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Common Fixed Point and Weak** Commuting Mappings

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Abstract
Existence of common fixed points of weak** commuting mappings which satisfies the contractive condition involving pair of mappings in a complete metric space under certain is shown.
Key words: commuting mappings, weak ** commuting mapping.

1. Introduction

A study of the common fixed points and weak** commuting mappings is fascinating field of research lying at the intersection of non-linear analysis. A wide spread interest in the domain and vast amount of mathematical activity have led to many remarkable new results.

In 1976, Jungck [4] investigated and found interdependence between commuting mappings and common fixed points and proved the followings:

Let T be a continuous mapping of a complete metric space (X, d) into itself. Then T has a fixed point in X, if and only if there exists an $\alpha \in (0, 1)$ and a mapping $S : X \to X$ which commutes with T and satisfies:

(1) $S(X) \subset T(X)$ and $d(Sx, Sy) \leq d(Tx, Ty)$

For all x, y in X. Indeed, S and T have a unique common fixed point if and only if (1) holds for some $\alpha \in (0, 1)$.

Further, in 1977, Singh [10] generalized the above result and proved that two continuous and commuting mappings from a complete metric space into itself satisfies some conditions, then two commuting mappings have a unique common fixed point.

Das and Vishwanathana Naik [1] have proved a theorem for two commuting mappings. Fisher [2] proved a common fixed point of commuting mappings, Rhoades and Seesa [8] established some fixed point theorems for three pair wise weakly commuting self maps satisfying a very general contractive definitions. Khan and Imdad [5], considering a pair of self maps {A, T} of metric space (X, d) satisfying a weaker condition the commutativity: namely weak* commuting pair of mappings, that is

$$d(ATx, TAx) \leq d(A^2x, T^2x)$$

For each x in X.

B. Fisher [2] has been proved following theorem for two commuting mappings T and S.

If S is a mapping and T is a continuous mapping of the complete metric space into itself and satisfying the inequality :

(2) $d(STx, TSy) \leq k \{d(Tx, TSy) + d(Sy, STx)\}$

for all x, y in X, where $0 \leq k \leq 1/2$, then S and T have a unique common fixed point.

In 1986, Pathak [7] has been further generalized a result of Khan and Imdad [5] by considering a pair of self maps {A, T} of a metric space (X, d) satisfying a weaker condition, then commutativity: namely, weak* commuting pair of mappings, that is

$$d(ATx, TAx) \leq d(A^2x, T^2x)$$

for each $x \in X$.

In 1995, Lohani and Badshah [6] further generalized the result of B. Fisher[2, 3]

The purpose of this note is to prove some results concerning fixed points of weak** commuting mappings defined on complete metric spaces and satisfying some new functional inequality.

Definition 1.1. According to Seesa [9] two self maps S and T defined on metric space (X, d) are said to be weakly commuting maps iff

$$d(STx, TSx) \leq d(Sx, Tx)$$

for all x in X.

Defintion 1. 2. Two self mappings S and T of metric space (X, d) is called weak** commuted, if $S(X) \subset T(X)$ and for any $x \in X$,

$$d(S^2T^2x, T^2S^2x) \leq d(S^2Tx, TS^2x) \leq d(ST^2x, T^2Sx) \leq d(STx, TSx) \leq d(S^2x, T^2x)$$

Definition 1.3. A map $S : X \to X$, X being a metric space, is called an idempotent, if $S^2 = S$.

We further generalize the result of Fisher [2, 3], Pathak [7] and Lohani & Badshah [6] by using another type of rational expression.

Theorem 1.1. If S is a mapping and T is a continuous mapping of complete metric space {S, T} is weak** commuting pair and the following condition :

$$d(S^2T^2x,T^2S^2y) \le \alpha \frac{d(T^2x,S^2T^2x)d(T^2x,T^2S^2y)+d(S^2y,T^2S^2y)d(S^2y,S^2T^2x)}{d(T^2x,T^2S^2y)+d(S^2y,S^2T^2x)}$$
$$+\beta d(T^2x,S^2y)$$

for all x, y in X, where $0 \le \alpha + \beta < 1$, then S and T have a unique common fixed point.

Proof. Let x be an arbitrary point in X. Define

$(S^2T^2)^n x = x_{2n}$ or $T^2(S^2T^2)^n x = x_{2n+1}$

Where n =0, 1, 2, 3…., by contractive condition (A),

$$d(x_{2n},x_{2n+1}) = d(S^2T^2(S^2T^2)x, T^2S^2(T^2(S^2T^2)^{n-1}x))$$

$$\le \alpha \frac{d\left(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x\right)d\left(T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1x}\right)+ d\left(S^2T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1}x\right)d\left(S^2T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x\right)}{d(\left(T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1}x)\right)+d(S^2T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)}$$
$$+\beta d\left(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x\right)$$

$$\le \alpha \frac{d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)d(T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1}x)}{d(T(ST)x, TS(T(ST)x)}$$
$$+\beta d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)$$

$$d(x_{2n},x_{2n+1}) \le \alpha d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x) + \beta d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x)$$

$$\le (\alpha+\beta)d(x_{2n-1},(S^2T^2)^n x)$$
$$\le (\alpha+\beta)d(x_{2n-1},x_{2n}).$$

Proceeding in the same manner

$d(x_{2n}, x_{2n+1}) < (\alpha + \beta)^{2n-1} d(x_1, x_2)$.

Also $d(x_n, x_m) \le \sum_{i=n}^{m} d(x_i, x_{i+1})$ for m > n.

Since k < 1, it follows that the sequence $\{x_n\}$ is Cauchy sequence in the complete metric space X and so it has a limit in X, that is

$\lim_{n\to\infty} x_{2n} = u = \lim_{n\to\infty} x_{2n+1}$

and since T is continuous, we have

$u = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} T^2(x_{2n}) = T^2u$.

Further,

$d(x_{2n+1}, S^2u) = d(T^2(S^2T^2)^{n+1}x, S^2u)$

$= d(T^2(S^2T^2)^{n+1}x, S^2(T^2u))$ for $u = T^2u$

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$$\leq \alpha \frac{d((S^2T^2)^{n+1}x, T^2(S^2T^2)^{n+1}x)d(S^2T^2)^{n+1}x, (S^2T^2u)) + d(T^2u, S^2T^2u)d(T^2u, T^2(S^2T^2)^{n+1}x))}{d((S^2T^2)^{n+1}x, S^2T^2u) + d(T^2u, T^2(S^2T^2)^{n+1}x)}$$

$$+ \beta d(S^2T^2)^{n+1}x, T^2u)$$

$$= \alpha \frac{[d(x_{2n+2}, x_{2n+3})d(x_{2n+2}, S^2u) + d(u, S^2u)d(u, x_{2n+3})]}{d(x_{2n+2}, S^2u) + d(u, x_{2n+3})} + \beta d(x_{2n+2}, u)$$

taking limit as n $\to \infty$, it follows that

$d(u, S^2u) = 0.$

which implies

$d(u, S^2u) = 0$ and so $u = S^2u = T^2u$.

Now consider weak** commutativity of pair {S, T} implies that $S^2T^2u = T^2S^2u$, $S^2Tu = TS^2u$, $ST^2u = T^2Su$ and so $S^2Tu = Tu$ and $T^2Su = Su$. Now

$d(u, Su) = d(S^2T^2u, T^2S^2(Su))$

$$\leq \alpha \frac{[d(T^2u, S^2T^2u)d(T^2u, T^2S^2(S\ u)) + d(S^2(S\ u), T^2S^2(S\ u))d(S^2(Su), S^2T^2u)]}{d(T^2u, T^2S^2(Su)) + d(S^2(Su), S^2T^2u)}$$

$$+ \beta d(T^2u, S^2(Su))$$

$$= \alpha \frac{d(u, S^2u)d(u, S^2T^2(Su)) + d(Su, Su)d(Su, u)}{d(u, S^2(Su)) + d(Su, u)} + \beta d(u, Su)$$

$$= \alpha \frac{d(u, u)d(u, Su) + d(Su, Su)d(Su, u)}{d(u, Su) + d(Su, u)} + \beta d(u, Su)$$

$$= 0$$

$$\Rightarrow (1-\beta)d(u, Su) \leq 0$$

$$\Rightarrow d(u, Su) \leq 0$$

Hence Su = u, similarly we can show that Tu = u. Hence u is a common fixed point of S and T.

Now suppose that x is an another common fixed point of S and T. Then

$$d(u, v) = d(S^2T^2u, T^2S^2v)$$

$$\leq \alpha \frac{d(T^2u, S^2T^2u)d(T^2u, T^2S^2v) + d(S^2v, T^2S^2v)d(S^2v, S^2T^2u)}{d(T^2u, T^2S^2v) + d(S^2v, S^2T^2u)} + \beta d(T^2u, S^2v)$$

$$\leq \alpha \frac{d(u, v)d(u, v) + d(u, v)d(u, v)}{d(u, v) + d(v, u)} + \beta d(u, v)$$

$$(1-\beta)d(u, v) \leq 0$$

$$d(u, v) \leq 0$$

Then it follows that $u = v$. Hence S and T have a unique common fixed point.

Theorem 1.2. If S is mapping and T is a continuous mapping of a complete metric space X into itself and satisfying {S, T} is weak** commuting pair and the following condition :

$$d(S^2T^2x, T^2S^2y) \leq \alpha \frac{[d(T^2x, S^2T^2x)] + [d(S^2y, T^2S^2y)]}{d(T^2x, S^2T^2x) + d(S^2y, T^2S^2y)} + \beta d(T^2x, S^2y) \qquad \text{(B)}$$

for all x, y in X, where α, $\beta \geq 0$ with $2\alpha + \beta < 1$, then S and T have a unique common fixed point.

Proof. Let x be an arbitrary point in X. Define
$(S^2T^2)^n x = x_{2n}$ or $T^2(S^2T^2)^n x = x_{2n+1}$
Where n =0, 1, 2, 3…., by contractive condition (B),

$$d(x_{2n},x_{2n+1}) = d((S^2T^2)^n x, T^2(S^2T^2)^n x))$$
$$= d(S^2T^2(S^2T^2)^{n-1}x,.T^2S^2(T^2(S^2T^2)^{n-1}x)$$
$$\frac{[d\{(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}\}]^2 + [d\{(S^2T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1})\}]^2,}{d(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x) + d(S^2T^2(S^2T^2)^{n-1}x, T^2S^2(T^2(S^2T^2)^{n-1}x))}$$
$$+\beta d\left(T^2(S^2T^2)^{n-1}x, S^2T^2(S^2T^2)^{n-1}x\right)$$
$$\leq \alpha\frac{[d(x_{2n-1},x_{2n})]^2 + d[(x_{2n},x_{2n+1})]^2}{d(x_{2n-1},x_{2n}) + d(x_{2n},x_{2n+1})} + \beta d(x_{2n-1},x_{2n})$$
$$\leq \alpha[d(x_{2n-1},x_{2n}) + d(x_{2n},x_{2n+1})] + \beta d(x_{2n-1},x_{2n})$$
$$\leq (\alpha+\beta)d(x_{2n-1},x_{2n}) + \alpha d(x_{2n},x_{2n+1})$$
$$(1-\alpha)d(x_{2n},x_{2n+1}) \leq (\alpha+\beta)d(x_{2n-1},x_{2n})$$
$$d(x_{2n},x_{2n+1}) \leq \frac{\alpha+\beta}{1-\alpha}d(x_{2n-1},x_{2n})$$
$$\leq kd(x_{2n-1},x_{2n})$$

where $k = \dfrac{\alpha+\beta}{1-\alpha}$.

Proceeding in the same manner, we have
$d(x_{2n}, x_{2n+1}) \leq k^{2n-1}d(x_1,x_2)$.
Also

$d(x_n, x_m) \leq \sum_{i=n}^{m} d(x_i,x_{i+1})$ for m > n.

Since k < 1, it follows that the sequence $\{x_n\}$ is Cauchy sequence in the complete metric space X and so it has a limit in X, that is

$\lim_{n\to\infty}x_{2n} = u = \lim_{n\to\infty}x_{2n+1}$

and since T is continuous, we have

$u = \lim_{n\to\infty}x_{2n+1} = \lim_{n\to\infty}T^2(x_{2n})m = T^2u$.

Further

$d(x_{2n+3}, S^2u) = d(T^2(S^2T^2)^{n+1}x, S^2u)$
$= d(T^2(S^2T^2)^{n+1}x, S^2(T^2u))$ (since $u = T^2u$)

$$\leq \alpha\frac{[d(T^2u,S^2T^2u)]^2 + d[(S^2T^2)^{n+1}x, T^2(S^2T^2)^{n+1}x]^2}{d(T^2u,S^2T^2u) + d\{(S^2T^2)^{n+1}x, T^2(S^2T^2)^{n+1}x\}} + \beta d(T^2u,(S^2T^2)^{n+1}x)$$
$$\leq \alpha[d(T^2u,S^2(T^2u)) + d(x_{2n+2},x_{2n+3})] + \beta d(x_{2n+2},T^2u)$$
$$\leq \alpha(d(u,S^2u) + d(x_{2n+2},x_{2n+3})) + \beta d(x_{2n+2},u)$$

Taking limit as $n \to \infty$, it follows that

$d(u, S^2u) \leq 0$,

which implies that $d(u, S^2u) = 0$ and so that $u = S^2u = T^2u$.
Now consider weak** commutativity of pair {S, T}, implies that $S^2T^2u = T^2S^2u$, S^2Tu, TS^2u, $ST^2u = T^2Su$ and so $S^2Tu = Tu$ and $T^2Su= Su$.
Now $d(u, Su) = d(S^2T^2u, T^2S^2(Su))$

$$\leq \alpha\frac{[d(T^2u,S^2T^2u)]^2 + d[S^2(Su),T^2S^2(Su)]^2}{d(T^2u,S^2T^2u) + d(S^2(Su),T^2S^2(Su))} + \beta d(T^2u,S^2(Su))$$

$$\leq \alpha \frac{[d(u,u)]+[d(Su,Su)]}{d(u,u)+d(Su,Su)} + \beta d(u,Su)$$

$(1-\beta)\, d(u, Su) \leq 0$

this implies that $(1-\beta) \neq 0$. Hence d(u, Su) = 0 or Su = u.
Similarly we can show that Tu = u. Hence u is a common fixed point of S and T. Now suppose that v is another common fixed point of S and T, then

$$d(u, v) = d(S^2T^2u, T^2S^2v)$$

$$\leq \alpha \frac{[d(T^2u,S^2T^2u)]^2 + [d(S^2v,T^2S^2v)]^2}{d(T^2u,S^2T^2u)+d(S^2v,T^2S^2v)} + \beta d(T^2u,S^2v)$$

$$\leq \alpha[d(u,u)] + d[(v,v)] + \beta d(u,v)$$

$(1-\beta)\, d(u, v) \leq 0$.

Since $(1-\beta) \neq 0$, then $d(u, v) = 0$. Thus it follows that $u = v$. Hence S and T have a unique common fixed point.

Example 1.1. Let X = [0,1] with Euclidean metric space and define S and T by $Sx = \frac{x}{x+2}, Tx = \frac{x}{2}$ for all x ∈ X, then [0, 1/5] ⊂ [0,1/4], where Sx = [0, 1/5] and Tx = [0, 1/4]

$$d(S^2T^2x, T^2S^2x) = \frac{x}{3x+16} - \frac{x}{8x+16}$$

$$= \frac{5x^2}{(3x+16)-(8x+16)}$$

$$\leq \frac{2x}{(2x+8)(4x+8)}$$

$$= \frac{x}{2x+8} - \frac{x}{4x+8}$$

$$= d(S^2Tx, TS^2x)$$

$\Rightarrow d(S^2T^2x, T^2S^2x) \leq d(S^2Tx, TS^2x)$

$$d(S^2Tx, TS^2x) = \frac{x}{2x+8} - \frac{x}{4x+8}$$

$$= \frac{2x^2}{(2x+8)(4x+8)}$$

$$\leq \frac{3x^2}{(x+8)(4x+8)}$$

$$= \frac{x}{x+8} - \frac{x}{4x+8}$$

$$= d(ST^2x, T^2Sx)$$

$\Rightarrow d(S^2Tx, TS^2x) \leq d(ST^2x, T^2Sx)$

$$d(ST^2x, T^2Sx) = \frac{x}{x+8} - \frac{x}{4x+8}$$

$$= \frac{3x^2}{(x+8)((x+8)}$$

$$\leq \frac{x^2}{(x+4)(2x+4)}$$

$$= \frac{x}{x+4} - \frac{x}{2x+4}$$

$$= d(STx, TSx)$$

$\Rightarrow d(ST^2 x, T^2 Sx) \le d(STx, TSx)$

$$d(STx,\ TSx) = \frac{x}{x+4} - \frac{x}{2x+4}$$
$$= \frac{x^2}{(x+4)(2x+4)}$$
$$\le \frac{3x^2}{4(3x+4)}$$
$$= \frac{x}{4} - \frac{x}{3x+4}$$
$$= d(T^2x,\ S^2x)$$

$\Rightarrow d(STx, TSx) \le d(T^2x, S^2x)$

using [0, 1] for $x \in$X, we conclude that definition (1.2) as follows :

$d(S^2T^2x,\ T^2S^2x) \le d(S^2Tx,\ TS^2x) \le d(ST^2 x,\ T^2 Sx) \le d(STx,\ TSx) \le d(T^2x,\ S^2x)$ for any $x \in$ X.

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