

Interval Valued intuitionistic Fuzzy Homomorphism of BF-algebras

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Abstract
The notion of interval-valued intuitionistic fuzzy sets was first introduced by Atanassov and Gargov as a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Satyanarayana et. al., applied the concept of interval-valued intuitionistic fuzzy ideals to BF-algebras. In this paper, we introduce the notion of interval-valued intuitionistic fuzzy homomorphism of BF-algebras and investigate some interesting properties.
Keywords: BF-algebras, interval valued intuitionistic fuzzy sets, i-v intuitionistic fuzzy ideals

1. Introduction and preliminaries

For the first time Zadeh (1965) introduced the concept of fuzzy sets and also Zadeh (1975) introduced the concept of an interval-valued fuzzy sets, which is an extension of the concept of fuzzy set. Atanassov and Gargov, 1989 introduced the notion of interval-valued intuitionistic fuzzy sets, which is a generalization of both intuitionistic fuzzy sets and interval-valued fuzzy sets. On other hand, Satyanarayana et al., (2012) applied the concept of interval-valued intuitionistic fuzzy ideals. In this paper we introduce the notion of interval-valued intuitionistic fuzzy homomorphism of BF-algebras and investigate some interesting properties.

By a BF-algebra we mean an algebra satisfying the axioms:
(1). $x * x = 0$,
(2). $x * 0 = x$,
(3). $0 * (x * y) = y * x$, for all $x, y \in X$
Throughout this paper, X is a BF-algebra.

Example 1.1 Let R be the set of real number and let $A = (R, *, 0)$ be the algebra with the operation $*$ defined by

$$x * y = \begin{cases} x, \text{if } y = 0 \\ y, \text{if } x = 0 \\ 0, \text{otherwise} \end{cases}$$

Definition 1.2 The subset I of X is said to be an ideal of X, if (i) $0 \in I$ and (ii) $x * y \in I$ and $y \in I \Rightarrow x \in I$.

Definition 1.3 A mapping $f : X \to Y$ of BF-algebra is called a homomorphism if $f(x * y) = f(x) * f(y)$, for all $x, y \in X$. Note that if f is a homomorphism of BF-algebras, then f(0)=0.

An intuitionistic fuzzy set (shortly IFS) in a non-empty set X is an object having the form $A = \{(x, \mu_A(x), \lambda_A(x)) : x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $\lambda_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non membership (namely $\lambda_A(x)$) of each element $x \in X$. For the sake of simplicity we use the symbol form $A = (X, \mu_A, \lambda_A)$ or $A = (\mu_A, \lambda_A)$

By interval number D we mean an interval $[a^-, a^+]$ where $0 \le a^- \le a^+ \le 1$. The set of all closed subintervals of $[0,1]$ is denoted by $D[0,1]$. For interval numbers $D_1 = [a_1^-, b_1^+]$, $D_2 = [a_2^-, b_2^+]$.
We define

- $\min(D_1, D_2) = D_1 \cap D_2 = \min([a_1^-, b_1^+], [a_2^-, b_2^+])$
$= [\min\{a_1^-, a_2^-\}, \min\{b_1^+, b_2^+\}]$

- $\max(D_1, D_2) = D_1 \cup D_2 = \max([a_1^-, b_1^+], [a_2^-, b_2^+])$
  $= [\max\{a_1^-, a_2^-\}, \max\{b_1^+, b_2^+\}]$
- $D_1 + D_2 = [a_1^- + a_2^- - a_1^-.a_2^-, b_1^+ + b_2^+ - b_1^+.b_2^+]$

And put

- $D_1 \leq D_2 \Leftrightarrow a_1^- \leq a_2^-$ and $b_1^+ \leq b_2^+$
- $D_1 = D_2 \Leftrightarrow a_1^- = a_2^-$ and $b_1^+ = b_2^+$,
- $D_1 < D_2 \Leftrightarrow D_1 \leq D_2$ and $D_1 \neq D_2$
- $mD = m[a_1^-, b_1^+] = [ma_1^-, mb_1^+]$, where $0 \leq m \leq 1$.

Let L be a given nonempty set. An interval-valued fuzzy set B on L is defined by $B = \{(x, [\mu_B^-(x), \mu_B^+(x)] : x \in L\}$, Where $\mu_B^-(x)$ and $\mu_B^+(x)$ are fuzzy sets of L such that $\mu_B^-(x) \leq \mu_B^+(x)$ for all $x \in L$. Let $\tilde{\mu}_B(x) = [\mu_B^-(x), \mu_B^+(x)]$, then $B = \{(x, \tilde{\mu}_B(x)) : x \in L\}$
Where $\tilde{\mu}_B : L \to D[0, 1]$.

A mapping $A = (\tilde{\mu}_A, \tilde{\lambda}_A) : L \to D[0,1] \times D[0,1]$ is called an interval-valued intuitionistic fuzzy set (i-v IF set, in short) in L if $0 \leq \mu_A^+(x) + \lambda_A^+(x) \leq 1$ and $0 \leq \mu_A^-(x) + \lambda_A^-(x) \leq 1$ for all $x \in L$ (that is, $A^+ = (X, \mu_A^+, \lambda_A^+)$ and $A^- = (X, \mu_A^-, \lambda_A^-)$ are intuitionistic fuzzy sets), where the mappings $\tilde{\mu}_A(x) = [\mu_A^-(x), \mu_A^+(x)] : L \to D[0,1]$ and $\tilde{\lambda}_A(x) = [\lambda_A^-(x), \lambda_A^+(x)] : L \to D[0, 1]$ denote the degree of membership (namely $\tilde{\mu}_A(x)$)and degree of non-membership(namely $\tilde{\lambda}_A(x)$ of each element $x \in L$ to A respectively.

2. MAIN RESUTS

Definition 2.1: An interval-valued IFS $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is called interval-valued intuitionistic fuzzy ideal (shortly i-v IF ideal) of BF-algebra X if it satisfies

(i-v IF1) $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$
(i-v IF2) $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$
(i-v IF3) $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y)\}$, for all $x, y \in X$.

Example 2.2 Consider a BF-algebra $X = \{0, 1, 2, 3\}$ with following table

*	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Let A be an interval-valued intuitionistic fuzzy set in X defined by $\tilde{\mu}_A(0) = \tilde{\mu}_A(1) = [0.6, 0.7]$ and $\tilde{\mu}_A(2) = \tilde{\mu}_A(3) = [0.2, 0.3]$, $\tilde{\lambda}_A(0) = \tilde{\lambda}_A(1) = [0.1, 0.2]$, $\tilde{\lambda}_A(2) = \tilde{\lambda}_A(3) = [0.5, 0.7]$. It is easy to verify that A is an interval-valued intuitionistic fuzzy ideal of X.

Definition 2.3 Let $f : X \to X'$ be a homomorphism of BF-algebras. For any interval valued intuitionistic fuzzy set $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ in X' we define a new interval valued intuitionistic fuzzy set $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ in X, by $\tilde{\mu}_A^f(x) = \tilde{\mu}_A(f(x))$, $\tilde{\lambda}_A^f(x) = \tilde{\lambda}_A(f(x))$ for all $x \in X$.

Theorem 2.4 Let X and X' be BF-algebras and f is a homomorphism from X onto X'.

(i). If $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X', then $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an i-v intuitionistic fuzzy ideal of X.

(ii). If $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an i-v intuitionistic fuzzy ideal of X, then $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X'.

Proof: (i) Suppose $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X'. For $x' \in X'$ there exist $x \in X$ such that $f(x) = x'$, we have

$$\begin{aligned}\tilde{\mu}_A^f(0) &= \left[\mu^{-f}_A(0), \mu^{+f}_A(0)\right] \\ &= \left[\mu_A^-(f(0)), \mu_A^+(f(0))\right] \\ &= \left[\mu_A^-(0'), \mu_A^+(0')\right] \\ &\geq \left[\mu_A^-(x'), \mu_A^+(x')\right] \\ &= \left[\mu_A^-(f(x)), \mu_A^+(f(x))\right] \\ &= \left[\mu^{-f}_A(x), \mu^{+f}_A(x)\right] \\ &= \tilde{\mu}_A^f(x)\end{aligned}$$

and

$$\begin{aligned}\tilde{\lambda}_A^f(0) &= \left[\lambda^{-f}_A(0), \lambda^{+f}_A(0)\right] \\ &= \left[\lambda_A^-(f(0)), \lambda_A^+(f(0))\right] \\ &= \left[\lambda_A^-(0'), \lambda_A^+(0')\right] \\ &\leq \left[\lambda_A^-(x'), \lambda_A^+(x')\right] \\ &= \left[\lambda_A^-(f(x)), \lambda_A^+(f(x))\right] \\ &= \left[\lambda^{-f}_A(x), \lambda^{+f}_A(x)\right] \\ &= \tilde{\lambda}_A^f(x)\end{aligned}$$

Let $x, z \in X, y' \in X'$ then there exists $y \in X$ such that $f(y) = y'$. We have $\tilde{\mu}_A^f(x) = \tilde{\mu}_A(f(x))$

$$\geq \min\{\tilde{\mu}_A(f(x) * y'), \tilde{\mu}_A(y')\}$$
$$= \min\{\tilde{\mu}_A(f(x * y)), \tilde{\mu}_A(f(y))\}$$
$$= \min\{\tilde{\mu}_A^f(x * y), \tilde{\mu}_A^f(y)\}$$

and $\tilde{\lambda}_A^f(x) = \tilde{\lambda}_A(f(x))$

$$\leq \max\{\tilde{\lambda}_A(f(x) * y'), \lambda_A(y')\}$$
$$= \max\{\tilde{\lambda}_A(f(x) * f(y)), \tilde{\lambda}_A(f(y)\}$$
$$= \max\{\tilde{\lambda}_A(f(x * y)), \tilde{\lambda}_A(f(y)\}$$
$$= \max\{\tilde{\mu}_A^f(x * y), \tilde{\mu}_A^f(y)\}$$

Hence $A^f = (X, \tilde{\mu}_A^f, \tilde{\lambda}_A^f)$ is an i-v intuitionistic fuzzy ideal of X.

(ii) Since $f : X \to X'$ is onto, for $x, y \in X'$ there exist $a, b \in X$ such that $f(a) = x, f(b) = y$.

Now $\tilde{\mu}_A(x) = \tilde{\mu}_A(f(a)) = \tilde{\mu}_A^f(a) \geq \min\{\tilde{\mu}_A^f((a * b), \tilde{\mu}_A^f(b)\}$

$$= \min\{\tilde{\mu}_A(f(a * b)), \tilde{\mu}_A(f(b))\}$$
$$= \min\{\tilde{\mu}_A(f(a) * f(b)), \tilde{\mu}_A(f(b))\}$$
$$= \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\}$$

and $\tilde{\lambda}_A(x) = \tilde{\lambda}_A(f(a)) = \tilde{\lambda}_A^f(a) \leq \max\{\tilde{\lambda}_A^f(a * b), \tilde{\lambda}_A^f(b)\}$

$$= \max\{\tilde{\lambda}_A(f(a * b)), \tilde{\lambda}_A(f(b))\}$$
$$= \max\{\tilde{\lambda}_A(f(a) * f(b)), \tilde{\lambda}_A(f(b))\}$$
$$= \max\{\tilde{\lambda}_A(x * y), \tilde{\lambda}_A(y)\}$$

Hence $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal X'.

Definition 2.5 Let f be a mapping on set X and $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X. Then the i-v fuzzy sets $\tilde{u}$ and $\tilde{v}$ on $f(X)$ is defined by $\tilde{u}(y) = \sup_{x \in f^{-1}(y)} \tilde{\mu}_A(x)$ and $\tilde{v}(y) = \inf_{x \in f^{-1}(y)} \tilde{\lambda}_A(x)$ for all $y \in f(X)$ is called image of A under f. If $\tilde{u}$ and $\tilde{v}$ are i-v fuzzy sets in $f(X)$, then the fuzzy set $\tilde{\mu}_A = \tilde{u} \circ f$ and $\tilde{\lambda}_A = \tilde{v} \circ f$ is called the pre –image of $\tilde{u}$ and $\tilde{v}$ respectively under f.

Definition 2.6 An i-v IFS $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ in X is said to satisfy the "sup-inf" property if for any sub –set $T \subseteq X$ there exist $x_0, y_0 \in T$ such that $\tilde{\mu}_A(x_0) = \sup_{t \in T} \tilde{\mu}_A(t)$ and $\tilde{\lambda}_A(y_0) = \inf_{s \in T} \tilde{\lambda}_A(s)$.

Theorem 2.7 Let $f : X \to X'$ be onto homomorphism of BF- algebras. If $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X with "sup-inf" property. Then the image of A under f is also an i-v intuitionistic fuzzy ideal of X'.

Proof: For any $x \in X$ we have $\tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$.

Suppose $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X with "sup-inf" property. The image of A under f is defined by

$$\tilde{u} : X' \to [0,1] \text{ by } \tilde{u}(y') = \sup_{x \in f^{-1}(y')} \tilde{\mu}_A(x) \text{ for all } y' \in X'$$

and

$$\tilde{v} : X' \to [0,1] \text{ by } \tilde{v}(y') = \inf_{x \in f^{-1}(y')} \tilde{\lambda}_A(x) \text{ for all } y' \in X'.$$

Since $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X.

Thus $\tilde{u}(0') = \sup_{t \in f^{-1}(0')} \tilde{\mu}_A(t) = \tilde{\mu}_A(0) \geq \tilde{\mu}_A(x)$, for all $x \in X$.

Therefore $\tilde{u}(0') \geq \tilde{\mu}_A(x)$ for all $x \in X$.

Further more we have $\tilde{u}(x') = \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t)$ for all $x' \in X'$.

Hence $\tilde{u}(0') \geq \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t) = \tilde{u}(x')$. Therefore $\tilde{u}(0') \geq \tilde{u}(x')$ for all $x' \in X'$

And $\tilde{v}(0') = \inf_{t \in f^{-1}(0')} \tilde{\lambda}_A(t) = \tilde{\lambda}_A(0) \leq \tilde{\lambda}_A(x)$ for all $x \in X$.

Therefore $\tilde{v}(0') \leq \tilde{\lambda}_A(x)$ for all $x \in X$. Further more we have $\tilde{v}(x') = \inf_{t \in f^{-1}(x')} \tilde{\lambda}_A(t)$ for all $x' \in X'$.

Hence $\tilde{v}(0') \leq \inf_{t \in f^{-1}(x')} \tilde{\lambda}_A(t) = \tilde{v}(x')$, $\forall x' \in X'$. Thus $\tilde{v}(0') \leq \tilde{v}(x')$, $\forall x' \in X'$.

Since f is onto mapping then for any $x', y' \in X'$. Since $X' = f(X)$, then there exist $x, y \in X$ such that $x' = f(x), y' = f(y)$. Let $x_0 \in f^{-1}(x')$ be such that $\tilde{\mu}_A(x_0) = \sup_{t \in f^{-1}(x')} \tilde{\mu}_A(t)$ and

hence

$$\begin{aligned}
\tilde{u}(x') = \tilde{u}(f(x)) &= \sup_{t \in f^{-1}(f(x))} \tilde{\mu}_A(t) \\
&= \tilde{\mu}_A(x_0) \\
&\geq \min\{\tilde{\mu}_A((x_0 * y)), \tilde{\mu}_A(y)\} \\
&= \min\{\tilde{u}(f(x_0 * y)), \tilde{u}(f(y))\} \\
&= \min\{\tilde{u}(f(x_0) * f(y)), \tilde{u}(f(y))\} \\
&= \min\{\tilde{u}(x' * y'), \tilde{u}(y')\}
\end{aligned}$$

Therefore $\tilde{u}(x') \geq \min\{\tilde{u}(x' * y'), \tilde{u}(y')\}$ for all $x', y' \in X$.

Let $x_0 \in f^{-1}(x')$ be such that $\tilde{\lambda}_A(x_0) = \inf_{t \in f^{-1}(x')} \tilde{\lambda}_A(t)$.

Now $\tilde{v}(x') = \tilde{v}(f(x)) = \inf_{t \in f^{-1}(f(x))} \tilde{\lambda}_A(t)$

$$
\begin{aligned}
&= \tilde{\lambda}_A(x_0) \\
&\le \max\{\tilde{\lambda}_A(x_0 * y), \tilde{\lambda}_A(y)\} \\
&= \max\{\tilde{v}(f(x_0 * y)), \tilde{v}(f(y))\} \\
&= \max\{\tilde{v}(f(x_0) * f(y)), \tilde{v}(f(y))\} \\
&= \max\{\tilde{v}(x' * y'), \tilde{v}(y')\}
\end{aligned}
$$

Therefore $\tilde{v}(x') \le \max\{\tilde{v}(x' * y'), \tilde{v}(y')\}$ for all $x', y' \in X$

Thus $A = (X', \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X'.

Definition 2.8 Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X and let $\tilde{\alpha}, \tilde{\beta} \in [0, 1]$ be such that $\tilde{\alpha} + \tilde{\beta} \le [1,1]$. Then the set $X_A^{(\tilde{\alpha}, \tilde{\beta})} = \{x \in X / \tilde{\mu}_A(x) \ge \tilde{\alpha}, \tilde{\lambda}_A(x) \le \tilde{\beta}\}$ is called an $(\tilde{\alpha}, \tilde{\beta})$ -level sub set of $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$.

Theorem 2.9 Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an interval-valued intuitionistic fuzzy ideal of X. Then $X_A^{(\tilde{\alpha}, \tilde{\beta})}$ is an ideal of X, for every $(\tilde{\alpha}, \tilde{\beta}) \in \mathrm{Im}(\tilde{\mu}_A) \times \mathrm{Im}(\tilde{\lambda}_A)$ with $\tilde{\alpha} + \tilde{\beta} \le [1,1]$

Proof: Let $x \in X_A^{(\tilde{\alpha}, \tilde{\beta})}$, then $x \in X, \tilde{\mu}_A(x) \ge \tilde{\alpha}$ and $\tilde{\lambda}_A(x) \le \tilde{\beta} \Rightarrow$

$x \in X, \tilde{\mu}_A(0) \ge \tilde{\mu}_A(x) \ge \tilde{\alpha}$ and $\tilde{\lambda}_A(0) \le \tilde{\lambda}_A(x) \le \tilde{\beta}$. Therefore $0 \in X_A^{(\tilde{\alpha}, \tilde{\beta})}$.

Let $x, y \in X$ be such that $x * y$ and $y \in X_A^{(\tilde{\alpha}, \tilde{\beta})}$ then $\tilde{\mu}_A(x * y) \ge \tilde{\alpha}$, $\tilde{\lambda}_A(x * y) \le \tilde{\beta}$ and $\tilde{\mu}_A(y) \ge \tilde{\alpha}$, $\tilde{\lambda}_A(y) \le \tilde{\beta}$. It follows from (i-v IF2) and (i-v IF3) that

$$\tilde{\mu}_A(x) \ge \min\{\tilde{\mu}_A(x * y), \tilde{\mu}_A(y)\} \ge \min\{\tilde{\alpha}, \tilde{\alpha}\} = \tilde{\alpha} \text{ and}$$

$$\tilde{\lambda}_A(x) \le \max\{\tilde{\lambda}_A(x * y), \lambda_A(y)\} \le \max\{\tilde{\beta}, \tilde{\beta}\} = \tilde{\beta}$$

So that $x \in X_A^{(\tilde{\alpha}, \tilde{\beta})}$. Hence $X_A^{(\tilde{\alpha}, \tilde{\beta})}$ is an ideal of X.

Theorem 2.10 Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X such that $X_A^{(\tilde{\alpha}, \tilde{\beta})}$ is an ideal of X. If $(\tilde{\alpha}, \tilde{\beta}) \in \mathrm{Im}(\tilde{\mu}_A) \times \mathrm{Im}(\tilde{\lambda}_A)$ with $\tilde{\alpha} + \tilde{\beta} \le [1,1]$, then $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued IF ideal of X.

Proof: Let $A(x) = (\tilde{\alpha}, \tilde{\beta})$ for all $x \in X$, that is, $\tilde{\mu}_A(x) = \tilde{\alpha}$ and $\tilde{\lambda}_A(x) = \tilde{\beta}$ for all $x \in X$. Since $0 \in X_A^{(\tilde{\alpha}, \tilde{\beta})}$, we have $\tilde{\mu}_A(0) \ge \tilde{\alpha} = \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) \le \tilde{\beta} = \tilde{\lambda}_A(x)$ for all $x \in X$. Let $x, y \in X$ be

such that $A(x*y)=(\tilde{\alpha}_1,\tilde{\beta}_1)$ and $A(y)=(\tilde{\alpha}_2,\tilde{\beta}_2)$, that is, $\tilde{\mu}_A(x*y)=\tilde{\alpha}_1, \tilde{\lambda}_A(x*y)=\tilde{\beta}_1$ and $\tilde{\mu}_A(y)=\tilde{\alpha}_2$, $\tilde{\lambda}_A(y)=\tilde{\beta}_2$. Then $x*y\in X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$ and $y\in X_A^{(\tilde{\alpha}_2,\tilde{\beta}_2)}$. We may assume that $(\tilde{\alpha}_1,\tilde{\beta}_1)\le(\tilde{\alpha}_2,\tilde{\beta}_2)$, that is, $\tilde{\alpha}_1\le\tilde{\alpha}_2$ and $\tilde{\beta}_1\ge\tilde{\beta}_2$, with out loss of generality. It follows that $X_A^{(\tilde{\alpha}_2,\tilde{\beta}_2)}\subseteq X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$. So that $x*y\in X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$ and $y\in X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$. Since $X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$ is an ideal of X, we have $x\in X_A^{(\tilde{\alpha}_1,\tilde{\beta}_1)}$. Thus $\tilde{\mu}_A(x)\ge\tilde{\alpha}_1=\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}$
$\tilde{\lambda}_A(x)\le\tilde{\beta}_1=\max\{\tilde{\lambda}_A(x*y),\tilde{\lambda}_A(y)\}$, for all $x,y\in X$. Consequently $A=(X,\tilde{\mu}_A,\tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X.

Note that: $X_A^{(\tilde{\alpha},\tilde{\beta})}=\{x\in x/\tilde{\mu}_A(x)\ge\tilde{\alpha},\ \tilde{\lambda}_A(x)\le\tilde{\beta}\}=\{x\in X/\tilde{\mu}_A(x)\}$ and $\{x\in X/\tilde{\lambda}_A(x)\}$
$=U(\tilde{\mu}_A;\tilde{\alpha})\cap L(\tilde{\lambda}_A;\tilde{\beta})$.
Hence we have the following corollary.

Corollary 2.11 Let $A=(X,\tilde{\mu}_A,\tilde{\lambda}_A)$ be an i-v IFS in X. Then A is an i-v intuitionistic fuzzy ideal of X if and only if $U(\tilde{\mu}_A;\tilde{\alpha})$ and $L(\tilde{\lambda}_A;\tilde{\beta})$ are ideals of X, for every $\alpha\in[0,\tilde{\mu}_A(0)]$ and $\beta\in[\tilde{\lambda}_A(0),1]$ with $\tilde{\alpha}+\tilde{\beta}\le[1,1]$.

Theorem 2.12 Let $I\subseteq X$ and $A=(X,\tilde{\mu}_A,\tilde{\lambda}_A)$ be an i-v IFS in X defined by

$$\tilde{\mu}_A(x)=\begin{cases}\tilde{\alpha}_0 & \text{if } x\in I\\ \tilde{\alpha}_1 & \text{otherwise}\end{cases} \quad\text{and}\quad \tilde{\lambda}_A(x)=\begin{cases}\tilde{\beta}_0 & \text{if } x\in I\\ \tilde{\beta}_1 & \text{otherwise}\end{cases}$$

for all $x\in X$ where $0\le\tilde{\alpha}_0<\tilde{\alpha}_1$, $0\le\tilde{\beta}_0<\tilde{\beta}_1$ and $\tilde{\alpha}_i+\tilde{\beta}_i\le 1$ for $i=0,1$.
Then the following conditions are equivalent:
(1). A is an i-v intuitionistic fuzzy ideal of X.
(2). I is an ideal of X.

Proof: Assume (1), that is, A is an i-v intuitionistic fuzzy ideal of X.
Let $x,y\in I$. Now $\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)=\tilde{\alpha}_0$ and so $\tilde{\mu}_A(0)\ge\tilde{\alpha}_0$ implies $0\in I$.
Let $x,y\in X$ be such that $x*y$ and $y\in I$. We have $\tilde{\mu}_A(x)\ge\min\{\tilde{\mu}_A(x*y),\tilde{\mu}_A(y)\}=\min\{\tilde{\alpha}_0,\tilde{\alpha}_0\}=\tilde{\alpha}_0$ and so $x\in I$. Hence I is an ideal of X.
Assume (2), Let $x\in X$. If $x\in I$ implies $\tilde{\mu}_A(x)=\tilde{\alpha}_0$, since $0\in I$ we have $\tilde{\mu}_A(0)=\tilde{\alpha}_0$ and so $\tilde{\mu}_A(0)=\tilde{\mu}_A(x)$. Also $\tilde{\lambda}_A(x)=\tilde{\beta}_0$ and so $\tilde{\lambda}_A(0)=\tilde{\lambda}_A(x)$. If $x\notin I$ implies $\tilde{\mu}_A(x)=\tilde{\alpha}_1$ and $\tilde{\lambda}_A(x)=\tilde{\beta}_1$. Now $\tilde{\mu}_A(0)=\tilde{\alpha}_0>\tilde{\alpha}_1=\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0)=\tilde{\beta}_0<\tilde{\beta}_1=\tilde{\lambda}_A(x)$.
Therefore, in either cases $\tilde{\mu}_A(0)\ge\tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0)\le\tilde{\lambda}_A(x)$ for all $x\in X$.

Let $x, y \in X$ be such that $x * y$ and $y \in X$. If $x * y \in I$ and $y \in I$ since I is an ideal of X. We have that $x \in I$ and so $\tilde{\mu}_A(x) = \tilde{\alpha}_0 = \min\{\tilde{\alpha}_0, \tilde{\alpha}_0\} = \min\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$ and

$\tilde{\lambda}_A(x) = \tilde{\beta}_0 = \max\{\tilde{\beta}_0, \tilde{\beta}_0\} = \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$

If $x * y \in I$ and $y \notin I \Rightarrow x \notin I$ and so $\tilde{\mu}_A(x) = \tilde{\alpha}_1 = \min\{\tilde{\alpha}_0, \tilde{\alpha}_1\} = \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and

$\tilde{\lambda}_A(x) = \tilde{\beta}_1 = \max\{\tilde{\beta}_0, \tilde{\beta}_1\} = \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$

If $x * y \notin I$ and $y \in I \Rightarrow x \notin I$ and so $\tilde{\mu}_A(x) = \tilde{\alpha}_1 = \min\{\tilde{\alpha}_1, \tilde{\alpha}_0\} = \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$

$\tilde{\lambda}_A(x) = \tilde{\beta}_1 = \max\{\tilde{\beta}_1, \tilde{\beta}_0\} = \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$

If $x * y \notin I$ and $y \notin I \Rightarrow x \notin I$ and so $\tilde{\mu}_A(x) = \tilde{\alpha}_1 = \min\{\tilde{\alpha}_1, \tilde{\alpha}_1\} = \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$

$\tilde{\lambda}_A(x) = \tilde{\beta}_1 = \max\{\tilde{\beta}_1, \tilde{\beta}_1\} = \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$

Therefore $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$, for all $x, y \in X$

Hence $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X.

Corollary 2.13 Let $I \subseteq X$ and $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X defined by

$$\tilde{\mu}_A(x) = \begin{cases} \tilde{1}, & \text{if } x \in I \\ \tilde{0}, & \text{otherwise} \end{cases} \text{ and } \tilde{\lambda}_A(x) = \begin{cases} \tilde{1}, & \text{if } x \in I \\ \tilde{0}, & \text{otherwise} \end{cases}, \text{ for all } x \in X.$$

Then the following conditions are equivalent:

(1) A is an i-v intuitionistic fuzzy ideal of X.

(2) I is an ideal of X.

Proposition 2.14 Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v intuitionistic fuzzy ideal of X and $(\tilde{\alpha}_1, \tilde{\beta}_1), (\tilde{\alpha}_2, \tilde{\beta}_2) \in Im(\tilde{\mu}_A) \times Im(\tilde{\lambda}_A)$ with $\tilde{\alpha}_i + \tilde{\beta}_i \leq 1$ for $i = 1,2$. Then

$X_A^{(\tilde{\alpha}_1, \tilde{\beta}_1)} = X_A^{(\tilde{\alpha}_2, \tilde{\beta}_2)}$ if and only if $(\tilde{\alpha}_1, \tilde{\beta}_1) = (\tilde{\alpha}_2, \tilde{\beta}_2)$.

Proof: If $(\tilde{\alpha}_1, \tilde{\beta}_1) = (\tilde{\alpha}_2, \tilde{\beta}_2)$ then clearly $X_A^{(\tilde{\alpha}_1, \tilde{\beta}_1)} = X_A^{(\tilde{\alpha}_2, \tilde{\beta}_2)}$. Assume that $X_A^{(\tilde{\alpha}_1, \tilde{\beta}_1)} = X_A^{(\tilde{\alpha}_2, \tilde{\beta}_2)}$. Since $(\tilde{\alpha}_1, \tilde{\beta}_1) \in Im(\tilde{\mu}_A) \times Im(\tilde{\lambda}_A)$ then there exist $x \in X$ such that $\tilde{\mu}_A(x) = \tilde{\alpha}_1, \tilde{\lambda}_A(x) = \tilde{\beta}_1$. It follows that $x \in X_A^{(\tilde{\alpha}_1, \tilde{\beta}_1)} = X_A^{(\tilde{\alpha}_2, \tilde{\beta}_2)}$, so that $\tilde{\alpha}_1 = \tilde{\mu}_A(x) \geq \tilde{\alpha}_2$ and $\tilde{\beta}_1 = \tilde{\lambda}_A(x) \leq \tilde{\beta}_2$. Similarly we have $\tilde{\alpha}_1 \leq \tilde{\alpha}_2$ and $\tilde{\beta}_1 \geq \tilde{\beta}_2$. Hence $(\tilde{\alpha}_1, \tilde{\beta}_1) = (\tilde{\alpha}_2, \tilde{\beta}_2)$.

Theorem 2.15 Let $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ be an i-v IFS in X and $Im(A) = \{(\tilde{\alpha}_0, \tilde{\beta}_0), (\tilde{\alpha}_1, \tilde{\beta}_1), \ldots, (\tilde{\alpha}_k, \tilde{\beta}_k)\}$ where $(\tilde{\alpha}_i, \tilde{\beta}_i) < (\tilde{\alpha}_j, \tilde{\beta}_j)$ whenever $i > j$. Let $\{G_r / r = 0,1,2,\ldots,k\}$ be family of i-v ideals of X such that $G_0 \subset G_1 \subset \ldots \subset G_k = X$ and $A(G_r^*) = (\tilde{\alpha}_r, \tilde{\beta}_r)$, that is, $\tilde{\mu}_A(G_r^*) = \tilde{\alpha}_r$ and $\tilde{\lambda}_A(G_r^*) = \tilde{\beta}_r$, where $G_r^* = G_r \setminus G_{r-1}$ and $G_{-1} = \varphi$ for $r = 0,1,2,3,\ldots..k$. Then $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an i-v intuitionistic fuzzy ideal of X.

Proof: Since $0 \in G_0$, we have $\tilde{\mu}_A(0) = \tilde{\alpha}_0 \geq \tilde{\mu}_A(x)$ and $\tilde{\lambda}_A(0) = \tilde{\beta}_0 \leq \tilde{\lambda}_A(x)$ for all $x \in X$. Let $x, y \in X$. To prove that $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ satisfies the conditions (i - v IF 2) and (i - v IF 3). We discuss the following cases:

If $x * y \in G_r^*$ and $y \in G_r^* = G_r \setminus G_{r-1}$ then $x \in G_r$, because G_r is an ideal of X.

Thus $\tilde{\mu}_A(x) \geq \tilde{\alpha}_r = \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\tilde{\lambda}_A(x) \leq \tilde{\beta}_r = \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$.

If $x * y \notin G_r^*$ and $y \notin G_r^*$, then the following four cases will be arise:

1. $x * y \in X \setminus G_r$ and $y \in X \setminus G_r$, 2. $x * y \in G_{r-1}$ and $y \in G_{r-1}$,
3. $x * y \in X \setminus G_r$ and $y \in G_{r-1}$, 4. $x * y \in G_{r-1}$ and $y \in X \setminus G_r$.

But, in either case, we know that

$$\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\} \text{ and } \tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$$

If $x * y \in G_r^*$ and $y \notin G_r^*$ that either $y \in G_{r-1}$ or $y \in X \setminus G_r$. It follows that either $x \in G_r$ (or) $x \in X \setminus G_r$. Thus $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$

If $x * y \notin G_r^*$ and $y \in G_r^*$, then by similar processes, we have $\tilde{\mu}_A(x) \geq \min\{\tilde{\mu}_A(x*y), \tilde{\mu}_A(y)\}$ and $\tilde{\lambda}_A(x) \leq \max\{\tilde{\lambda}_A(x*y), \tilde{\lambda}_A(y)\}$ for all $x, y \in X$. Thus $A = (X, \tilde{\mu}_A, \tilde{\lambda}_A)$ is an interval-valued intuitionistic fuzzy ideal of X.

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