows that the convergence ratio (β) hovers round the average figure of $\beta = \mathbf{0.120543}$ for increasing values of the penalty parameter with longer processing time which makes the convergence linear though close to being super-linear because of its proximity to zero. This convergence is satisfactory for optimization algorithms since the convergence is not close to one.

7.0 Conclusion

This research has enabled us to develop an efficient numerical method for computing the optimal state and control variables of an optimal proportional control problem with high level of accuracy. We present a discretization method using the Simpson's rule whereby the control problem is transcribed into a high-dimensional nonlinear programming problem using the augmented Lagrangian function. Excellent result was being obtained using the MATLAB subroutines when result is compared with the analytical result from the method of steps. All the excellent computational results obtained were from the computations performed on a DELL processor of 1.67 GHz Intel® Atom (TM) CPU under Window 7 operating system.

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