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Characterization of Student's T- Distribution with some Application to Finance

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Abstract
The distributional form of the returns on the underlying assets plays a key role in finance under valuation theories for derivative securities. Among them, Student t-distributions are generally applied in financial studies as heavy-tailed substitute to the normal distribution. Therefore, distributions of logarithmic asset returns can often be fitted extremely well using Student t-distribution with υ degree of freedom, such that $3 \leq \upsilon \leq 5$. The aim of this paper is to investigate the characterization behavior of Student t-distributions and its related properties into finance which are based on computational aspects using Mathematica. Furthermore, convolution, infinity divisibility and self-decomposability properties of Lévy-Student process are considered as background to the option pricing. Finally, applications of modeling high frequency price returns are discussed.
Keywords: Characterization behavior, degree of freedom, heavy-tail, Lévy-Student process

1. Introduction

The distributional form of the returns on the underlying assets plays a key role in finance under valuation theories for derivative securities. Among them, Stochastic processes with heavy-tailed marginal distributions, including Student's t-distribution, are used commonly for modeling in logarithmic stock returns and stochastic volatility in finance, econometrics, insurance, turbulence, communication networks, etc (Aas & Haff 2006; Heyde & Leonenko 2005; Kumari & Tan 2013). Hence, it is frequently used to model the asset returns for which the tails of the normal distribution are almost invariably found to be thin.

This distribution is increasing important in classical as well as in Bayesian modeling in Statistics. Both Normal and T-distribution are members of the general family of symmetric distributions. Due to its central importance in statistical inference, the Student-t offers a more viable alternative particularly when it comes to real market data. The applications of Student-t can be classified into the fields of empirical modeling, cluster analysis, discriminant analysis, multiple regression, risk and dependence modeling, Lévy processes & derivatives, portfolio selection and many more. Therefore in this paper, we concentrate on characterization of Student t-distribution and its related properties into finance.

2. Characterization of Student T-Distribution

Considering the sample of independent observation $X_1 \ldots\ldots\ldots X_n$ from the normal population with mean μ and variance σ^2 for testing the null hypothesis $H_0 : \mu = \mu_0$ against the alternative $H_1 : \mu \neq \mu_0$, Gosset (Student 1908) suggested the 't' test statistic

$$t_n = \frac{(\bar{x}_n - \mu_0)}{s_n/\sqrt{n}}, \qquad n \geq 2, \tag{01}$$

where $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2$.

He derived that the distribution law $\mathcal{L}(t_n) = T_1(n-1,\ 1, 0)$, where $T_1(\upsilon, \sigma, \mu)$ denotes the univariate Student's t-distribution(ST) with with $\upsilon > 0$ degree of freedom/ tail parameter, a scaling parameter $\sigma^2 > 0$ and a location parameter $\mu \in \mathbb{R}^1$. The t-test and the associated theory became well-known through the work of Fisher (1927), who called the distribution "Student's distribution".

A random variable T_1 has a standard Student's t distribution (ST) with υ degrees of freedom and it can be written as a ratio: $T_1 = \frac{Y}{\sqrt{\chi^2_\upsilon/\upsilon}}$, between a standard normal random variable and the square root of a Chi-square random variable with υ degrees of freedom (dividing by υ a Chi-square random variable with degrees of freedom). This is similar to obtaining a Gamma random variable with parameters $\upsilon/2$ and 2. Then its Probability Density function (PDF) can be defined as;

$$f_{\upsilon}(x) = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi\,\upsilon}\;\Gamma\left(\frac{\upsilon}{2}\right)}\left[1+\frac{1}{\upsilon}x^2\right]^{-\left(\upsilon+1/2\right)}, \quad t \in \mathbb{R}^1, \qquad (02)$$

and $\Gamma(z)$ is the Euler's gamma.

Normal distribution is recovered as $\upsilon \to \infty$. This reflects the fact that the estimated variance converges in probability to the true variance when υ grows without limit. The tails of the t distribution become more pronounced for smaller values of υ. Whereas for finite υ the tails of the density function decay as an inverse power of order $\upsilon + 1$, and is therefore fat-tailed relative to the Normal. Therefore, extreme outcomes are more likely than Normal distribution. It is implied by the Fig.1. When $\upsilon = 1$, the t distribution is identical to another classical distribution, the Cauchy distribution. Its tails are so fat that they prevent it from having a mean (as well as any higher order moment).

A random variable (X) has a general Student's t-distribution and it can be written as a linear transformation of a standard Student's t random variable (T_1) as, $X = \mu + \sigma\, T_1$. The PDF of X can be obtained using the formula for the density of a function of an absolutely continuous variable ($X = g(T) = \mu + \sigma\, T_1$ is a strictly increasing function of T_1, since σ is strictly positive) as;

$$f_X(x) = f_{T_1}\left(g^{-1}(x)\right)\frac{dg^{-1}(x)}{dx} = f_{T_1}\left(\frac{x-\mu}{\sigma}\right)\frac{1}{\sigma}$$

$$f_X(x) = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sqrt{\pi\,\sigma\,\upsilon}\;\Gamma\left(\frac{1}{2}\upsilon\right)}\left[1+\frac{1}{\upsilon}\left(\frac{x-\mu}{\sigma}\right)^2\right]^{-\left(\upsilon+1/2\right)}, x \in \mathbb{R}^1. \qquad (03)$$

This distribution results from compounding a Gaussian distribution with mean μ and unknown precision (the reciprocal of the variance), with a gamma distribution with parameters $\upsilon/2$ and $\upsilon/2$. In other words, the random variable x is assumed to have a normal distribution with an unknown precision distributed as gamma. This is marginalized over the gamma distribution. The reason for the usefulness of this characterization is that the gamma distribution is the conjugate prior distribution of the precision of a Gaussian distribution. As a result, the three-parameter Student's t distribution arises naturally in many Bayesian inference problems.) Therefore, the standard ST-distribution can be formulated using the following integration as;

$$f_{\mu,\sigma^2,\upsilon}(x) = \int_0^\infty \frac{\Gamma\left(\frac{\upsilon}{2}\right)^{\frac{\upsilon}{2}}}{\Gamma\left(\frac{\upsilon}{2}\right)\sqrt{2\pi\, y\sigma^2}}\, e^{-\frac{1}{2y}\left(\frac{x-\mu}{\sigma}\right)^2-\frac{\upsilon}{2y}}\, y^{-\frac{\upsilon}{2}-1}\, dy\ . \qquad (04)$$

Fig. 2 illustrates the structural behavior of the $ST(\upsilon, \sigma, \mu)$ distribution when parameters are changing. (refer the 'Appendix A' for Mathematica coding).

2.1 Simulation of General ST-distribution

We use four methods to simulate, 4 different samples from ST distribution. Fig. 3 shows the results of sampling from ST distribution with 4 degrees of freedom, using following approaches;

I. 10000 independent samples are taken from five standard normal variables $Z_1, Z_2, Z_3, Z_4 and\ Z_5$ and then calculate the $T_{(4)}$ as;

$T_{(4)} = \frac{Z_1}{(Z_2+Z_3+Z_4+Z_5)/\sqrt{4}}$, which has the ST distribution with 4 degrees of freedom. (Fig. 3 (a)).

II. Take 10000 samples from independent standard normal variable Z_1 and Chi-squared variable χ_4^2, with 4 degrees of freedom and calculate the $T_{(4)}$ as; $T_{(4)} = \frac{Z_1}{\sqrt{\chi_4^2/4}}$, which is the sample from the ST distribution with 4 degrees of freedom, (Fig. 3(b)).

III. Take 10000 samples from independent standard normal variable Z_1 and Inverse-gamma variable $I\Gamma_{\left(\frac{\upsilon}{2},\frac{\upsilon}{2}\right)}$, with $\upsilon = 4$, and calculate sample from the ST distribution with 4 degree of freedom as; $T_{(4)} = Z_1 * \sqrt{I\Gamma_{(2,2)}}$, (Fig. 3 (c)).

IV. Bailey (1994) discovered that the T distribution could be sampled by a very elegant modification to the well known Box-Muller method, and its polar variant, for the Normal distribution. With some modification of polar algorithm for generating $T_{(\upsilon)}$-variates can be summarized as follows;
   a) Generate two uniform variates u_1 and u_2 from [0, 1] and replace u_1 and u_2 as $u_1 = 2u_1 - 1$ and $u_2 = 2u_2 - 1$.
   b) Let $w = u_1^2 + u_2^2$, If $w > 1$ return to step (a.) for resample.
   c) Then calculate T as; $T = u_1\sqrt{\frac{n\left(w^{-2/n}-1\right)}{w}}$ (Fig. 3(d)).

(see 'Appendix B' for Mathematica cording)

2.2 Moments

The general ST distribution is symmetrical around zero whereby all odd moments vanish. The k^{th} moment exists if and only if $k < \upsilon$;

$$E(X^k) = \begin{cases} 0 & k \text{ odd};\ 0 < k < v \\ \dfrac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{\upsilon-k}{2}\right)\upsilon^{k/2}}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\upsilon\right)} & k \text{ even};\ 0 < k < v \\ \text{Not Define} & k \text{ odd};\ 0 < v < k \\ \text{Infinite} & k \text{ even};\ 0 < v < k \end{cases} \quad (05)$$

In this case, the variance of the distribution is $Var(T) = E(X^2) = \upsilon/\upsilon - 2$ and it is finite for $\upsilon > 2$. Due to symmetry, the skewness of the distribution is zero and kurtosis is given by $K(T) = \frac{E(X^4)}{[E(X^2)]^2} = \frac{3(\upsilon-2)}{\upsilon-4}$ and it is finite for $\upsilon > 4$.

Kurtosis measures the "fatness" of the tails of a distribution. Positive excess kurtosis means that distribution has fatter tails than a normal distribution. Fat tails means the probability of big positive and negative returns realizations is higher than that suggested by normal distribution. When calculating kurtosis, a result of +3.00 indicates the absence of kurtosis (distribution is mesokurtic). For simplicity in its interpretation, some statisticians adjust this result to zero (i.e. kurtosis minus 3 equals zero), and then any reading other than zero is referred to as excess kurtosis. Negative numbers indicate a platykurtic distribution while positive numbers indicate a leptokurtic distribution. The likelihood of large gains or large losses on an investment is high when fatness of the tails is higher. Excess kurtosis indicates that the volatility of the investment is itself highly volatile.
The k^{th} moment of the standard ST distribution is given in equation (06) and it can be illustrated by using the following Fig. 4.

$$E(X^k) = \begin{cases} 4^{-n}\nu^n\sigma^{2n}(1+2n)!\,\Gamma\left(\dfrac{-2n+\nu}{2}\right)\dfrac{\left[\begin{array}{c}(\mu^2+\nu\sigma^2)\ {}_2F_1[-n,\frac{1}{2}(-2n+\nu),-\frac{1}{2},-\frac{\mu^2}{\nu\sigma^2}]+ \\ (4n\mu^2-\nu(\mu^2+\sigma^2))\ {}_2F_1[-n,\frac{1}{2}(-2n+\nu),\frac{1}{2},-\frac{\mu^2}{\nu\sigma^2}])\end{array}\right]}{(1+2n)\mu(1+2n-\nu)\Gamma(1+n)\Gamma(\frac{\nu}{2})} & k \text{ odd};\ 0 < k < v \\ \dfrac{4^{-n}\nu^n\sigma^{2n}(2n)!\,\Gamma[\frac{1}{2}(-2n+\nu)]\ {}_2F_1[-n,\frac{1}{2}(-2n+\nu),\frac{1}{2},-\frac{\mu^2}{\nu\sigma^2}]}{\Gamma[1+n]\Gamma[\frac{\nu}{2}]} & k \text{ even};\ 0 < k < v \\ \text{Not Define} & k \text{ odd};\ 0 < v < k \\ \text{Infinite} & k \text{ even};\ 0 < v < k \end{cases}$$

(06)

(where, ${}_2F_1$ is a Hypergeometric function)

2.3 Characteristic Function

The characteristic function (CF) $\psi(u)$, of the random variable $X \sim ST(\mu,\ \sigma^2,\ \upsilon)$ is given by;

$$\psi(u) = e^{iu\mu}2^{1-\frac{v}{2}}\frac{K_{\frac{v}{2}}(\sqrt{v}\sigma|u|)}{\Gamma(\frac{v}{2})}\left(\sqrt{v}\sigma|u|\right)^{\frac{v}{2}}; \quad u \in \mathbb{R} \quad (07)$$

(where $K_{\frac{v}{2}}(\dots)$ is the modified Bessel function of the second kind) and it can be derived as follows with Lemma 01and 02.

$\psi(u) = E[e^{iux}] = E[e^{iu(\mu+\delta\epsilon)}] = e^{iu\mu}E[e^{iu(\delta\epsilon)}]$[1]

[1] **Lemma 01**: The random variable $X \sim ST(\mu,\sigma,\nu)$ has the representation: $X \overset{D}{\Rightarrow} \mu + \delta\epsilon$.

Where; the independent random variables ϵ and δ^2 are standard normal N(0,1) and inverse gamma distribution $R\Gamma(\frac{1}{2}\upsilon, \frac{1}{2}\sigma^2)$ respectively and μ is a constant. The probability density function of f_{δ^2} is given by:

Let, $X = \delta\epsilon$

$$\psi(u) = e^{iu\mu}\int_{-\infty}^{\infty} e^{iux} f_x(x) = e^{iu\mu}\int_{-\infty}^{\infty} e^{iux}\int_{-\infty}^{\infty}\frac{1}{|t|} f_\delta(t) f_\epsilon\left(\frac{x}{t}\right) dt\, dx^2$$

$$= e^{iu\mu}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{iux}\frac{1}{|t|}\frac{1}{\sqrt{2\pi}} e^{-(x/t)^2/2} dx\right] f_\delta(t)dt \qquad = e^{iu\mu}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} e^{iux}\frac{1}{|t|} f_\epsilon\left(\frac{x}{t}\right) dx\right] f_\delta(t)dt$$

$$= e^{iu\mu}\int_{-\infty}^{\infty}\left[2\int_0^{\infty}\mathrm{Cos}(ux)\frac{1}{|t|}\frac{1}{\sqrt{2\pi}} e^{-(x/t)^2/2} dx\right] f_\delta(t)dt = e^{iu\mu}\int_{-\infty}^{\infty} e^{-\left(u^2/2\right)t^2} f_\delta(t)dt$$

$$= e^{iu\mu}\int_0^{\infty} e^{-\left(u^2/2\right)t^2}\, 2t f_{\delta^2}(t^2)dt$$

Let $t^2 = \bar{t}$ then $= e^{iu\mu}\int_0^{\infty} e^{-\left(u^2/2\right)t^2} f_{\delta^2}(\bar{t})d\bar{t} \quad = e^{iu\mu}\int_0^{\infty} e^{-\left(u^2/2\right)\bar{t}}\frac{\left(\frac{1}{2}\sigma^2\right)^{\upsilon/2}}{\Gamma\left(\frac{1}{2}\upsilon\right)}\bar{t}^{-\upsilon/2-1} e^{\sigma^2/(2\bar{t})} d\bar{t}$

$$= e^{iu\mu}\frac{\left(\frac{1}{2}\sigma^2\right)^{\upsilon/2}}{\Gamma\left(\frac{1}{2}\upsilon\right)}\int_0^{\infty} e^{-\left(u^2/2\right)\bar{t}} e^{\sigma^2/(2\bar{t})}\bar{t}^{(-\upsilon/2-1)} d\bar{t}$$

$$\psi(u) = e^{iu\mu} 2^{(1-\upsilon/2)}\frac{K_{\upsilon/2}(\sqrt{v}\sigma|u|)}{\Gamma\left(\frac{\upsilon}{2}\right)}\left(\sqrt{v}\sigma|u|\right)^{\upsilon/2}$$ □

3. Empirical Evidence

Some empirical results as given ~~in~~ below is an indication of the suitability of the Lévy student process for financial data. Daily closing prices of Gold Future index and S&P 500 index are considered from Janaury-2002 to December-2012, which are traded in the NYMEX, with a total of 2870 and 2769 observations respectively. In the analysis, logarithmic returns were used.

The maximum likelihood estimates of the parameters are given in table 01. For the assessment of goodness of fit, the Anderson-Darling (AD) and Person χ^2 tests are utilized with the log-likelihood estimator and results are recorded in Table 01. The smaller the value of AD and χ^2 means that closer to empirical distribution and fitted one. Obviously, the statistics for Lévy process are smaller than the value for Brownian motion. Higher log-likelihood value gives better fit. Further, the corresponding empirical densities, Gaussian and Student T density, for Gold Future Index and S&P 500 index are shown in Fig. 5 in separately.

Therefore, Student-t distribution is more realistic distribution in describing the financial data than Gaussian, for the historical data investigated. As a result, we can be used to student process for asset price modeling.

4. Financial Applications

4.1 Convolution

Convolutions and related operations are found in many applications of mathematics and engineering. The concept of convolution is needed to many derivatives in Lévy processes applications.

Corollary 1: The PDF for n-fold self-convolution can be found by; either convolution integrals:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots f(x_1, x_2, \ldots) g(y_1 - x_1, y_2 - x_2, \ldots)\, dx_1\, dx_2 \cdots \qquad (08)$$

or the inverse Fourier transform of the nth power of the characteristic function of the original function as,

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$$f_{\delta^2}(x) = \begin{cases}\frac{\left(\frac{1}{2}\sigma^2\right)^{\upsilon/2}}{\Gamma\left(\frac{1}{2}\upsilon\right)} x^{-\upsilon/2-1} e^{\sigma^2/(2x)} & \text{If } x > 0\\ 0 & \text{If } x \le 0\end{cases}$$

For > 0 , $f_\delta(x) = 2x f_{\delta^2}(x)$, the density function of δ.

[2] **Lemma 02**: Let X_1 and X_2 be independent stochastic random variable (absolutely continuous) with density function f_{X_1} and f_{X_2}. Then the random variable $X_1.X_2$ is absolutely continuous with density function (consider product of r. v.) as:

$$f_{X_1.X_2}(x) = \int_{-\infty}^{\infty}\frac{1}{|t|} f_{X_1}(t) f_{X_2}\left(\frac{x}{t}\right) dt\, ; x \in \mathbb{R}$$

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ixt}\psi(x)^n dx.$ (09)

In general, the Student t-distributions are not closed under convolution (Nadarajah & Dey 2005), because the sums of independent identically distributed (iid) t variables are not 't-distributed'. Therefore, analytical solutions can be derived in some special cases only. Nadarajah & Dey (2005) provided analytical solutions for the density function $f_{\upsilon_1,\upsilon_2}$ of the 2 fold-convolution for any odd integer values of υ_1 and υ_2. They showed that there are no similar analytical solutions for even υ. However, numerical solution can be found for any integer υ using the characteristic function technique.

The n-fold self-convolution of a Student's t-distribution for odd υ (=3, 5, 7, ...) can be formulated simply and their expression are given in below. However, n-fold convolution do not hold for even υ (=4, 6, ...) except $\upsilon = 2$ case.

The PDF for a 2-fold self-convolution:

$$f_u^{(2)}(u) = \frac{1}{\sqrt{\nu}\Gamma\left[\frac{\nu}{2}\right]^2} 2^{\frac{1}{2}-\nu}\sqrt{\pi}\Gamma\left[\frac{1+2\nu}{2}\right]\Gamma\left[\frac{1+\nu}{2}\right] {}_2F_1\left[\frac{1}{2}+\nu, \frac{1+\nu}{2}, \frac{2+\nu}{2}, -\frac{u^2}{4\nu}\right]$$

$$\nu = 2; \qquad f_u^{(2)}(u) = \frac{3}{32}\pi^{3/2}\, {}_2F_1\left[\frac{3}{2}, \frac{5}{2}, 2, -\frac{u^2}{8}\right]$$

$$\nu = 5; \qquad f_u^{(2)}(u) = \frac{400\sqrt{5}(8400+120u^2+u^4)}{3\pi(20+u^2)^5} \tag{10}$$

The PDF for a 4-fold self-convolution with $\nu = 5$:

$$f_u^{(4)}(u) = \frac{800\sqrt{5}}{3\pi(80+u^2)^9}\begin{pmatrix}118450858700000 + 1521664000000u^2 + 12456960000u^4 + 73728000u^6 + \\ 308400u^8 + 810u^{10} + u^{12}\end{pmatrix} \tag{11}$$

The PDF for a 6-fold self-convolution with $\nu = 5$

$$f_u^{(6)}(u) = \frac{400\sqrt{5}}{\pi(180+u^2)^{13}}(9796994465739264X10^{10} + 101231217186048X10^{10}u^2 + 650898019296X10^{10}u^4 + 324358903872X10^8u^6 + 1315565064X10^8u^8 + 4314479616X10^5u^{10} + 11115436X10^5u^{12} + 2153X10^6u^{14} + 2934X10^3u^{16} + 2500u^{18} + u^{20}) \tag{12}$$

The tails of the n-fold self-convolution of a Student's t-distribution maintain the character of the original t-distribution. For large t, the n-fold $\nu = 5$ pdf goes as u^{-6}. Thus, n-day returns will maintain the fat tails of the distribution of the daily returns, which is known to be described well by a Student's t-distribution. This is in agreement with the analysis of Bouchaud and Potters (2003, p. 33). Fig. 6 shows the shapes of the convoluted probability density functions for different n (=1, 2, 4, 6, 10 & 15) fold self convolution with $\upsilon = 5$. The tails of the distributions show a u^{-6} behavior that is characteristic of a Student's t-distribution with $\upsilon = 5$ degrees of freedom.

4.2 Infinite Divisibility & Self-decomposability

Relating student's t distributions to the Lévy processes, the crucial role are paid the properties of infinite divisibility or self-decomposability (Grigelionis 2013; Sato 1999). Grosswald (1976) proved that the standard student's t-distribution of any degree of freedom is infinitely divisible, by deriving the following formula:

$$K_{\upsilon-1}(x) = xK_\upsilon(x)\int_0^\infty \frac{g_\upsilon(u)}{x^2+u}du, \ \upsilon \geq -1, \ x > 0 \tag{13}$$

where, $g_\upsilon(u) = 2[\pi^2 x(J_\upsilon^2(\sqrt{x}) + Y_\upsilon^2(\sqrt{x}))]^{-1}, \ x > 0,$

$J_\upsilon(x)$ and $Y_\upsilon(x)$ are the Bessel functions of the first kind and second kind, respectively while Jurek (2001) proved that the general student t-distribution is infinite divisible(ID) and self-decomposable(SD) by using Lemma 1 and Lemma 3[3]. That means, the class of all self-decomposable characteristic (or probability distributions) are infinitely divisible,

i.e., $$\forall(n \geq 1)\, \exists\, (\psi_n), \forall(u \in \mathbb{R}) \text{ s.t. } \psi(u) = [\psi_n(u)]^n \quad (14)$$

The characteristic function of every infinitely divisible distribution can be represented in a very special form, the Lévy–Khintchineformula:

$$\psi_\mu(u) = e^{\Phi(u)} = \exp\left(iub - \frac{1}{2}cu^2 + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|>1\}}\right) \upsilon(dx)\right) \quad (15)$$

where the coefficient $b \in \mathbb{R}$, $c \geq 0$ and the L évy measure $\upsilon(dx)$ which satisfies $\upsilon(dx) = 0$ and $\int_{\mathbb{R}} (x^2 \wedge 1)\, \upsilon(dx) < \infty$ are unique.

The Lévy–Khintchine representation of the characteristic function of the general $ST(\upsilon, \sigma, \mu)$ can be obtained from the results of Halgreen (2005) by choosing $\alpha = |\beta| = 0$, $\sigma > 0$ and $\lambda = -\frac{\upsilon}{2} < 0$ as follows;

$$\psi(u) = \exp\left(iu\mu + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux1_{\{|x|>1\}}\right) g(x)dx\right), \quad (16)$$

With $g(x) = \frac{1}{|x|}\int_0^\infty \frac{e^{-|x|\sqrt{2y}}dy}{\pi^2 y\left(J^2_{\upsilon/2}(\sigma\sqrt{2y}) + Y^2_{\upsilon/2}(\sigma\sqrt{2y})\right)}$

4.3 L évy- Student Processes

Student t-distribution is infinitely divisible and therefore generate a L évy processes $L = (L(t))_{t\geq 0}$, such that the distribution of $L(1)$, has symmetric $ST(\upsilon, \sigma, \mu)$ distribution (for simplicity and without loss of generality, we consider $t = 1$ with Student marginal. However, as ST distributions are not closed under convolution as above and Nadarajah & Dey (2005), the marginals of these L évy processes are Student distributed only for one time horizon t_0. On other time horizons, the marginal are not Student distributed. Their distribution can be derived analytically only for some special cases (Heyde & Leonenko 2005; Cufaro 2007). Apart from these cases, they have to be derived numerically in quite an involved manner (Barndorff & Shephard 2004). For this reason, only few research papers have covered L évy Student processes, so far.

The law of $L(t)$ is determined by the law of $L(1)$, which is ID. The independent and stationary of the increments of the L évy process leads to the cumulant transform $\varkappa(u) = \log(\psi(u))$ is given by;

$$\varkappa_{L(t)}(u) = t\,\varkappa_{L(1)}(u) \qquad u \in \mathbb{R}. \quad (17)$$

For $\upsilon > 1$ and $E[L(t)] = t\mu$, $t \geq 0$, the process can be split into $L(t) = t\mu + L^0(t)$, $t \geq 1$, with $E[L^0(t)] = 0$. By Eq. (16), the characteristic function of the random variable $L^0(t)$, $t \geq 0$ with the characteristic function $\psi(u)$, of the process at time t and

$$\psi_{L^0(t)}(u) = (\psi_{L^0(1)}(u))^t, \qquad u \in \mathbb{R}. \quad (18)$$

As the characteristic function of a random variable equals its Fourier transformation up to some constant factors, the inverse Fourier transformation reproduces the density from the characteristic function. Let $\varphi_{L(1)}$ be the characteristic function of the Student distribution, the PDF of $L^0(t)$ is given;

$$f_{L(t)}(x) = \int_{-\infty}^{\infty} e^{iux}\psi_{L^0(t)}(u)\,du$$

$$= \int_{-\infty}^{\infty} e^{iux}\left(e^{iu\mu}2^{1-\frac{v}{2}}\frac{K_{\frac{v}{2}}(\sqrt{v}\sigma|u|)}{\Gamma(\frac{v}{2})}\left(\sqrt{\upsilon}\sigma|u|\right)^{\frac{v}{2}}\right)^t du$$

$$= \frac{2^{t(1-\upsilon/2)}}{\pi\Gamma^t(\frac{\upsilon}{2})}\int_0^\infty \cos(ux)e^{iu\mu t}(\sqrt{\upsilon}\sigma|u|)^{\upsilon t/2}K^t_{\frac{\upsilon}{2}}(\sqrt{\upsilon}\sigma|u|)du\ . \quad (19)$$

When $t = 1$ the expression (19) can be exactly calculated and coincides with the PDF (03) of a $ST(\upsilon, \sigma, \mu)$. Furthermore, the principal drawback for not being stable is in the subsequent definition of the L évy–Student process. In fact the CF of the Student

[3] **Lemma 3**: Let $L(t)$be a levy process with having the strong Markov and scaling (the process $L(t)$ has the scaling property if for each 0<c<1 there exists a constant h(c) such that $L(ct) \stackrel{d}{=} h(c)L(t)$) properties. For any independent random variable $T \geq 0$, there is $L(t+T) - L(T) \stackrel{d}{=} L(t)$. If T is SD and the $h(c)$ is homeomorphism of unit interval, then $L(T)$ is SD.

Lévy process $\varphi_{L^{o}(t)}(u)$ coincides with (06) only for only for $t = 1$, while for $t \neq 1$ it is no more the CF of a Student law (Petronia *el al.* 2006).

The integral on the right hand side of (18) can be computed numerically for specific values of υ. The results are shown in Fig. 6, where the densities of the convolution semi-group are represented for values of t varying from 1 to 3, with the parameters of $\upsilon = 3$, $\mu = 0$ and unit value of σ. The integral is difficult to evaluate for even υ. Therefore, it is better to see the behavior of Student Lévy process when $\upsilon = 3$ situation due to results in section 3 (value of degree of freedom in both indexes are close to 3).

5. Discussion

The Student t-distribution has strong reason to be regarded as an alternative model of first choice particularly when the benchmark normal or Black Scholes, model is found to be inadequate. It implies an Inverse gamma distribution for the marginal distribution of the squared volatility.

Student distributions are infinitely divisible and self-decomposable; there exist Lévy processes with Student marginal's for a certain point in time. The densities of the Lévy Student processes for any point in time are given by integrals (18) which can be solved analytically only for few special cases. There are some problem solving the problem numerically due to slow convergence and heavy tails of the distributions. Therefore, a model can be developed for a specific values of υ to price the option. This distributional property can be exploited to identify possible dynamics of the volatility process and hence the evolution of the asset price process can be derived

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Appendix

Appendix A

```
st[μ_, σ_, ν_] := (ν/((x-μ)^2/σ^2 + ν))^((1+ν)/2) / (σ √ν Beta[ν/2, 1/2]);
stM[n_, μ_, σ_, ν_] := (σ^n ν^(n/2) Pochhammer[1/2, n/2]) / Pochhammer[-n/2 + ν/2, n/2];
stExp[μ_, σ_, ν_] := Derivative[1, 0, 0, 0][stM][0, μ, σ, ν];
stVar[μ_, σ_, ν_] := Derivative[2, 0, 0, 0][stM][0, μ, σ, ν];
stKurt[μ_, σ_, ν_] := Derivative[4, 0, 0, 0][stM][0, μ, σ, ν] / Derivative[2, 0, 0, 0][stM][0, μ, σ, ν]^2;
stSkew[μ_, σ_, ν_] := Derivative[3, 0, 0, 0][stM][0, μ, σ, ν] / Derivative[2, 0, 0, 0][stM][0, μ, σ, ν]^(3/2);
Manipulate[
 Module[{gr, tx, mean = stExp[μ, σ, υ], std = Sqrt[stVar[μ, σ, υ]]},
  gr = Quiet@DiscretePlot[st[μ, σ, υ], {x, mean - 6 std, mean + 6 std, 0.01},
     PlotRange → All, Joined → True, ImageSize → Medium, ImagePadding → 20];
  tx =
   Text@Column[{Row[{"mean: ", mean}], Row[{"variance: ", std^2}],
      Row[{"skewness: ", stSkew[μ, σ, υ]}],
      Row[{"excess kurtosis: ", stKurt[μ, σ, υ]}]}];
  Pane[Grid[{{gr, tx}}], ImageSize → {500, 250}, Alignment → Center]],
 {{μ, 0, "μ"}, -2, 2, Appearance → "Labeled"},
 {{υ, 3, "υ"}, 2, 5, Appearance → "Labeled"},
 {{σ, 0.5, "σ"}, 0.01, 1, Appearance → "Labeled"}, SaveDefinitions → True,
 SynchronousUpdating → False]
```

Appendix B

```
studentrand = Compile[{{n, _Real}},
        Module[{u1 = 0.0, u2 = 0.0, w = 2.0, den},
            While[w >= 1.00,
                  (u1 = 2.0 * Random[] - 1.0;
                  u2 = 2.0 * Random[] - 1.0;
                  w = u1 * u1 + u2 * u2)];
            u1 * Sqrt[(n / w) * (w^(-2 / n) - 1)]]];
testdata = Table[studentrand[4.0], {10 000}];
skdist4 = SmoothKernelDistribution[testdata];
Show[Histogram[testdata, Automatic, "ProbabilityDensity"],
 Plot[PDF[skdist4, x], {x, -10, 15}, PlotRange → All],
 AxesLabel → {t_values, Probability_Density}]
```

Appendix C

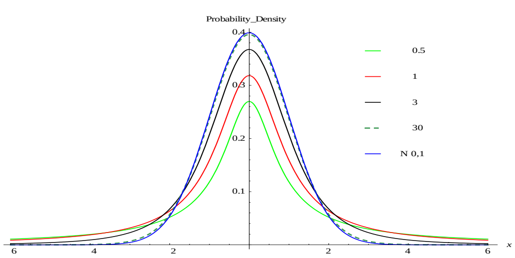


Fig.1. Plot of the standard normal density and student t-densities with 0.5,1,3 and 30 degrees of freedom

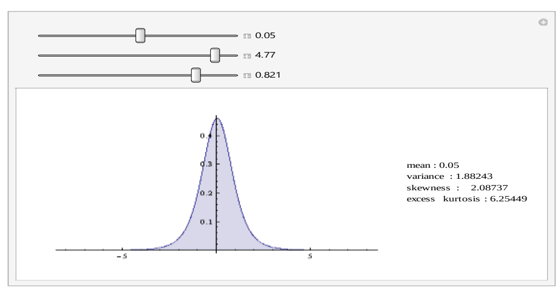
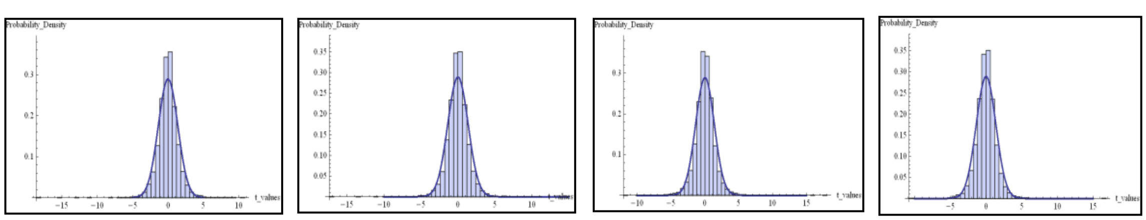


Fig.2. Structural behavior of the $ST(\upsilon, \sigma, \mu)$ distribution with different parameter values



(a).Sampling distribution from Method I

(b).Sampling distribution from Method II

(c).Sampling distribution from Method III

(d).Sampling distribution from Method IV

Fig. 3. The results of above four methods, (which each one has ST distribution with 4 degrees of freedom) are compared with the normal distribution

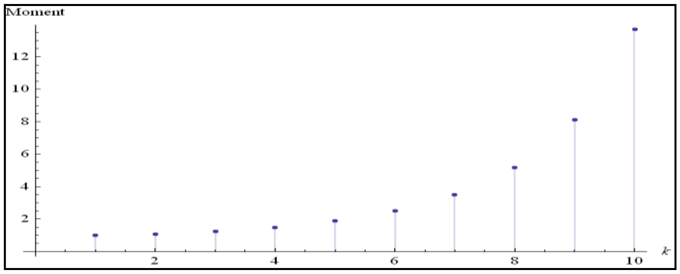
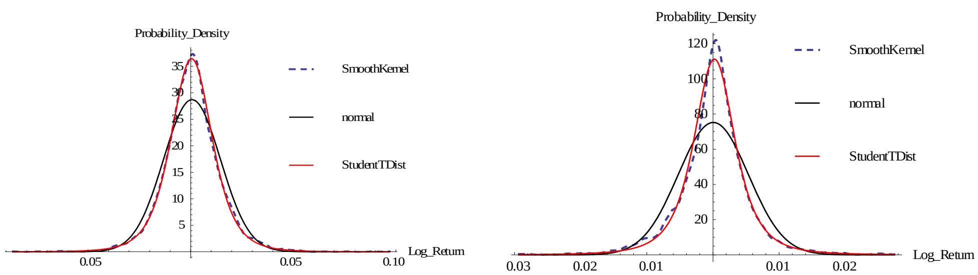


Fig.4. First 10 moments of the Standard ST Distribution



(a).Gold Returns

(b).S&P 500 Returns

Fig. 5. Densities Estimation for return, from Smooth Kernel, Student T and Normal Distribution

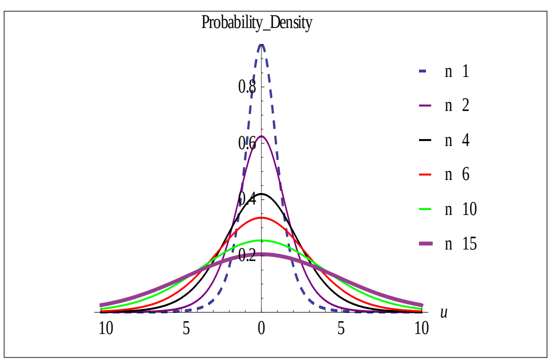


Fig.6. Shapes of the convoluted PDFs for different n (=1, 2, 4, 6, 10 & 15) with $\upsilon = 5$

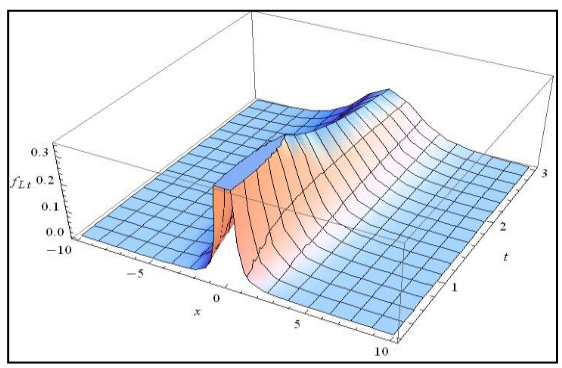


Fig. 7. Convolution Semi-group Densities

Table 1: Parameter estimates and goodness of fit-tests for the gold future index & S&P 500 index

Index	Model	υ	μ	σ	AD	χ^2	Log-Likeli-hood
Gold	Gaussian	-	0.00063	0.0139	14.386	176.12	6642
					(0.000)	(0.000)	
	Student T	3.45	0.00044	0.0103	0.4712	47.14	6886
					(0.841)	(0.243)	
S&P 500	Gaussian	-	0.00005	0.0053	20.145	196.45	5614
					(0.000)	(0.000)	
	Student T	2.89	0.00023	0.0032	0.6532	32.15	5941
					(0.521)	(0.154)	