

Common Fixed Point Theorems for Random Operators in Hilbert Space

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Abstract

Our main aim of this paper is introduced some new unique common random fixed point theorems of random operators in Hilbert Space by considering a sequence of measurable functions satisfying conditions A or B and C. Our results are motivated from [3, 5, 6, 7, 8].

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1. Introduction and preliminaries:

In recent years, the study of random fixed points has attracted much attention; some of the recent literatures in random fixed point may be noted in [1, 2, 3]. In this paper we construct a sequence of measurable function and consider its convergence to the common unique random fixed point of two continuous random operators defined on a non- empty closed subset of a separable Hilbert space. For the purpose of obtaining the random fixed point of the two continuous random operators. We have introduced a rational inequality and used the parallelogram law.

Throughout this paper, (Ω, Σ) denotes a measurable space consisting of a set Ω and sigma algebra Σ of subset of Ω . H stands for a separable Hilbert space and C is nonempty closed subset of H .

1.1 Definition: A function $f: \Omega \rightarrow C$ is said to be measurable if $f^{-1}(B \cap C) \in \Sigma$ for every Borel subset B of H .

1.2. Definition: A function $F: \Omega \times C \rightarrow C$ is said to be a random operator if $F(., x) : \Omega \rightarrow C$ is measurable for every $x \in C$.

1.3. Definition: A measurable function $g: \Omega \rightarrow C$ is said to be a random fixed point of the random operator $F: \Omega \times C \rightarrow C$ if $F(t, g(t)) = g(t)$ for all $t \in \Omega$.

1.4. Definition: A random operator $F: \Omega \times C \rightarrow C$ is said to be continuous if for fixed $t \in \Omega$, $F(t, .) : C \rightarrow C$ is continuous.

1.5. Theorem: Let C be a non-empty closed subset of a separable Hilbert space H . Let S and T be two continuous random operators defined on C such that for $t \in \Omega$, $S(t, .), T(t, .) : C \rightarrow C$ satisfy condition (C). Then S and T have a common unique random fixed point in C . and satisfy the following condition

$$\|S_x - T_y\|^2 \leq \frac{a\|S_x - T_y\|^2\|x - S_x\|^2}{1 + \|x - y\|^2} + b[\|x - S_x\|^2 + \|y - T_y\|^2] + C\|x - y\|^2$$

For each x, y in C , a, b , being positive real number such that $0 < a + b < 1/2$.

Proof : We define a sequence of function $\{g_n\}$ as $g_0 : \Omega \rightarrow C$ is arbitrary measurable function for $t \in \Omega$ and $n = 0, 1, 2, 3, \dots$

$$g_{2n+1}(t) = S(t, g_{2n}(t)) \quad g_{2n+2}(t) = T(t, g_{2n+1}(t)) \tag{1}$$

If $g_{2n}(t) = g_{2n+1}(t) = g_{2n+2}(t)$ for $t \in \Omega$, for some n then we see that $g_{2n}(t)$ is a random fixed point of S and T . therefore we suppose that no two consecutive terms of sequence $\{g_n\}$ are equal. Now consider for $t \in \Omega$

$$\begin{aligned} \|g_{2n+1}(t) - g_{2n+2}(t)\|^2 &= \|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2 \\ &\leq \frac{a\|S(t, g_{2n}(t)) - T(t, g_{2n+1}(t))\|^2\|g_{2n}(t) - S(t, g_{2n}(t))\|^2}{1 + \|g_{2n}(t) - g_{2n+1}(t)\|^2} \\ &\quad + b[\|g_{2n}(t) - S(t, g_{2n}(t))\|^2 + \|g_{2n+1}(t) - T(t, g_{2n+1}(t))\|^2] \\ &\quad + c[\|g_{2n}(t) - g_{2n+1}(t)\|^2] \\ &= \frac{a\|g_{2n+1}(t) - g_{2n+2}(t)\|^2\|g_{2n}(t) - g_{2n+1}(t)\|^2}{1 + \|g_{2n}(t) - g_{2n+1}(t)\|^2} \\ &\quad + b[\|g_{2n}(t) - g_{2n+1}(t)\|^2 + \|g_{2n+1}(t) - g_{2n+2}(t)\|^2] \end{aligned}$$

$$\begin{aligned}
 &+c[\|g_{2n}(t)-, g_{2n+1}(t)\|^2] \\
 &= (a + b) \| (g_{2n+1}(t) - g_{2n+2}(t)) \|^2 + (b + c) \|g_{2n}(t)-, g_{2n+1}(t)\|^2 \\
 &\Rightarrow [1 - (a + b)] \| (g_{2n+1}(t) - g_{2n+2}(t)) \|^2 \leq (b + c) \|g_{2n}(t)-, g_{2n+1}(t)\|^2 \\
 &\Rightarrow \| (g_{2n+1}(t) - g_{2n+2}(t)) \| \leq \frac{(b + c)}{[1 - (a + b)]} \|g_{2n}(t)-, g_{2n+1}(t)\| \\
 &\Rightarrow \| (g_{2n+1}(t) - g_{2n+2}(t)) \| \leq K \|g_{2n}(t)-, g_{2n+1}(t)\| \quad \text{Where } k = \left[\frac{(b+c)}{[1-(a+b)]} \right]^{\frac{1}{2}} \leq \frac{1}{2}
 \end{aligned}$$

In general

$$\begin{aligned}
 &\Rightarrow \|g_n(t)-, g_{n+1}(t)\| \leq K \|g_{n-1}(t)-, g_n(t)\| \\
 &\Rightarrow \|g_n(t)-, g_{n+1}(t)\| \leq K^n \|g_0(t)-, g_1(t)\| \quad \text{for all } t \in \Omega \tag{2}
 \end{aligned}$$

Now, we shall prove that for $t \in \Omega$, $\{g_n(t)\}$ is a Cauchy sequence. For this for every positive integer p we have, for $t \in \Omega$

$$\begin{aligned}
 &\Rightarrow \|g_n(t) - g_{n+p}(t)\| = \|g_n(t) - g_{n+1}(t) + g_{n+1}(t) - \dots + g_{n+p-1}(t) - g_{n+p}(t)\| \\
 &\leq \|g_n(t) - g_{n+1}(t)\| + \|g_{n+1}(t) - g_{n+2}(t)\| + \dots + \|g_{n+p-1}(t) - g_{n+p}(t)\| \\
 &\leq [K^n + K^{n+1} + \dots + K^{n+p-1}] \|g_0(t)-, g_1(t)\| \quad \text{by (2)} \\
 &= K^n [1 + k + K^2 \dots + K^{p-1}] \|g_0(t) - g_1(t)\| \\
 &\leq \frac{K^n}{(1-k)} \|g_0(t) - g_1(t)\| \quad \text{for all } t \in \Omega
 \end{aligned}$$

as $n \rightarrow \infty$, $\|g_n(t) - g_{n+p}(t)\| \rightarrow 0$ it follows that for $t \in \Omega$, $\{g_n(t)\}$ is a Cauchy sequence and hence is convergent in Hilbert space H.

For $t \in \Omega$, let

$$\{g_n(t)\} \rightarrow g(t) \text{ as } n \rightarrow \infty \tag{3}$$

Since C is closed, g is a function from C to C.

Existence of random fixed point: For $t \in \Omega$,

$$\begin{aligned}
 \|g(t) - T(t, g(t))\|^2 &= \|g(t) - g_{2n+1}(t) + g_{2n+1}(t) - \dots - T(t, g(t))\|^2 \\
 &\leq 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|g_{2n+1}(t) - T(t, g(t))\|^2 \\
 \text{[by parallelogram law } \|x + y\|^2 &\leq 2[\|x\|^2 + \|y\|^2] \\
 &= 2\|g(t) - g_{2n+1}(t)\|^2 + 2\|S(t, g_{2n}(t)) - T(t, g(t))\|^2
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2\|g(t) - g_{2n+1}(t)\|^2 + \frac{2a\|S(t, g_{2n}(t)) - T(t, g(t))\|^2 \|g_{2n}(t) - S(t, g_{2n}(t))\|^2}{1 + \|g_{2n}(t) - g(t)\|^2} + 2b[\|g_{2n}(t) - \\
 &S(t, g_{2n}(t)) + g(t) - T(t, g(t)) + 2c \|g_{2n}(t) - g(t)\|^2] \\
 &= 2\|g(t) - g_{2n+1}(t)\|^2 + \frac{2a\|g_{2n+1}(t) - T(t, g(t))\|^2}{1 + \|g_{2n}(t) - g(t)\|^2} + 2b[\|g_{2n}(t) - g_{2n+1}(t)\|^2 \\
 &+ \|g(t) - T(t, g(t))\|^2] + 2c [\|g_{2n+1}(t) - T(t, g(t))\|^2]
 \end{aligned}$$

As $\{g_{2n+1}(t)\}$ and $\{g_{2n+2}(t)\}$ are subsequences of $\{g_n(t)\}$ as $n \rightarrow \infty$, $\{g_{2n+1}(t)\} \rightarrow g(t)$ and $\{g_{2n+2}(t)\} \rightarrow g(t)$

Therefore,

$$\begin{aligned}
 &\Rightarrow \|g(t) - T(t, g(t))\|^2 \\
 &\leq 2\|g(t) - g(t)\|^2 + \frac{2a\|g(t) - g(t)\|^2 [1 + \|g(t) - g(t)\|^2]}{1 + \|g(t) - g(t)\|^2} + 2b[\|g(t) - g(t)\|^2 + \|g(t) - T(t, g(t))\|^2] \\
 &\quad + 2c [\|g(t) - T(t, g(t))\|^2] \\
 &\Rightarrow [1 - 2b] \|g(t) - T(t, g(t))\|^2 \leq 0
 \end{aligned}$$

$$\begin{aligned}
 &\Rightarrow \|g(t) - T(t, g(t))\|^2 = 0 \quad (\text{as } 2b < 1) \\
 &\Rightarrow T(t, g(t)) = g(t) \quad \forall t \in \Omega \tag{4}
 \end{aligned}$$

In an exactly similar way we can prove that for all $t \in \Omega$,

$$\Rightarrow S(t, g(t)) = g(t) \tag{5}$$

Again if $A: \Omega \times C \rightarrow C$ is a continuous random operator on a non empty subset C of a separable Hilbert space H, then for any measurable function $f: \Omega \rightarrow C$, the function $h(t) = A(t, f(t))$ is also measurable [1]

It follows from the construction of $\{g_n\}$ (by (1)) and the above consideration that $\{g_n\}$ is a sequence of measurable function. From (3), it follow that g is also a measurable function. This fact along with (4) and (5) shows that $g: \Omega \rightarrow C$ is common random fixed point of S and T.

Uniqueness:

Let $h: \Omega \rightarrow \mathcal{C}$ be another random fixed point common to S and T that is for $t \in \Omega$.

$$S(t, h(t)) = h(t)$$

$$T(t, h(t)) = h(t)$$

Then for $t \in \Omega$.

$$\begin{aligned} \Rightarrow \|g(t) - h(t)\|^2 &= \|S(t, g(t)) - T(t, h(t))\|^2 \\ &\leq \frac{a\|S(t, g(t)) - T(t, h(t))\|^2 \|g(t) - S(t, g(t))\|}{1 + \|g(t) - h(t)\|^2} + b[\|g(t) - S(t, g(t))\|^2 \\ &\quad + \|h(t) - T(t, h(t))\|^2] + c[\|g(t) - h(t)\|^2] \\ &\leq \frac{a\|h(t) - h(t)\|^2 \|g(t) - g(t)\|}{1 + \|g(t) - h(t)\|^2} + b[\|g(t) - g(t)\|^2 + \|h(t) - h(t)\|^2] + c[\|g(t) - h(t)\|^2] \end{aligned}$$

$$\Rightarrow (1 - c)\|g(t) - h(t)\|^2 \leq 0$$

$$\Rightarrow g(t) = h(t) \text{ for all } t \in \Omega.$$

This complete proof of theorem.

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