A Class of A-Stable Order Four and Six Linear Multistep Methods for Stiff Initial Value Problems

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Abstract
A new three and five step block linear methods based on the Adams family for the direct solution of stiff initial value problems (IVPs) are proposed. The main methods together with the additional methods which constitute the block methods are derived via interpolation and collocation procedures. These methods are of uniform order four and six for the three and five step methods respectively. The stability analysis of the two methods indicates that the methods are A–stable, consistent and zero stable. Numerical results obtained using the proposed new block methods show that they are attractive for the solutions of stiff problems and compete favorably with the well-known Matlab stiff ODE solver ODE23S.
Keywords: Linear multistep methods, initial value problems, interpolation and collocation.

1. Introduction

In this paper, we shall be concerned with the approximate numerical integration of the stiff initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0, \quad x \in [a,b], \quad y \in R \tag{1}$$

Numerical analysts have focused most of their works on the development of more efficient and accurate methods for the solution of stiff problems, and as such many methods have been proposed. Notable among these methods is the Backward Differentiation formulae (BDF), because of its wide region of absolute stability. Several researchers such as Kim (2010), Chollom *et. al.* (2012), Kumleng *et.al* (2011, 2012) have proposed methods for the numerical solution of (1) which were shown to be very efficient.

In this paper, we describe the construction of a new three and five step methods based on the Adams methods similar to the generalized Adams methods of Brugnano and Trigiante (1998). These methods are constructed using the interpolation concept where the continuous interpolants provide the block methods through evaluation at some grid points. This approach produce two self- starting new A – stable block methods which provide the solutions of stiff initial value problems on non-overlapping intervals.

2. The New Method

In this section, the new A – stable block methods shall be constructed based on the continuous finite difference approximation approach using the interpolation and collocation criteria described by Lie and Norsett (1981) called multistep collocation (MC) and block multistep methods by Onumanyi *et al.* (1994).

We define based on the interpolation and collocation methods the continuous form of the k- step new method as

$$y_{n+v} - \alpha_{v-1}(x) y_{n+v-1} = h \sum_{j=0}^{m} \beta_j(x) f_{n+j} \tag{2}$$

where α_{v-1} and $\beta_j(x)$ are the continuous coefficients of the method, m is the number of distinct collocation points, h is the step size and $v = \frac{k-1}{2}$, $k = 3,5,7...$

From Onumanyi et. al (1994), we obtain our matrices D and $C = D^{-1}$ as

$$D = \begin{bmatrix} 1 & x_{n+v-1} & x_{n+v-1}^2 & \cdots & x_{n+v-1}^m \\ 0 & 1 & 2x_0 & \cdots & mx_0^{m-1} \\ 0 & 1 & 2x_1 & \cdots & mx_1^{m-1} \\ \vdots & & & \cdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & mx_{m-1}^{m-1} \end{bmatrix} \tag{3}$$

and

$$C = \begin{bmatrix} \alpha_{v-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{v-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & & & \vdots \\ \alpha_{v-1,m+1} & h\beta_{0,m+1} & \cdots & h\beta_{m-1,m+1} \end{bmatrix} \qquad (4)$$

respectively.

2.1 Derivation of the Three Step New Block Method

In this case, k = 3, v=1, t = 1 and m = 4 and its continuous form expressed in the form of (2) is;

$$y(x) = \alpha_1(x)y_n + h\left(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3}\right) \qquad (5)$$

The matrix D, in (3) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 \end{pmatrix} \qquad (6)$$

Using the Maple software, the inverse of the matrix in (6) is obtained and this yields the elements of the matrix C. The elements of the matrix C substituted into (5) yields the continuous formulation of the method as:

$$y(x_n+\lambda) = y_n + \left(-\lambda - \tfrac{11\lambda^2}{12h} + \tfrac{\lambda^3}{3h^2} - \tfrac{\lambda^4}{24h^3}\right)f_n + \left(\tfrac{3\lambda^2}{2h} - \tfrac{5\lambda^3}{6h^2} + \tfrac{\lambda^4}{8h^3}\right)f_{n+1}$$
$$+\left(-\tfrac{3\lambda^2}{4h} + \tfrac{2\lambda^3}{3h^2} - \tfrac{\lambda^4}{8h^3}\right)f_{n+2} + \left(\tfrac{\lambda^2}{6h} - \tfrac{\lambda^3}{6h^2} + \tfrac{\lambda^4}{24h^3}\right)f_{n+3} \qquad (7)$$

Evaluating (7) at the following points $\lambda = h,\ \lambda = 2h,\ \lambda = 3h$ yields the following discrete methods which constitute the new three step block method.

$$\begin{aligned} y_{n+1} - y_n &= \tfrac{h}{24}\left(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right) \\ y_{n+2} - y_n &= \tfrac{h}{3}\left(f_n + 4f_{n+1} + f_{n+2}\right) \\ y_{n+3} - y_n &= \tfrac{h}{8}\left(3f_n + 9f_{n+1} + 9f_{n+2} + 3f_{n+3}\right) \end{aligned} \qquad (8)$$

The main new discrete scheme for the three step method is

$$y_{n+1} - y_n = \tfrac{h}{24}\left(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3}\right) \qquad (9)$$

This new method is consistent since its order is 4, it is also zero-stable, above all, it is A – stable as can be seen in figure 1. The new three step discrete methods that constitute the block method (8) have the following orders and error constants as can be seen in Table 1.

2.2 Derivation of the Five –Step New Block Method

In this case, k = 5, $v=2$, t = 1 and m = 6 and its continuous form expressed in the form of (2) is;

$$y(x) = \alpha_1(x)y_{n+1} + h\left(\beta_0(x)f_n + \beta_1(x)f_{n+1} + \beta_2(x)f_{n+2} + \beta_3(x)f_{n+3} + \beta_4(x)f_{n+4} + \beta_5(x)f_{n+5}\right) \qquad (10)$$

Similarly, we generate the continuous formulation of the new five step method as;

$$
\begin{aligned}
y(x_n+\lambda) = y_{n+1} &+ \left(-\tfrac{95}{288}h+\lambda-\tfrac{137\lambda^2}{120h}+\tfrac{5\lambda^3}{8h^2}-\tfrac{17\lambda^4}{96h^3}+\tfrac{\lambda^5}{40h^4}-\tfrac{\lambda^6}{720h^5}\right)f_n \\
&+\left(-\tfrac{1427}{1440}h+\tfrac{5\lambda^2}{2h}-\tfrac{77\lambda^3}{36h^2}+\tfrac{71\lambda^4}{96h^3}-\tfrac{7\lambda^5}{60h^4}+\tfrac{\lambda^6}{144h^5}\right)f_{n+1} \\
&+\left(\tfrac{133}{240}h-\tfrac{5\lambda^2}{2h}+\tfrac{107\lambda^3}{36h^2}-\tfrac{59\lambda^4}{48h^3}+\tfrac{13\lambda^5}{60h^4}+\tfrac{\lambda^6}{144h^5}\right)f_{n+2} \\
&+\left(-\tfrac{241}{720}h+\tfrac{5\lambda^2}{3h}-\tfrac{13\lambda^3}{6h^2}+\tfrac{49\lambda^4}{48h^3}-\tfrac{\lambda^5}{5h^4}+\tfrac{\lambda^6}{720h^5}\right)f_{n+3} \\
&+\left(\tfrac{173}{1440}h-\tfrac{5\lambda^2}{8h}+\tfrac{61\lambda^3}{72h^2}-\tfrac{41\lambda^4}{96h^3}+\tfrac{11\lambda^5}{120h^4}-\tfrac{\lambda^6}{144h^5}\right)f_{n+4} \\
&+\left(-\tfrac{3}{160}h+\tfrac{\lambda^2}{10h}-\tfrac{5\lambda^3}{36h^2}+\tfrac{7\lambda^4}{90h^3}-\tfrac{\lambda^5}{60h^4}+\tfrac{\lambda^6}{720h^5}\right)f_{n+5}
\end{aligned}
\tag{11}
$$

Evaluating (11) at the following points $\lambda=0,\ \lambda=2h,\ \lambda=3h, \lambda=4h,\ \lambda=5h$ yields the following discrete methods which constitute the new five step block method.

$$
\begin{aligned}
y_{n+1}-y_n &= \tfrac{h}{1440}\left(475f_n+1427f_{n+1}-798f_{n+2}+482f_{n+3}-173f_{n+4}+27f_{n+5}\right) \\
y_{n+2}-y_{n+1} &= \tfrac{h}{1440}\left(-27f_n+637f_{n+1}+1022f_{n+2}-258f_{n+3}+77f_{n+4}-11f_{n+5}\right) \\
y_{n+3}-y_{n+1} &= \tfrac{h}{90}\left(-f_n+34f_{n+1}+114f_{n+2}+34f_{n+3}-f_{n+4}\right) \\
y_{n+4}-y_{n+1} &= \tfrac{h}{160}\left(-3f_n+69f_{n+1}+174f_{n+2}+174f_{n+3}+69f_{n+4}-3f_{n+5}\right) \\
y_{n+5}-y_{n+1} &= \tfrac{h}{45}\left(14f_{n+1}+64f_{n+2}+24f_{n+3}+64f_{n+4}+14f_{n+5}\right)
\end{aligned}
\tag{12}
$$

The main new discrete scheme for the five step method is

$$
y_{n+2}-y_{n+1}=\tfrac{h}{1440}\left(-27f_n+637f_{n+1}+1022f_{n+2}-258f_{n+3}+77f_{n+4}-11f_{n+5}\right) \tag{13}
$$

This new method is consistent since its order is 6, its also zero-stable, above all, it is A – stable as can be seen in figure 1.
The new five step discrete methods (12) have the following orders and error constants as shown in Table 1.

3. Analysis of the New Methods
In this section, we consider the analysis of the newly constructed methods. Their convergence is determined and their regions of absolute stability are plotted.
3.1 Convergence
The convergence of the new block methods is determined using the approach by Fatunla (1991) and Chollom *et. al* (2007) for linear multistep methods, where the block methods are represented in a single block, r point multistep method of the form

$$
A^{(0)}y_{m+1}=\sum_{i=1}^{k}A^{(i)}y_{m+1}+h\sum_{i=0}^{k}B^{(i)}f_{m-1} \tag{14}
$$

where h is a fixed mesh size within a block, $A^i, B^i, i=0,1,2,\dots k$ are $r\times r$ identity matrix while y_m, y_{m-1} and y_{m+1} are vectors of numerical estimates.

Definition 1: A block method is zero stable provided the roots $R_{ij}=1,2,\dots k$ of the first characteristic polynomial $\rho(R)$ specified as

$$
\rho(R)=\det\left[\sum_{i=0}^{k}A^{(i)}R^{k-i}\right]=0 \tag{15}
$$

satisfies $|R_j|\le 1$, the multiplicity must not exceed two. Fatunla (1994)

The block method (8) expressed in the form of (14) gives

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{pmatrix}=\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}\begin{pmatrix}y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix}+h\left[\begin{pmatrix}\frac{19}{24}&\frac{-5}{24}&\frac{1}{24}\\\frac{4}{3}&\frac{1}{3}&0\\\frac{9}{8}&\frac{9}{8}&\frac{3}{8}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\end{pmatrix}+\begin{pmatrix}0&0&\frac{9}{24}\\0&0&\frac{1}{3}\\0&0&\frac{3}{8}\end{pmatrix}\begin{pmatrix}f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix}\right] \tag{16}$$

where

$$A^{(0)}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ A^{(1)}=\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix},B^{(0)}=\begin{pmatrix}\frac{19}{24}&\frac{-5}{24}&\frac{1}{24}\\\frac{4}{3}&\frac{1}{3}&0\\\frac{9}{8}&\frac{9}{8}&\frac{3}{8}\end{pmatrix},\ B^{(1)}=\begin{pmatrix}0&0&\frac{9}{24}\\0&0&\frac{1}{3}\\0&0&\frac{3}{8}\end{pmatrix}$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (15) gives the characteristic polynomial of the block method (8) as

$$\rho(\lambda)=\det(\lambda A^{(0)}-A^{(1)})$$

$$=\det\left[\lambda\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}-\begin{pmatrix}0&0&1\\0&0&1\\0&0&1\end{pmatrix}\right]$$

$$=\det\begin{bmatrix}\lambda&0&-1\\0&\lambda&-1\\0&0&\lambda-1\end{bmatrix}$$

$$=\lambda^2(\lambda-1)=0$$

Therefore, $\lambda_1=1,\lambda_2=\lambda_3=0$. The block method (8) by definition1 is zero stable and by Henrici (1962), the block method is convergent since it is also consistent.

Similarly, the block method (12) expressed in the form of (14) gives

$$\begin{pmatrix}1&0&0&0&0\\-1&1&0&0&0\\-1&0&1&0&0\\-1&0&0&1&0\\-1&0&0&0&1\end{pmatrix}\begin{pmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\end{pmatrix}=\begin{pmatrix}0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_{n}\end{pmatrix}+$$

$$h\left[\begin{pmatrix}\frac{1427}{1440}&\frac{-798}{1440}&\frac{482}{1440}&\frac{-173}{1440}&\frac{27}{1440}\\\frac{637}{1440}&\frac{1022}{1440}&\frac{-258}{1440}&\frac{77}{1440}&\frac{-11}{1440}\\\frac{34}{90}&\frac{114}{90}&\frac{34}{90}&\frac{-1}{90}&0\\\frac{69}{160}&\frac{174}{160}&\frac{174}{160}&\frac{69}{160}&\frac{-3}{160}\\\frac{14}{45}&\frac{64}{45}&\frac{24}{45}&\frac{64}{45}&\frac{14}{45}\end{pmatrix}\begin{pmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\end{pmatrix}+\begin{pmatrix}0&0&0&0&\frac{475}{1440}\\0&0&0&0&\frac{-27}{1440}\\0&0&0&0&\frac{-1}{90}\\0&0&0&0&\frac{-3}{160}\\0&0&0&0&0\end{pmatrix}\begin{pmatrix}f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_{n}\end{pmatrix}\right] \tag{17}$$

where

$$A^{(0)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix}, \; A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} \frac{1427}{1440} & \frac{-798}{1440} & \frac{482}{1440} & \frac{-173}{1440} & \frac{27}{1440} \\ \frac{637}{1440} & \frac{1022}{1440} & \frac{-258}{1440} & \frac{77}{1440} & \frac{-11}{1440} \\ \frac{34}{90} & \frac{114}{90} & \frac{34}{90} & \frac{-1}{90} & 0 \\ \frac{69}{160} & \frac{174}{160} & \frac{174}{160} & \frac{69}{160} & \frac{-3}{160} \\ \frac{14}{45} & \frac{64}{45} & \frac{24}{45} & \frac{64}{45} & \frac{14}{45} \end{pmatrix}, \; B^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{475}{1440} \\ 0 & 0 & 0 & 0 & \frac{-27}{1440} \\ 0 & 0 & 0 & 0 & \frac{-1}{90} \\ 0 & 0 & 0 & 0 & \frac{-3}{160} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Substituting $A^{(0)}$ and $A^{(1)}$ into (15) gives the characteristic polynomial of the block method (12) as

$$\rho(\lambda) = \det(\lambda A^{(0)} - A^{(1)})$$

$$= \det\left[\lambda \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= \det \begin{bmatrix} \lambda & 0 & 0 & 0 & -1 \\ -\lambda & \lambda & 0 & 0 & 0 \\ -\lambda & 0 & \lambda & 0 & 0 \\ -\lambda & 0 & 0 & \lambda & 0 \\ -\lambda & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$= \lambda^5 - \lambda^4 = 0$$

Therefore, $\lambda_1 = 1, \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 0$. The block method (12) by definition1 is zero stable and by Henrici (1962), it is convergent since it is also consistent.

3.2 Regions of Absolute Stability of the Methods

The absolute stability regions of the newly constructed block methods (8) and (12) are plotted using Chollom (2005) by reformulating the methods as general linear methods. The regions of absolute stability of the methods of the main discrete schemes and the new block methods are as shown in the figures 1 and 2 below. These absolute stability regions are all A –stable since they consist of the set of points in the complex plane outside the enclosed figures.

4 Numerical Examples

We report here a few numerical examples on some stiff problems taken from the literature. For comparisons, we also report the performance of the new block methods and the well-known Matlab stiff ODE solver ODE23S on the same problems.

Problem 1: We consider the Robertson's problem

$$
\begin{aligned}
y'_1 &= -0.04y_1 + 10000y_2y_3 && y_1(0) = 1 \\
y'_2 &= 0.04y_1 - 10000y_2y_3 - 30000000y_2^2 && y_2(0) = 0 \\
y'_3 &= 30000000y_2^2 && y_3(0) = 0 \\
&0 \le x \le 70, \quad h = 0.1
\end{aligned}
$$

Problem 2: We consider the Van der Pol's equation

$$
\begin{aligned}
y'_1 &= y_2 && y_1(0) = 2 \\
y'_2 &= -y_1 + \mu y_2(1 - y_1^2)\,, && y_2(0) = 0 \\
&0 \le x \le 70, \quad h = 0.01, \quad \mu = 10,
\end{aligned}
$$

5. Conclusion

The new block methods were applied to two well-known stiff problems from the literature, the numerical results suggest that the proposed new block methods (8) and (12) are suitable for solving stiff problems and perform competitively with the well-known ODE23s. This success is achieved because of the good stability properties of the proposed new block methods.

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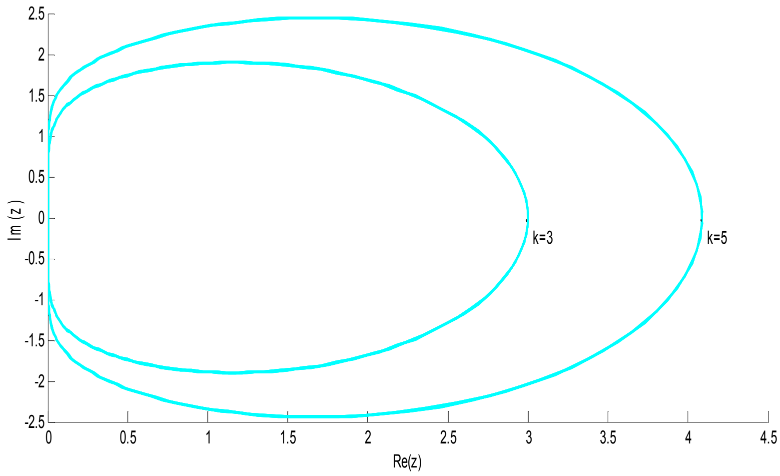

Figure2. Absolute Stability Regions of the new Three and Five step discrete methods

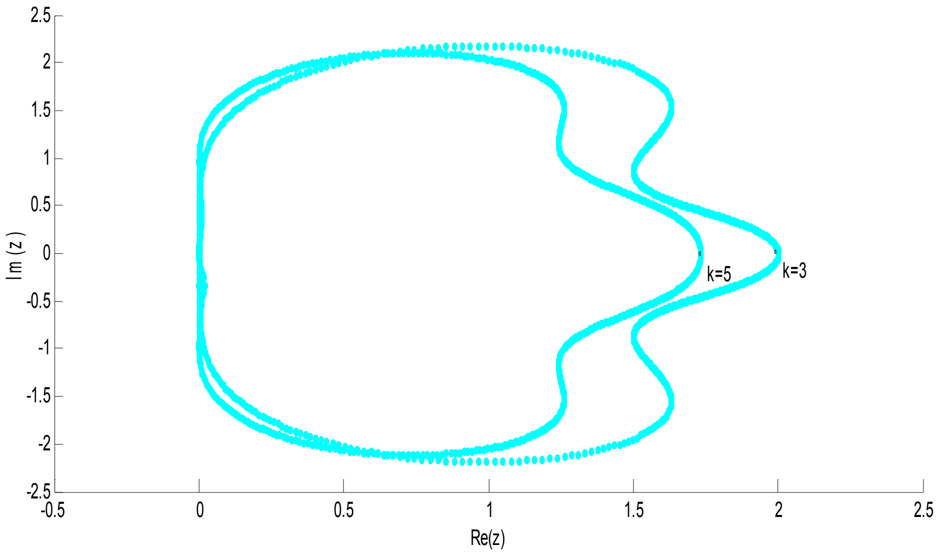

Figure1. Absolute Stability Regions of the Three and Five step New block methods

The absolute stability regions consists of the set of points in the complex plane outside the enclosed figures. Therefore, both the discrete and block methods are all A - stable since the left –half complex plane is contained in S., where $S = \{z \in C : R(z) \leq 1\}$.

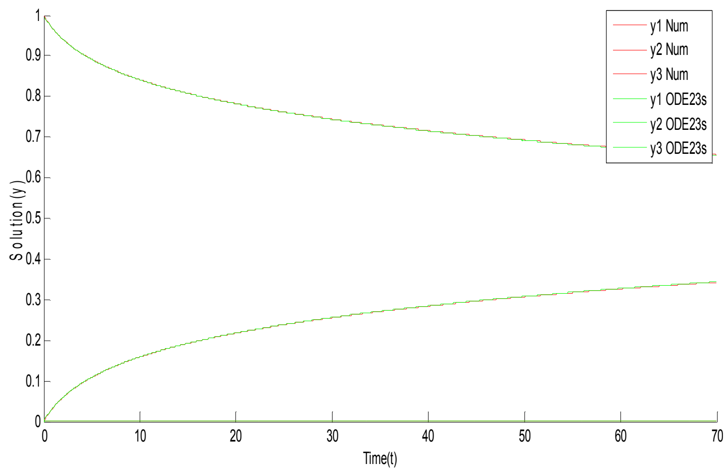

Figure 1. Solution to Problem 1 using the Three Step Block Method and ODE23S

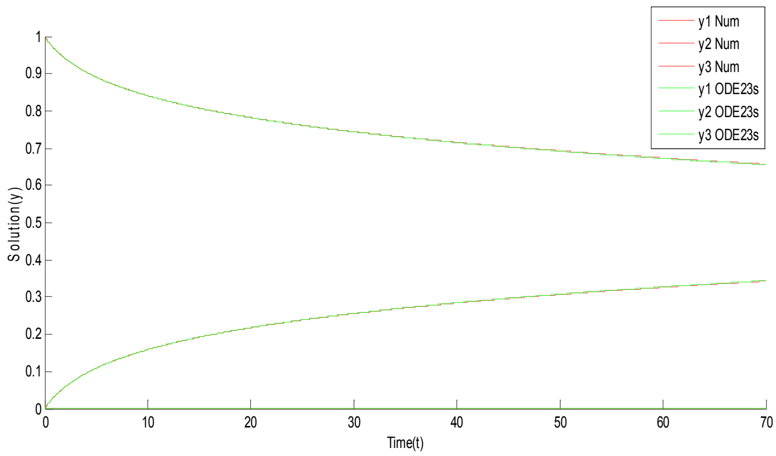

Figure 2. Solution to Problem 1 using the Five Step Block Method and ODE23S

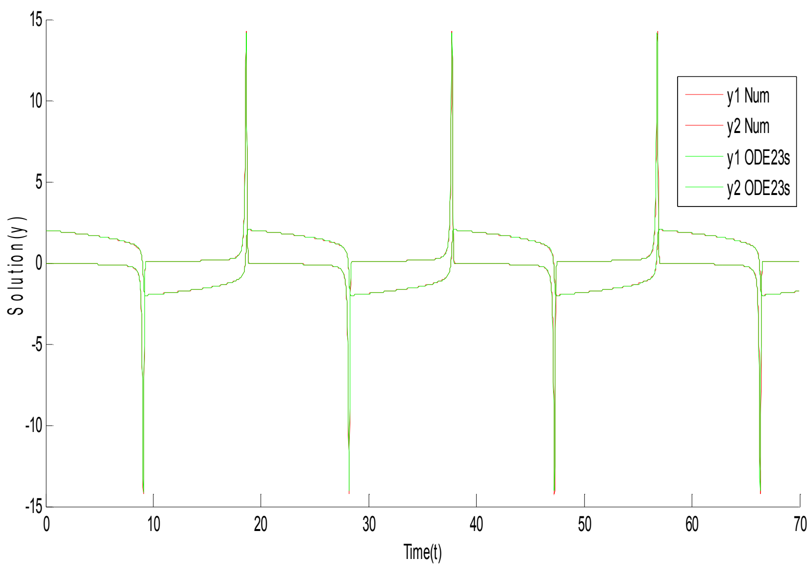

Figure 3. Solution to Problem 2 using the Three Step Block Method and ODE23S

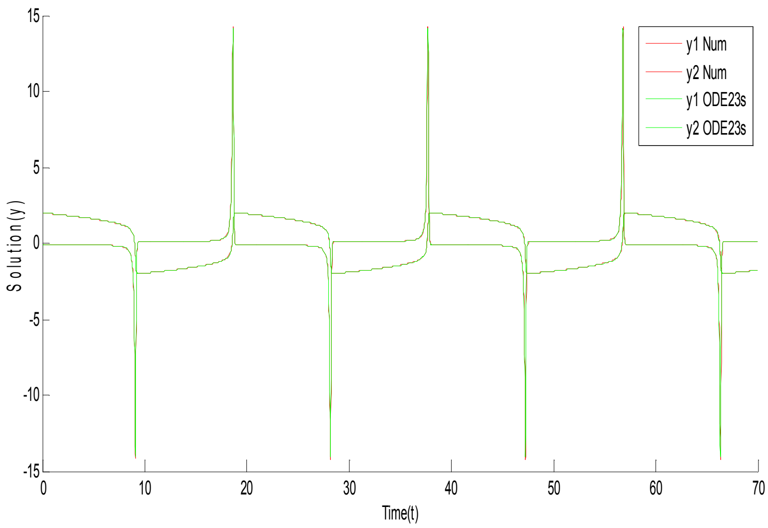

Figure 4. Solution to Problem 2 using the Five Step Block Method and ODE23S

Table 1: Order and Error Constants of the new three step method

Method	Order	Error constant
y_{n+1}	4	$-\frac{19}{720}$
y_{n+2}	4	$-\frac{1}{90}$
y_{n+3}	4	$-\frac{3}{80}$

Table 2: Order and Error Constant of the new five step method

Method	Order	Error constant
y_{n+1}	6	$-\frac{863}{60480}$
y_{n+2}	6	$\frac{271}{60480}$
y_{n+3}	6	$\frac{1}{756}$
y_{n+4}	6	$\frac{13}{2240}$
y_{n+5}	6	$-\frac{8}{945}$