Common Fixed Point Theorem for Occasionally Weakly Compatible Mapping in Q-Fuzzy Metric Spaces

Kamal Wadhwa[1] , Farhan Beg[2]
1. Govt. Narmada Mahavidhyalaya, Hoshangabad (M.P.)
2. Truba College Of Science & Technology Bhopal,(M.P.) India
E-mail: beg_farhan26@yahoo.com (corresponding author)

Abstract
This Paper present some common fixed point theorem for Occasionally Weakly Compatible mapping in Q-fuzzy metric spaces under various conditions.
Keywords: Fixed point , Occasionally Weakly Compatible mapping, Q-fuzzy metric spaces , t-norm

1. Introduction:

The concept of fuzzy sets introduced by Zadeh [12] in 1965 plays an important role in topology and analysis. Since then, there are many author to study the fuzzy set with application. Especially Kromosil and Michalek [10] put forward a new concept of fuzzy metric spaces. George and Veermani [6] revised the notion of fuzzy metric spaces with the help of continuous t-norm. As a result of many fixed point theorem for various forms of mapping are obtained in fuzzy metric spaces. Dhage [5] introduced the definition of D-metric spaces and proved many new fixed point theorem in D-metric spaces. Recently, Mustafa and Sims[13] presented a new definition of G-metric space and made great contribution to the development of Dhage theory.
On the other hand ,Lopez-Rodrigues and Romaguera [11] introduced the concept of Hausdorff fuzzy metric in a more general space .
The Q-fuzzy metrics spaces is introduced by Guangpeng Sun and kai Yang[7] which can be cosider as a Generalization of fuzzy metric spaces. Sessa [18] improved commutativity condition in fixed point theorem by introducing the notion of weakly commuting maps in metric space. R.Vasuki[14] proved fixed point theorems for R-weakly commuting mapping Pant [14,15,16] introduced the new concept of reciprocally continuous mappings and established some common fixed point theorems. The concept of compatible maps by [10] and weakly compatible maps by [8] in fuzzy metric space is generalized by A.Al Thagafi and Naseer Shahzad [1] by introducing the concept of occasionally weakly compatible mappings. Recent results on fixed point in Q-fuzzy metric space can be viewed in[7]. In this paper we prove some fixed point theorems for four occasionally weakly compatible *owc* mappings which improve the result of Ganpeng Sun and Kai Yang [7] in Q-fuzzy metric spaces.

2. Preliminary Notes:

Definition:2.1[2] A binary operation
$*:[0,1]\times[0,1]\to[0,1]$ is a continuous t-norm if it satisfy the following condition:
(i) $*$ is associative and commutative .
(ii) $*$ is continous function.
(iii) $a*1=a$ for all $a\in[0,1]$
(iv) $a*b\le c*d$ whenever $a\le c$ and $b\le d$ and $a,b,c,d \in [0,1]$
Definition 2.2[7] : A 3-tuple $(X,Q,*)$ is
called a Q-fuzzy metric space if X is an arbitrary (non-empty) set ,* is a continuous t -norm, and Q is a fuzzy set on
$X^3\times(0,\infty)$,satisfying the following conditions for each $x,y,z,a \in X$ and $t,s>0$:
(i) $Q(x,x,y,t)>0$ and $Q(x,x,y,t)\le Q(x,y,z,t)$ for all $x,y,z \in X$ with $z\neq y$
(ii) $Q(x,y,z,t)=1$ if and only if $x=y=z$
(iii) $Q(x,y,z,t) = Q(p(x,y,z),t)$,(symmetry) where p is a permutation function ,
(iv) $Q(x,a,a,t) * Q(a,y,z,s)\le Q(x,y,z,t+s)$,
(v) $Q(x,y,z,.):(0,\infty)\to[0,1]$ is continuous
A Q-fuzzy metric space is said to be symmetric if $Q(x,y,y,t)= Q(x,x,y,t)$ for all $x,y \in X$.
Example : Let X is a non empty set and G is the G-metric on X. Denote $a*b = a.b$ for all $a, b \in[0,1]$.For each $t > 0$:

$$Q(x,y,z,t) =\frac{t}{t+G(x,y,z)}$$

Then $(X,Q,*)$ is a Q-fuzzy metric .
Definition 2.3[6] Let $(X,Q,*)$ be a Q-fuzzy metric space.For $t>0$, the open ball $B_Q(x,r,t)$ with center $x\in X$ and radius $0<r<1$ is defined by $B_Q(x,r,t)=\{y\in X:Q(x,y,y,t)>1-r\}$

A subset A of X is called open set if for each x∈A there exist t>0 and 0<r<1 such that $B_Q(x,r,t) \subseteq A$.
A sequence { x_n} in X converges to x if and only if Q (x_m,x_n,x ,t) →1 as n→∞,for each t>0. It is called a Cauchy sequence if for each 0 <ε<1 and t>0,there existn_0∈N such that Q(x_m,x_n,x_l)>1- ε for each l, n ,m ≥ n_0.The Q-fuzzy metric space is called to be complete if every Cauchy sequence is convergent. Following similar argument in G-metric space, the sequence {x_n} in X also converges to x if and only if Q(x_n,x_n,x,t) →1 as n→∞, for each t >0 and it is a Cauchy sequence if for each 0 <g<1 and t>0, there exist n_0∈N suc h that Q(x_m,x_n,x_n) > 1-ε for each n,m ≥ n_0.

Lemma2.4[7] : If (X,Q, ∗) be a Q-fuzzy metric space,then Q(x,y,z,t) is non-decreasing with respect to t for all x,y,z in X .

Proof: Proof is this is implicated in [7]

Lemma2.5[7] : Let (X,Q, ∗) be a Q-fuzzy metric space.(a) If there exists a positive number k<1 such that : Q(y_{n+2},y_{n+1},y_{n+1},kt)≥Q(y_{n+1},y_n,y_n, t),t >0, n ∈N then{y_n} is a Cauchy sequence in X.
(b) if there exists k ∈ (0,1) such that Q(x,y,y,kt) ≥ Q(x,y,y,t) for all x, y ∈X and t > 0 then x= y .
Proof: By the assume $\lim_{n\to\infty} Q(x,y,z,t)$ =1 and the property of non-decreasing, it is easy to get the results .

Definition 2.6[3]: Let X be a set, f and g Self maps of X. A point x ∈ X is called a coincidence point of f and g iff fx=gx .We shall call w=fx=gx a point of coincidence of f and g.

Definition 2.7[7]: Let f and g be self maps on a Q-fuzzy metric space (X ,Q,*) .Then the mappings are said to be weakly compatible if they commute at their coincidence point, that is, f x =gx implies that fgx = gfx .

Definition 2.8 [7]: Let f and g be self maps on a Q-fuzzy metric space (X,Q, *) . The pair (f ,g) is said to be compatible if
$\lim_{n\to\infty} Q(fgx_n, gfx_n, gfx_n, t)$=1 whenever {$x_n$} is a sequence in X such that
$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n$=z for some z∈ X

Definition 2.9[3]: Two self maps f and g of a set X are occasionally weakly compatible (owc) iff there is a point x in X which is coincidence point of f and g at which f and g commute.

Lemma 2.10 [9]: Let X be a set, f , g owc self maps of X. If f and g have unique point of coincidence, w = f x = g x , then w is the unique common fixed point of f and g .

3. Main Result

Theorem 3.1 : Let (X,Q,∗) be complete symmetric Q-fuzzy metric space and f ,g ,S and T be a self mapping of X. Let the pair {f, T} and {g, S} be owc. If there exist k ∈ (0,1) such that
Q(fx,gy,gy,kt) ≥ Q(Tx,Sy,Sy,t)∗Q(Tx,gy,gy,t) ∗Q(fx.Sy,Sy,t) ∗Q(fx,Tx,Tx,t)…(1)
For all x ,y ∈ X and for all $t > 0$,then there exist a unique point w ∈ X such that f w =Tw=w and a unique point z ∈ X such that g z = S z =z, Moreover, z = w so that there is a unique common fixed point of f , g ,S and T.

Proof:Let the pair {f,T} and {g ,S} be owc, so there are point x ,y ∈ X such that fx =Tx and gy = Sy. We claim that f x= gy. If not by inequality (1)

Q(fx,gy,gy,kt)≥Q(fx,gy,gy,t)∗Q(fx,gy,gy, t)∗Q(fx,gy,gy,t)∗Q(fx,fx,fx,t)
≥Q(fx, gy, gy ,t)∗1
≥Q(fx, gy, gy ,t)

Therefore f x = g y i.e. fx=Tx=gy=Sy.
Suppose that there is another point z such that f z=T z then by (1) we have f z = Tz =g y = S y , So f x = f z and w = f x = T x is the unique point of coincidence of f and g by Lemma 2.10 w is the only common fixed point of f and g. Similarly there is a unique point z∈ X. such that z= gz =Sz.
Assume that w≠ z . We have

Q(w,z,z,kt) = Q(fw,gz,gz,kt)
≥Q(Tw,Sz,Sz,t)∗Q(Tw,gz,gz,t) ∗Q(fw.Sz,Sz,t) ∗ Q(fw,Tw,Tw,t)
≥Q(w,z ,z, t)∗Q(w ,z ,z ,t) ∗Q(w, z, z, t) ∗Q(w, w, w, t)
≥Q(w,z ,z, t)∗1
≥Q(w,z ,z, t)

Therefore we have z = w by Lemma 2.10 and z is unique common fixed point of f ,g, S and T. The uniqueness of the fixed point holds from (1).

Theorem 3.2 : Let (X,Q,∗) be complete symmetric Q-fuzzy metric space and f ,g ,S and T be a self mapping of X. Let the pair {f, T} and {g ,S} be owc. If there exist k ∈ (0,1) such that

$$Q(fx,gy,gy,kt) \geq \emptyset \left[min \begin{Bmatrix} Q(Tx,Sy,Sy,t), Q(Tx,gy,gy,t) \\ , Q(fx.Sy,Sy,t), Q(fx,Tx,Tx,t) \end{Bmatrix} \right] \quad ....(2)$$

for all x,y ∈ X and $\emptyset: [0,1] \to [0,1]$ such that $\emptyset(t) > t$ for all 0< t< 1,then there exist a unique common fixed point of f ,g ,S and T.

Proof: The proof follows from Theorem 3.1 .

Theorem 3.3 Let (X,Q,∗) be complete symmetric Q-fuzzy metric space and f ,g ,S and T be a self mapping of X. Let the pair {f, T} and {g, S} be owc. If there exist $k \in (0,1)$ such that

$$Q(fx, gy, gy, kt) \geq \emptyset \begin{Bmatrix} Q(Tx, Sy, Sy, t), Q(Tx, gy, gy, t), \\ Q(fx. Sy, Sy, t), Q(fx, Tx, Tx, t) \end{Bmatrix} \ldots..(3)$$

For all $x,y \in X$, t>0 and $\emptyset[0,1]^4 \rightarrow [0,1]$ such that $\emptyset(t,t,t,1) > t$ for all 0< t< 1 then there exist a unique common fixed point of f ,g ,S and T.

Proof: Let the pair {f,T} and {g ,S} be owc, so there are point $x, y \in X$ such that f x =Tx and gy = Sy. We claim that f x= gy .If not by inequality (3)

$$\begin{aligned} Q(fx, gy, gy, kt) &\geq \emptyset\{Q(fx, gy, gy, t), Q(fx, gy, gy, t), Q(fx, gy, gy, t), Q(fx, fx, fx, t)\} \\ &\geq \emptyset \ \{Q(fx,gy,gy,t), Q(fx,gy,gy,t), Q(fx,gy,gy,t),1\} \\ &> Q(fx,gy,gy,t) \end{aligned}$$

Therefore f x = g y i.e. fx=Tx=gy=Sy.

Suppose that there is another point z such that f z=Tz then by (3) we have f z =T z =g y = S y , So f x = f z and w = f x = T x is the unique point of coincidence of f and g by Lemma 2.10 w is the only common fixed point of f and g. Similarly there is a unique point $z \in X$. such that z=gz=Sz.

$$Q(fx, gy, gy, kt) \geq \emptyset \begin{Bmatrix} Q(Tx, Sy, Sy, t), Q(Tx, gy, gy, t), \\ Q(fx. Sy, Sy, t), Q(fx, Tx, Tx, t) \end{Bmatrix}$$

Assume that $w \neq z$. We have

$$\begin{aligned} Q(w,z,z,t) &= Q(fw,gz,gz,kt) \\ &\geq \emptyset\{Q(Tw, Sz, Sz, t), Q(Tw, gz, gz, t), Q(fw. Sz, Sz, t), Q(fw, Tw, Tw, t)\} \\ &\geq \emptyset\{Q(w, z, z, t), Q(w, z, z, t), Q(w, z, z, t), Q(w, w, w, t)\} \\ &\geq \emptyset\{Q(w, z, z, t), Q(w, z, z, t), Q(w, z, z, t), 1\} \\ &> Q(w,z,z,t) \end{aligned}$$

Therefore we have z = w by Lemma 2.10 and z is unique common fixed point of f ,g, S and T. The uniqueness of the fixed point holds from (3).

References

[1] A.Al -Thagafi and Naseer Shahzad, "Generalized I-NonexpansiveSelfmaps and Invariant Approximation", Acta Mathematica Sinica, English Series May, 2008, Vol.24, No.5, pp.867-876.

[2] B.Schweizer ans A.Sklar, "*Statistical metric spaces*", Pacific J. Math. 10(1960), 313-334

[3] C.T. Aage, J.N.Salunke,"*On fixed point theorem in Fuzzy Metric Spaces*" Int. J.Open Problem Compt. Math., Vol. 3, No. 2, June 2010 ,pp 123-131.

[4] C.T. Aage, J.N.Salunke, "*On fixed point theorem in Fuzzy Metric Spaces Using A Control Function*"Submitted.

[5] Dhage, B.C., Generalised metric spaces and mappings with fixed point. Bull. Calcutta M ath. Soc.,84(4) 1992.:329-336

[6] George, A. and P. Veermani, On some results in fuzzy metric spaces. Fuzzy Sets Sys., 64,1994. 395-399.

[7] Guangpeng Sun and Kai Yang,*"Generalized Fuzzy Metric Spaces with Properties***"** Research Journal of Applied Sciences, Engineering and Technology 2(7) 2010,: 673-678,

[8] G. Jungck and B.E. Rhoades, " *Fixed Point for Occasionally Weakly Compatible Mappings*" , Erratum, Fixed Point Theory, Volume 9, No. 1,2008,383-384.

[9] G. Jungck and B. E. Rhoades, "*Fixed Point Theorems for Occasionally Weakly compatible Mappings*", Fixed Point Theory, Volume 7, No. 2, 2006, 287-296.

[10] Kramosil. J and J. Michalek, Fuzzy metric and statistical metric spaces. Kybernetica, 11,1975. : 326-334

[11] Lopez-Rodrigues,J.andS.Romaguera, The Hausdorff fuzzy metric on compact sets. Fuzzy Sets Sys., 147, 2004.: 273-283

[12] L.A. Zadeh," *Fuzzy sets"*, Inform and Control 8 (1965), 338-353.

[13] Mustafa, Z. and B. Sims,. A new approach to generalized metric spaces. J. Nonlinear Convex Anal., 7, 2006: 289-297.

[14] R .Vasuki, "*Common fixed points for R-weakly commuting maps in fuzzy metric spaces*", Indian J. Pure Appl. Math. 30(1999), 419-423.

[15] R.P. Pant ,"*Common fixed point of Four Mappings",* Bull. Cal. Math. Soc.90 (1998), 281-286.

[16] R.P. Pant ,"*Common fixed point Theorems for contractive Mappings",* J. Math. Anal. Appl. 226 (1998), 251-258.

[17] R.P. Pant ,"A remark on *Common fixed point of Four Mappings in a fuzzy metric space",J*. Fuzzy. Math. 12(2) (2004), 433-437.

[18] S.Sessa, *"on a weak commutative condition in fixed point consideration*", Publ.Inst. Math(Beograd), 23(46)(1982), 146-153.