Complete L-Fuzzy Metric Spaces and Common Fixed Point Theorems

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Abstract
In this paper we work out generalized complete L-Fuzzy Metric Space and Common Fixed Point Theorems which is a generalization of results in Aibi et al [1].
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1. Introduction and Preliminaries:

The concept of fuzzy set was introduced initially by Zadesh [**23**] in 1965.Which are a generalization of fuzzy metric and intuitionist fuzzy metric space. Various concepts of fuzzy metric space were considered in [7, 8, 13, 14].In this sequel we shall adopt the usual terminology.

Definition 1.1: [11] *Let* $L=(L,\le_L)$ be a complete lattice, and U a non-empty set called a universe. An L-fuzzy set A on U is defined a mapping $A:U\to L$. For each u in U, $A(u)$ represents the degree $(in\ L)$ to which u satisfies A.

Lemma 1.1: [5, 6] consider the set L^* and the operation $\le_{L^*}$ defined by:

$$L^*=\{(x_1,x_2):(x_1,x_2)\in[0,1]^2 \ and\ x_1+x_2\le 1\},$$

$(x_1,x_2)\le_{L^*}(y_1,y_2)\Leftrightarrow x_1\le y_1\ and\ x_2 y_2$, for every $(x_1,x_2),(y_1,y_2)\in L^*$. Then $(L^*,\le_{L^*})$ is a complete lattice and convention of L- fuzzy metric spaces introduced by saadatiel. **[19]**

Classically, a triangular norm T on $([0,1],\le)$ is defined as an increasing, commutative, associative mapping $T:[0,1]^2\to[0,1]$ satisfying $T(1,x)=x$, for all $x\in[0,1]$.

These definitions can be straightforwardly extended to any lattice $L=(L,\le_L)$. Define first $0_L=\inf L$ and $1_L=\sup L$.

Definition 1.2: A triangular norm $(t-norm)$ on L is a mapping $T:L^2\to L$ satisfying the following conditions:

1. $(\forall x\in L)(T(x,1_L)=x)$; (Boundary condition)
2. $(\forall (x,y)\in L^2)(T(x,y)=T(y,x))$; (Commutativity)
3. $(\forall (x,y,z)\in L^3)(T(x,T(y,z))=T(T(x,y),z))$; (Associativity)
4. $(\forall (x,x'y,y')\in L^4)(x\le_L x'\ and\ y\le_L y'\Rightarrow T(x,y)\le_L T(x',y'))$. (Monotonicity)

A t–norm T on L is said to be continuous if for any $x,y\in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ which converge x to and y we have

$$\lim_n T(x_n,y_n)=T(x,y)$$

For example, $T(x,y)=\min(x,y)$ and $T(x,y)=xy$ are two continuous t-norms on $[0,1]$. A t–norm can also be defined recursively as an $(n+1)$–ary operation $(n\in N)$ by $T^1=T$ and

$$T^n(x_1,\dots,x_{n+1})=T(T^{n-1}(x_1,\dots,x_n),x_{n+1})$$

for $n \geq 2$ and $x_i \in L$.

Definition 1.3: A negation on L is any decreasing mapping $N : L \to L$ satisfying $N(0_L) = 1_L$ and $N(1_L) = 0_L$. It $N(N(x)) = x \forall x \in L$ Then N is called an involutive negation.

Definition 1.4: The 3-tuple $(X, \mathcal{M}, \mathcal{T})$ is said to be an L –fuzzy metric space if X is an arbitrary (non-empty) set, T is a continuous t-norm on L and M is an L -fuzzy set on $X^2 \times]0,+\infty[$ satisfying the following conditions for every x, y, z in X and $t, s \; in]0,+\infty[$:.

a) $M\ (x,y,t) >_L 0_L$;

b) $M\ (x,y,t) = 1_L$ for all $t > 0$ if and only if $x = y$;

c) $M\ (x,y,t) = M\ (x,y,t)$;

d) $T\ (M\ (x,y,t), M\ (y,z,s)) \leq_L M\ (x,z,t+s)$;

e) $M\ (x,y,.) :]0,\infty[\to L$ is continuous and $\lim_{t\to\infty} M\ (x,y,t) = 1_L$

Let $(X, \mathcal{M}, \mathcal{T})$ be an L –fuzzy metric space. For $t \in]0,+\infty[$, , we define the open ball $B(x,r,t) \subseteq A$ with center $x \in X$ and a fixed radius $r \in L \setminus \{0_L, 1_L\}$ as

$$B(x,r,t) = \{y \in X : M\ (x,y,t) >_L N\ (r)\}$$

A subset $A \subseteq X$ is called open if for each $x \in A$, there exist $t > 0$ and $r \in L \setminus \{0_L, 1_L\}$ such that $B(x,r,t) \subseteq A$. Let $\mathcal{T}_M$ denote the family of all open subsets of X. Then $\mathcal{T}_M$ is called the topology induced by the L –fuzzy metric $\mathcal{M}$.

Example 1.1: [21] Let (X,d) be a metric space. Denote $T\ (a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times]0,+\infty[$ be defined as follows:

$$M_{M,N}(x,y,t) = (M(x,y,t)) = \left(\frac{t}{t+d(x,y)} \; \frac{d(x,y)}{t+d(x,y)} \right)$$

Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.2: [1] Let (X,d) be a metric space. Denote $T\ (a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:

$$M_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \left(\frac{ht^n}{ht^n + md(x,y)}, \frac{d(x,y)}{ht^n + md(x,y)} \right)$$

for all $t, h, m, n \in R^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Lemma 1.2: [10] Let $(X, \mathcal{M}, \mathcal{T})$ be an L -fuzzy metric space. Then, $\mathcal{M}\ (x,y,t)$ is nondecreasing with respect to t, for all x, y in X.

Definition 1.5: A sequence $\{x_n\}_{n \in \mathbb{N}}$ in an L -fuzzy metric space $(X, \mathcal{M}, \mathcal{T})$ is called a Cauchy sequence, if for each $\varepsilon \in L \setminus \{0_L\}$ and $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $m \geq n_0$ $(n \geq m \geq n_0)$,

$$M\ (x_m, x_n, t) >_L N\ (\varepsilon)$$

The sequence $\{x_n\}_{n\in N}$ is said to be convergent to $x \in X$ in the L -fuzzy metric space (X,M,T) (denoted by $x_n \xrightarrow{M} x$) if $M(x_n,x,t) = M(x,x_n,t) \to 1_L$ whenever $n \to +\infty$ for every $t > 0$. A L -fuzzy metric space is said to be complete if and only if every Cauchy sequence is convergent.

Henceforth, we assume that T is a continuous t-norm on the lattice L such that for every $\mu \in L\setminus\{0_L, 1_L\}$, there is a $\lambda \in L\setminus\{0_L, 1_L\}$ such that

$$T^{n-1}(N(\lambda),...,N(\lambda)) >_L N(\mu)$$

For more information see [19].

Definition 1.6: Let (X,M,T) be an L –fuzzy metric space. M is said to be continuous on $X \times X \times]0,\infty[$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x,y,t)$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times]0,\infty[$ converges to a point $(x,y,t) \in X \times X \times]0,\infty[$ i.e., $\lim_n M(x_n,x,t) = \lim_n M(y_n,y,t) = 1_L$ and $\lim_n M(x,y,t_n) = M(x,y,t)$.

Lemma 1.3: Let (X,M,T) be an L -fuzzy metric space. Then, M is a continuous function on $X \times X \times]0,\infty[$.

Proof: The proof is the same as that for fuzzy spaces (see Proposition 1 of [15]).

Definition 1.7: Let A and S be mappings from an L –fuzzy metric space into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8: Let A and S be mappings from an L –fuzzy metric space into itself. Then the mappings are said to be weak compatible if

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = 1_L \ \forall t > 0$$

Whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = x \in X$$

Proposition 1.1: [22] If self-mappings A and S of an L –fuzzy metric space (X,M,T) are compatible, then they are weak compatible.

Lemma 1.4: [1,19] Let (X,M,T) be an L –fuzzy metric space. Define $E_{\lambda,M}$: $X^2 \to R + \cup\{0\}$ by

$$E_{\lambda,M}(x,y) = \inf\{t > 0 : (x,y,t) >_L N(\lambda)\}$$

For each $\lambda \in L\setminus\{0_L : 1_L\}$ and $x, y \in X$. Then we have

i) For any $\mu \in L\setminus\{0_L : 1_L\}$ there exists $\lambda \in L\setminus\{0_L : 1_L\}$ such that

$$E_{\mu,M}(x_1,x_n) \le E_{\lambda,M}(x_1,x_n) + E_{\lambda,M}(x_2,x_3) + ... + E_{\lambda,M}(x_{n-1},x_n)$$

for any $x_1,...,x_n \in X$;

ii) The sequence $\{x_n\}_{n\in N}$ is convergent to x w.r.t. L -fuzzy metric M if and only if $E_{\lambda,M}(x_n,x) \to 0$.

Also the sequence $\{x_n\}_{n\in N}$ is Cauchy w.r.t. L –fuzzy metric M if an only if it is Cauchy with $E_{\lambda,M}$.

Lemma 1.5: Let (X,M,T) be an L –fuzzy metric space. If

$$M(x_n, x_{n+1}, t) \ge_L M(x_0, x_1, k^n t)$$

for some $k>1$ and $n\in\mathrm{N}$. Then $\{x_n\}$ is a Cauchy sequence.

Definition 1.9: [9] We say that the L –fuzzy metric space $(X,\mathrm{M},\mathrm{T})$ has property (C), if it satisfies the following condition:

$$\mathrm{M}(x,y,t)=C, \text{ for } t>0 \text{ implies } C=1_{\mathrm{L}}$$

2. Main Results: Theorem 2.2:

Let A,B,S and T be self-mappings of a complete L-fuzzy metric space (X,M,T)which has property © satisfying

i) $A(X)\subseteq T(X), B(X)\subseteq S(X)$ and $T(X)$ or $S(X)$ is a closed subset of X.

ii) The pair (A,S) and (B,T) are weakly compatible and (A,S) or (B,T) satisfy the property$_{(c)}$.

iii)

$$M(Ax,By,Bz,t)$$

$$\geq \phi_L\begin{pmatrix} \mathrm{M}(Sx,Ty,Tz,kt),\mathrm{M}(Sx,By,Tz,kt),\mathrm{M}(Sx,Ty,Bz,kt),\mathrm{M}(Sx,By,By,kt) \\ \mathrm{M}(Ty,By,Bz,kt),\mathrm{M}(Ty,Ty,Bzkt),\mathrm{M}(Ty,By,By,kt),\mathrm{M}(Ty,Bz,Bz,kt) \\ \mathrm{M}(By,Ty,Tz,kt),\mathrm{M}(By,By,Tz,kt),\mathrm{M}(By,Tz,Tz,kt),\mathrm{M}(Tz,Bz,Bz,kt) \end{pmatrix}$$

The A,B,S and T have a unique common fixed point in X.

Proof: Let the pair (B,T) satisfy in property(E), hence there exist a sequence $\{x_n\}$ such that,

$$\lim_{n\to\infty}\mathrm{M}(Bx_n,u,u,t)=\lim_{n\to\infty}\mathrm{M}(Tx_n,u,u,t)=1$$

For some $u\in X$ and every $t>0$. there exist a sequence $\{y_n\}$ such that, $Bx_n=Sy_n$ hence $\lim_{n\to\infty}\mathrm{M}(Sy_n,u,u,t)=1$

We prove that $\lim_{n\to\infty}\mathrm{M}(Ay_n,u,u,t)=1$. Since

$$\mathrm{M}(Ay_n,Bx_{n+1},t)$$

$$\geq \phi_L\begin{pmatrix} \mathrm{M}(Sy_n,Tx_n,Tx_{n+1},kt),\mathrm{M}(Sy_n,Bx_n,Tx_{n+1z},kt),\mathrm{M}(Sy_n,Tx_n,Bx_{n+1},kt) \\ \mathrm{M}(Sy_n,Bx_n,Bx_n,kt),\mathrm{M}(Tx_n,Bx_{n+1},kt),\mathrm{M}(Tx_n,Tx_n,Bx_{n+1},kt) \\ \mathrm{M}(Tx_n,Bx_n,Bx_n,kt),\mathrm{M}(Tx_n,Bx_{n+1},Bx_{n+1},kt),\mathrm{M}(Bx_n,Tx_n,Tx_{n+1},kt) \\ \mathrm{M}(Bx_n,Bx_n,Tx_{n+1}kt),\mathrm{M}(Bx_n,Tx_{n+1},Tx_{n+1},kt)\mathrm{M}(Tx_{n+1z},Bx_{n+1},Bx_{n+1},kt) \end{pmatrix}$$

On making $n\to\infty$ the above inequality, we get

$$\lim_{n\to\infty}\mathrm{M}(Ay_n,Bx_n,Bx_{n+1},t)=1$$

$$\geq \phi_L(\mathrm{M}(u,u,u,kt)\mathrm{M}(u,u,u,kt),\ldots,\mathrm{M}(u,u,u,kt))=1$$

Therefore , $\lim_{n\to\infty}\mathrm{M}(Ay_n,u,u,t)=1$, hence

$$\lim_{n\to\infty}Ay_n=\lim_{n\to\infty}Sy_n=\lim_{n\to\infty}Bx_n=\lim_{n\to\infty}Tx_n=u$$

Let $S(x)$ be complete M –fuzzy metric space, then there exist $x\in X$ such that $Sx=u$. If $Ax\neq u$, then we have

$$\mathrm{M}\left(Ax,Bx_n,Bx_{n+1},t\right)$$
$$\geq \phi_L\begin{pmatrix} \mathrm{M}\left(Sx,Tx_n,Tx_{n+1},kt\right),\mathrm{M}\left(Sx,Bx_n,Tx_{n+1z},kt\right),\mathrm{M}\left(Sx,Tx_n,Bx_{n+1},kt\right) \\ \mathrm{M}\left(Sx,Bx_n,Bx_n,kt\right),\mathrm{M}\left(Tx_n,Bx_n,Bx_{n+1},kt\right),\mathrm{M}\left(Tx_n,Tx_n,Bx_{n+1},kt\right) \\ \mathrm{M}\left(Tx_n,Bx_n,Bx_n,kt\right),\mathrm{M}\left(Tx_n,Bx_{n+1},Bx_{n+1},kt\right),\mathrm{M}\left(Bx_n,Tx_n,Tx_{n+1},kt\right) \\ \mathrm{M}\left(Bx_n,Bx_n,Tx_{n+1},kt\right),\mathrm{M}\left(Bx_n,Tx_{n+1},Tx_{n+1},kt\right),\mathrm{M}\left(Tx_{n+1z},Bx_{n+1},Bx_{n+1},kt\right) \end{pmatrix}.$$

On making $n \to \infty$ we get $\mathrm{M}\left(Ax,u,u,t\right)=1$, hence $Ax=u=Sx$. By $\left(A,S\right)$ be weakly compatible, we have $ASx = SAx$, so

$$AAx = ASx = SAx = SSX$$

as $AX \subset TX$, there exist $\upsilon \in X$ such that $Ax = T\upsilon$. We prove that $T\upsilon = B\upsilon$. If $T\upsilon \neq B\upsilon$ then

$$\mathrm{M}\left(Ax,B\upsilon,B\upsilon,t\right)$$
$$\geq_L \phi\begin{pmatrix} \mathrm{M}\left(Sx,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(Sx,B\upsilon,T\upsilon,kt\right),\mathrm{M}\left(Sx,T\upsilon,B\upsilon,kt\right),\mathrm{M}\left(Sx,B\upsilon,B\upsilon,kt\right) \\ \mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,T\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right) \\ \mathrm{M}\left(B\upsilon,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(B\upsilon,B\upsilon,T\upsilon,kt\right),\mathrm{M}\left(B\upsilon,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right) \end{pmatrix}$$ If

$B\upsilon \neq u$ then

$$\mathrm{M}\left(Ax,B\upsilon,B\upsilon,t\right) > \mathrm{M}\left(Ax,B\upsilon,B\upsilon,t\right)$$

Is a contradiction. Thus $T\upsilon = B\upsilon = u$. By B and T be weakly compatible, we get $TT\upsilon = TB\upsilon = BT\upsilon = BB\upsilon$, so $Tu = Bu$. We prove $Au = u$, for

$$\mathrm{M}\left(Au,u,u,t\right) = \mathrm{M}\left(Au,B\upsilon,B\upsilon,t\right)$$
$$\geq_L \phi\begin{pmatrix} \mathrm{M}\left(Su,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(Su,B\upsilon,T\upsilon,kt\right),\mathrm{M}\left(Su,T\upsilon,B\upsilon,kt\right),\mathrm{M}\left(Su,B\upsilon,B\upsilon,kt\right) \\ \mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,T\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right) \\ \mathrm{M}\left(B\upsilon,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(B\upsilon,B\upsilon,T\upsilon,kt\right),\mathrm{M}\left(B\upsilon,T\upsilon,T\upsilon,kt\right),\mathrm{M}\left(T\upsilon,B\upsilon,B\upsilon,kt\right) \end{pmatrix}$$ If

$Au \neq u$ then

$$\mathrm{M}\left(Au,u,u,t\right) > \mathrm{M}\left(Au,u,kt\right)$$

Is a contradiction. Thus $Au = Su = u$. Now, we prove $Bu = u$. For

$$\mathrm{M}\left(u,Bu,Bu,t\right) = \mathrm{M}\left(Au,Bu,Bu,t\right)$$
$$\geq_L \phi\begin{pmatrix} \mathrm{M}\left(Su,Tu,Tu,kt\right),\mathrm{M}\left(Su,Bu,Tu,kt\right),\mathrm{M}\left(Su,Tu,Bu,kt\right),\mathrm{M}\left(Su,Bu,Bu\upsilon,kt\right) \\ \mathrm{M}\left(Tu,Bu,Bu,kt\right),\mathrm{M}\left(Tu,Tu,Bu,kt\right),\mathrm{M}\left(Tu,Bu,Bu,kt\right),\mathrm{M}\left(Tu,Bu,Bu,kt\right) \\ \mathrm{M}\left(Bu,Tu,Tu,kt\right),\mathrm{M}\left(Bu,Bu,Tu,kt\right),\mathrm{M}\left(Bu,Tu,Tu,kt\right),\mathrm{M}\left(Tu,Bu,Bu,kt\right) \end{pmatrix}$$ If

$Bu \neq u$ then

$$\mathrm{M}\left(u,Bu,Bu,t\right) > \mathrm{M}\left(u,Bu,Bu,kt\right)$$

Is a contradiction. Thus $Au = Bu = Su = Tu = u$. So, A,B,S and T have a fixed common point u.

Now to prove uniqueness, if possible $\upsilon \neq u$ be another common fixed point of A,B,S and T. Then

$$M(\upsilon,u,u,t) = M(A\upsilon,Bu,Bu,t)$$

$$\geq_L \phi\begin{pmatrix} M(S\upsilon,Tu,Tu,kt), M(S\upsilon,Bu,Tu,kt), M(S\upsilon,Tu,Bu,kt), M(S\upsilon,Bu,Bu,kt) \\ M(Tu,Bu,Bu,kt), M(Tu,Tu,Bu,kt), M(Tu,Bu,Bu,kt), M(Tu,Bu,Bu,kt) \\ M(Bu,Tu,Tu,kt), M(Bu,Bu,Tu,kt), M(Bu,Tu,Tu,kt), M(Tu,Bu,Bu,kt) \end{pmatrix}$$ is

$$>_L M(\upsilon,u,u,kt)$$

contradiction.