Common Fixed Point Theorems for Random Operators in Hilbert Space

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Abstract
Our main aim of this paper is introduced some new unique common random fixed point theorems of random operators in Hilbert Space by considering a sequence of measurable functions satisfying conditions A or B and C. Our results are motivated from [3, 5, 6, 7, 8].
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1. Introduction and preliminaries:
In recent years, the study of random fixed points has attracted much attention; some of the recent literatures in random fixed point may be noted in [1, 2, 3]. In this paper we construct a sequence of measurable function and consider its convergence to the common unique random fixed point of two continuous random operators defined on a non- empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the two continuous random operators. We have introduced a rational inequality and used the parallelogram law.

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra $\sum$ of subset of Ω. H stands for a separable Hilbert space and C is nonempty closed subset of H.

1.1 Definition: A function f: $\Omega \to C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H.

1.2. Definition: A function F: $\Omega \; X \; C \to C$ is said to be a random operator if F(., x) : $\Omega \to\to C$ is measurable for every x $\in C$.

1.3. Definition: A measurable function g: $\Omega \to C$ is said to be a random fixed point of the random operator F: $\Omega \; X \; C \to$ C if F(t(g(t))) = g(t) for all t $\in \Omega$.

1.4. Definition: A random operator F: $\Omega \; X \; C \to$ C is said to be continuous if for fixed t $\in \Omega$, F(t, .) : $\boldsymbol{C} \to$ C is continuous.

1.5. Theorem: Let C be a non-empty closed subset of a separable Hilbert space H. Let S and T be two continuous random operators defined on C such that for t $\in \Omega$, S(t, .), T(t,.): $C \to$ C satisfy condition (C). Then S and T have a common unique random fixed point in C. and satisfy the following condition

$$\|S_x - T_y\|^2 \le \frac{a\|S_x - T_y\|^2\|x - S_x\|^2}{1 + \|x - y\|^2} + b[\ \|x - S_x\|^2 + \ \|y - T_y\|^2] + C\|x - y\|^2$$

For each x, y in C, a, b, being positive real number such that $0 < a + b < 1/2$.

Proof : We define a sequence of function $\{g_n\}$ as $g_0 : \Omega \to C$ is arbitrary measurable function for t$\in \Omega$ and n = 0, 1, 2, 3… .

$$g_{2n+1}(t) = S(t, g_{2n}(t)) \qquad g_{2n+2}(t) = T(t, g_{2n+1}(t)) \qquad (1)$$

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for t $\in \Omega$, for some n then we see that $g_{2n}(t)$ is a random fixed point of S and T. therefore we suppose that no two consecutive terms of sequence $\{g_n\}$ are equal. Now consider for t $\in \Omega$

$$\|g_{2n+1}(t) - g_{2n+2}(t)\|^2 = \|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2$$

$$\le \frac{a\|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2\|g_{2n}(t) - S(t, g_{2n}(t))\|^2}{1 + \|g_{2n}(t) - g_{2n+1}(t)\|^2}$$

$$+b[\|g_{2n}(t) - S(t, g_{2n}(t))\|^2 + \|g_{2n+1}(t) - T(t, g_{2n+1}(t))\|^2]$$

$$+c[\|g_{2n}(t) - g_{2n+1}(t)\|^2]$$

$$= \frac{a\|g_{2n+1}(t) - g_{2n+2}(t))\|^2\|g_{2n}(t) - g_{2n+1}(t)\|^2}{1 + \|g_{2n}(t) - g_{2n+1}(t)\|^2}$$

$$+b[\|g_{2n}(t) -, g_{2n+1}(t))\|^2 + \|g_{2n+1}(t) -, g_{2n+2}(t))\|^2]$$

$+c[\|g_{2n}(\mathrm{t})-,g_{2n+1}(\mathrm{t}))\|^2]$

$= (\mathrm{a}+\mathrm{b})\left\|\left(g_{2n+1}(\mathrm{t})\right)-g_{2n+2}(\mathrm{t}))\right\|^2 + (\mathrm{b}+\mathrm{c})\|g_{2n}(\mathrm{t})-,g_{2n+1}(\mathrm{t}))\|^2$

$\Rightarrow [1-(a+b)]\left\|\left(g_{2n+1}(\mathrm{t})\right)-g_{2n+2}(\mathrm{t}))\right\|^2 \le (b+c)\|g_{2n}(\mathrm{t})-,g_{2n+1}(\mathrm{t}))\|^2$

$\Rightarrow \left\|\left(g_{2n+1}(\mathrm{t})\right)-g_{2n+2}(\mathrm{t}))\right\| \le \dfrac{(b+c)}{[1-(a+b)]}\|g_{2n}(\mathrm{t})-,g_{2n+1}(\mathrm{t}))\|$

$\Rightarrow \left\|\left(g_{2n+1}(\mathrm{t})\right)-g_{2n+2}(\mathrm{t}))\right\| \le \mathrm{K}\|g_{2n}(\mathrm{t})-,g_{2n+1}(\mathrm{t}))\|$ Where $k = [\frac{(b+c)}{[1-(a+b)]}]^{\frac{1}{2}} \le \frac{1}{2}$

In general

$\Rightarrow \|g_n(\mathrm{t})-,g_{n+1}(\mathrm{t}))\| \le K\,\|g_{n-1}(\mathrm{t})-,g_n(\mathrm{t}))\|$

$\Rightarrow \|g_n(\mathrm{t})-,g_{n+1}(\mathrm{t}))\| \le K^n\,\|g_0(\mathrm{t})-,g_1(\mathrm{t}))\|$ for all $t \in \Omega$ (2)

Now, we shall prove that for t ∈ Ω , $\{g_n(\mathrm{t})\}$ is a Cauchy sequence. For this for every positive integer p we have, for $t \in \Omega$

$\Rightarrow \|g_n(\mathrm{t})-g_{n+p}(\mathrm{t})\| = \|g_n(\mathrm{t})-g_{n+1}(\mathrm{t})+g_{n+1}(\mathrm{t})-\dots\dots+g_{n+p-1}(\mathrm{t})-g_{n+p}(\mathrm{t})\|$

$\le \|g_n(\mathrm{t})-g_{n+1}(\mathrm{t})\| + \|g_{n+1}(\mathrm{t})-,g_{n+2}(\mathrm{t}))\| + \cdots\ldots + \|g_{n+p-1}(\mathrm{t})-g_{n+p}(\mathrm{t})\|$

$\le [K^n+K^{n+1}+\cdots+K^{n+p-1}]\|g_0(\mathrm{t})-,g_1(\mathrm{t}))\|$ by (2)

$= K^n[1+k+K^2\ldots+K^{p-1}]\|g_0(\mathrm{t})-g_1(\mathrm{t}))\|$

$\le \frac{K^n}{(1-k)}\|g_0(\mathrm{t})-g_1(\mathrm{t}))\|$ for all $t \in \Omega$

as $n \to \infty$, $\|g_n(\mathrm{t})-g_{n+p}(\mathrm{t})\| \to 0$ it follows that for $t \in \Omega$, $\{g_n(\mathrm{t})\}$ is a Cauchy sequence and hence is convergent in Hilbert space H.

For $t \in \Omega$, let

$\{g_n(\mathrm{t})\} \to g(t)$ as n $\to \infty$ (3)

Since C is closed, g is a function from C to C.

Existence of random fixed point: For $t \in \Omega$,

$\|\mathrm{g(t)}-\mathrm{T(t.g(t))}\|^2 = \|g(t)-g_{2n+1}(\mathrm{t})+g_{2n+1}(\mathrm{t})-\dots\dots\dots\mathrm{T(t.g(t))}\|$

$\le 2\|\mathrm{g(t)}-g_{2n+1}(\mathrm{t})\|^2 + 2\|g_{2n+1}(\mathrm{t})-\mathrm{T(t.g(t))}\|^2$

[by parlallelogram law $\|x+y\|^2 \le 2[\|x\|^2 + 2\|y\|^2]$

$= 2\|\mathrm{g(t)}-g_{2n+1}(\mathrm{t})\|^2 + 2\left\|S\left(t.g_{2n}(\mathrm{t})\right)-\mathrm{T(t.g(t))}\right\|^2$

$\le 2\|\mathrm{g(t)}-g_{2n+1}(\mathrm{t})\|^2 + \frac{2a\left\|S(t.g_{2n}(\mathrm{t}))-\mathrm{T}(\mathrm{t},g(t))\right\|^2\left\|g_{2n}(\mathrm{t})-S(t.g_{2n}(\mathrm{t}))\right\|^2}{1+\|g_{2n}(\mathrm{t})-g(t)\|^2} +$ 2b $\left[\left\|g_{2n}(\mathrm{t})-\right.\right.$

St, *g2n*t*2 +gt*–Tt,*gt2* +2c *[g2n*t–*gt2]*

$= 2\|\mathrm{g(t)}-g_{2n+1}(\mathrm{t})\|^2 + \dfrac{2a\left\|g_{2n+1}(\mathrm{t})-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2}{1+\|g_{2n}(\mathrm{t})-g(t)\|^2} + 2\mathrm{b}[\|g_{2n}(\mathrm{t})-g_{2n+1}(\mathrm{t})\|^2$

$+ \left\|\mathrm{g(t)}-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2] + 2\mathrm{c}\,[\left\|g_{2n+1}(\mathrm{t})-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2$

As $\{g_{2n+1}(\mathrm{t})\}$ and $\{g_{2n+2}(\mathrm{t})\}$ are subsequences of $\{g_n(\mathrm{t})\}$ as $n \to \infty$, $\{g_{2n+1}(\mathrm{t})\} \to$ g(t) and $\{g_{2n+2}(\mathrm{t})\} \to g(t)$

Therefore,

$\Rightarrow \|\mathrm{g(t)}-\mathrm{T(t.g(t))}\|^2$

$\le 2\|\mathrm{g(t)}-\mathrm{g(t)}\|^2 + \dfrac{2a\|g(t)-g(t)\|^2[1+\|\mathrm{g(t)}-\mathrm{g(t)}\|^2]}{1+\|\mathrm{g(t)}-\mathrm{g(t)}\|^2} + 2\mathrm{b}[\|\mathrm{g(t)}-\mathrm{g(t)}\|^2 + \left\|\mathrm{g(t)}-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2]$

$+2C[\left\|\mathrm{g(t)}-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2]$

$\Rightarrow [1-2b]\left\|\mathrm{g(t)}-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2 \le 0$

$\Rightarrow \left\|\mathrm{g(t)}-\mathrm{T}\left(\mathrm{t},g(t)\right)\right\|^2 = 0$ (as $2\mathrm{b} < 1$)

$\Rightarrow \mathrm{T}\left(\mathrm{t},g(t)\right) = \mathrm{g(t)}$ $\forall\, t \in \Omega$ (4)

In an exactly similar way we can prove that for all t ∈ Ω,

$\Rightarrow \mathrm{S}\left(\mathrm{t},g(t)\right) = \mathrm{g(t)}$ (5)

Again if $A: \Omega X C \to$ is a continuous random operator on a non empty subset C of a separable Hilbert space H, then for any measurable function $f: \Omega \to C$, the function $\mathrm{h(t)} = A(t, f(t))$ is also measurable [1]

It follows from the construction of $\{g_n\}$ (by (1)) and the above consideration that $\{g_n\}$ is a sequence of measurable function. From (3), it follow that g is also a measurable function. This fact along with (4) and (5) shows that g: $\Omega \to C$ is common random fixed point of S and T.

Uniqueness:

Let $h: \Omega \to C$ be another random fixed point common to S and T that is for $t \in \Omega$.

$S(t, h(t)) = h(t)$

$T(t, h(t)) = h(t)$

Then for $t \in \Omega$.

$$\Rightarrow \quad \|g(t) - h(t)\|^2 = \|S(t, g(t) - T(t, h(t))\|^2$$

$$\leq \frac{a\|S(t, g(t)) - T(t, h(t))\|^2 \|g(t) - S(t, g(t))\|}{1 + \|g(t) - h(t)\|^2} + b[\|g(t) - S(t, g(t))\|^2 + \|h(t) - T(t, h(t))\|^2] + c[\|g(t) - h(t)\|]^2$$

$$\leq \frac{a\|h(t) - h(t)\|^2 \|g(t) - g(t)\|}{1 + \|g(t) - h(t)\|^2} + b[\|g(t) - g(t)\|^2 + \|h(t) - h(t)\|^2] + c[\|g(t) - h(t)\|]^2$$

$\Rightarrow (1 - c)\|g(t) - h(t)\|^2 \leq 0$

$\Rightarrow g(t) = h(t)$ for all $t \in \Omega$.

This complete proof of theorem.

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