An Investigation of Inference of the Generalized Extreme Value Distribution Based on Record

Omar I. O. Ibrahim[1], Adel O. Y. Mohamed[1], Mohamed A. El-Sayed[2] and Azhari A. Elhag[3]
[1]Mathematics Department, Faculty of Science, Taif University, Ranyah, Saudi Arabia.
[2]Mathematics Department, Faculty of Science, Fayoum University, 63514, Egypt,
Assistant Prof. of Computer Science, Taif University,21974,Taif, Saudi Arabia.
[3]Mathematics Department, Faculty of Science, Taif University, Taif, Saudi Arabia.
omer68_jawa@yahoo.com, drmasayed@yahoo.com

Abstract

In this article, the maximum likelihood and Bayes estimates of the generalized extreme value distribution based on record values are investigated. The asymptotic confidence intervals as well as bootstrap confidence are proposed. The Bayes estimators cannot be obtained in closed form so the MCMC method are used to calculate Bayes estimates as well as the credible intervals. A numerical example is provided to illustrate the proposed estimation methods developed here.

Keywords: Generalized extreme value distribution, Record values, Maximum likelihood estimation, Bayesian estimation.

1. Introduction

Record values arise naturally in many real life applications involving data relating to sport, weather and life testing studies. Many authors have been studied record values and associated statistics, for example, Ahsanullah ([1], [2], [3]), Arnold and Balakrishnan [4], Arnold, et al. ([5], [6]), Balakrishnan and Chan ([7], [8]) and David [9]. Also, these studies attracted a lot of attention see papers Chandler [10], Galambos [11].

In general, the joint probability density function (*pdf*) of the first m lower record values $X_{L(1)}, X_{L(2)},..., X_{L(m)}$ is given by

$$f_{X_{L(1)},X_{L(2)},...,X_{L(m)}}(x_1, x_2,..., x_m) = f(X_{L(m)})\prod_{i=1}^{m}\frac{f(X_{L(i)})}{F(X_{L(i)})} \tag{1}$$

The extreme value theory is blend of an enormous variety of applications involving natural phenomena such as rainfall, foods, wind gusts, air population and corrosion, and delicate mathematical results on point processes and regularly varying functions. Frechet [12] and Fisher [13] publishing result of an independent inquiry into the same problem. The Extreme lower bound distribution is a kind of general extreme value (the Gumbel-type I, extreme lower bound [Frechet]-typeII and Weibull distribution type III extreme value distributions). The applications of the extreme lower bound [Frechet]-type II turns out to be the most important model for extreme events the domain of attraction condition for the Frechet takes on a particularly easy from. In probability theory and statistics, the generalized extreme value (GEV) distribution is a family of continuous probability distributions developed within extreme value theory to combine the Gumbel, Fréchet and Weibull families also known as type I, II and III extreme value distributions. By the extreme value theorem the GEV distribution is the limit distribution of properly normalized maxima of a sequence of independent and identically distributed random variables. So, the GEV distribution is used as an approximation to model the maxima of long (finite) sequences of random variables. In some fields of application the generalized extreme value distribution is known as the Fisher--Tippett distribution, named after R. A. Fisher and L. H. C. Tippett who recognized three function forms outlined below. However usage of this name is sometimes restricted to mean the special case of the Gumbel distribution. The (*pdf*) and (*cdf*) of x are given respectively:

$$f(x) = \frac{1}{\beta}\left\{1+\xi\left(\frac{x-\theta}{\beta}\right)\right\}^{-1-\frac{1}{\xi}} \exp\left(-\left[1+\zeta\,\frac{(x-\theta)}{\beta}\right]^{-\frac{1}{\xi}}\right) \tag{2}$$

and

$$F(x) = \exp\left(-\left[1+\zeta\,\frac{(x-\theta)}{\beta}\right]^{-\frac{1}{\xi}}\right) \tag{3}$$

for $1+\zeta(x-\theta)/\beta>0$, where θ is the location parameter, β is the scale parameter and ζ is the shape parameter.

1.1 Markov chain Monte Carlo techniques

MCMC methodology provides a useful tool for realistic statistical modelling (Gilks et al.[14]; Gamerman,[15]), and has become very popular for Bayesian computation in complex statistical models. Bayesian analysis requires integration over possibly high-dimensional probability distributions to make inferences about model parameters or to make predictions. MCMC is essentially Monte Carlo integration using Markov chains. The integration draws samples from the required distribution, and then forms sample averages to approximate expectations (see Geman and Geman, [16]; Metropolis et al., [17]; Hastings, [18]).

1.2 Gibbs sampler

The Gibbs sampling algorithm is one of the simplest Markov chain Monte Carlo algorithms. The paper by Gelfand and Smith [19] helped to demonstrate the value of the Gibbs algorithm for a range of problems in Bayesian analysis. Gibbs sampling is a MCMC scheme where the transition kernel is formed by the full conditional distributions.

Algorithm 1.1.

1 - Choose an arbitrary starting point $q^{(0)}=\left(q_1^{(0)},\ldots,q_d^{(0)}\right)$ for which $g\left(\theta^{(0)}\right)>0$.

2 - Obtain $q_1^{(t)}$ from conditional distribution $g\left(q_1 \mid q_2^{(t-1)},q_3^{(t-1)},\ldots,q_d^{(t-1)}\right)$.

3- Obtain $q_2^{(t)}$ from conditional distribution $g\left(q_2 \mid q_1^{(t)},q_3^{(t-1)},\ldots,q_1^{(t-1)}\right)$.

. . . .

4 - Obtain $q_d^{(t)}$ from conditional distribution $g\left(q_d \mid q_1^{(t)},q_2^{(t)},q_3^{(t)},\ldots,q_{d-1}^{(t)}\right)$.

5 - Repeat steps 2 - 4.

1.3 The Metropolis-Hastings algorithm

The Metropolis algorithm was originally introduced by Metropolis et. al [17]. Suppose that our goal is to draw samples from some distributions $h(q \mid x)=\nu g(q)$, where ν is the normalizing constant which may not be known or very difficult to compute. The Metropolis-Hastings (MH) algorithm provides a way of sampling from $h(q \mid x)$ without requiring us to know ν. Let $Q\left(q^{(b)} \mid q^{(a)}\right)$ be an arbitrary transition kernel: that is the probability of moving, or jumping, from current state $q^{(a)}$ to $q^{(b)}$. This is sometimes called the proposal distribution. The following algorithm will generate a sequence of the values $q^{(1)},q^{(2)}$, ... which form a Markov chain with stationary distribution given by $h(q \mid x)$.

Algorithm 1.2.

1- Choose an arbitrary starting point $q^{(0)}$ for which $h\left(q^{(0)} \mid x\right)>0$.

2- At time t, sample a candidate point or proposal, q^*, from $Q\left(q^* \mid q^{(t-1)}\right)$, the proposal distribution.

3- Calculate the acceptance probability

$$\rho\left(q^{(t-1)},q^*\right)=\min\left[1,\frac{h\left(q^* \mid x\right)Q\left(q^{(t-1)} \mid q^*\right)}{h\left(q^{(t-1)} \mid x\right)Q\left(q^* \mid q^{(t-1)}\right)}\right]. \quad (4)$$

4- Generate $U \sim U(0,1)$.

5- If $U \le \rho\left(q^{(t-1)},q^*\right)$ accept the proposal and set $q^{(t)}=q^*$. Otherwise, reject the proposal and set $q^{(t)}=q^{(t-1)}$

6- Repeat steps 2 - 5.

If the proposal distribution is symmetric, so $Q(q \mid \phi)=Q(\phi \mid q)$ for all possible ϕ and q then, in particular, we have $Q\left(q^{(t-1)} \mid q^*\right)=q\left(q^* \mid q^{(t-1)}\right)$, so that the acceptance probability (5) is given by:

$$\rho\left(q^{(t-1)},q^*\right)=\min\left[1,\frac{h\left(q^* \mid x\right)}{h\left(q^{(t-1)} \mid x\right)}\right] \quad (5)$$

In this paper is organized in the following order: In Section 2 the point estimation of the parameters of generalized extreme value distribution based on record value and bootstrap confidence intervals based are investigated. In Section 3, we cover Bayes estimates of parameters and construction of credible intervals using MCMC approach. An illustrative example involving simulated records is given in Section 4.

2. Maximum Likelihood Estimation

Let $X_{L(1)}$, $X_{L(2)}, \dots, X_{L(m)}$ be m lower record values each of which has the generalized extreme value whose the pdf and cdf are, respectively, given by (2) and (3). Based on those lower record values and for simplicity of notation, we will use x_i instead of $X_{L(i)}$. The logarithm of the likelihood function may then be written as:

$$\ell(\beta,\xi\,|\,\underline{x}) = -m\log\beta - \{1+\xi T_m\}^{-\frac{1}{\xi}} - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\log\{1+\xi T_i\}, \tag{6}$$

where $T_i(\beta) = (X_i - \theta)/\beta$ with known θ. Calculating the first partial derivatives of Eq. (6) with respect to β and ξ and equating each to zero, we get the likelihood equations as:

$$\frac{\partial\ell(\beta,\xi\,|\,\underline{x})}{\partial\beta} = -\frac{m}{\beta} - \frac{T_m}{\beta}(1+\xi T_m)^{-\frac{1}{\xi}-1} + \frac{1+\xi}{\beta}\sum_{i=1}^{m}\frac{T_i}{1+\xi T_i} = 0, \tag{7}$$

and

$$\begin{aligned}\frac{\partial\ell(\beta,\xi\,|\,\underline{x})}{\partial\xi} &= -\frac{(1+\xi T_m)^{-\frac{1}{\xi}}}{\xi^2}\log(1+\xi T_m) + \frac{T_m}{\xi}(1+\xi T_m)^{-\frac{1}{\xi}-1} \\ &+ \frac{1}{\xi^2}\sum_{i=1}^{m}\log(1+\xi T_i) - \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{T_i}{1+\xi T_i}.\end{aligned} \tag{8}$$

By solving the two nonlinear equations (7) and (8) numerically, we obtain the estimates for the parameters ϵ and ξ, say $\hat{\beta}$ and $\hat{\xi}$.

Records are rare in practice and sample sizes are often very small, therefore, intervals based on the asymptotic normality of MLEs do not perform well. So two confidence intervals based on the parametric bootstrap and MCMC methods are proposed.

2.1 Approximate Interval Estimation

If sample sizes are not small. The Fisher information matrix $I(\beta,\zeta)$ is then obtained by taking expectation of minus of the second derivatives of the logarithm likelihood function. Under some mild regularity conditions, $(\hat{\beta},\hat{\zeta})$ is approximately bivariately normal with mean (β,ζ) and covariance matrix $I^{-1}(\beta,\zeta)$. In practice, we usually estimate $I^{-1}(\beta,\zeta)$ by $I^{-1}(\hat{\beta},\hat{\zeta})$. A simpler and equally veiled procedure is to use the approximation

$$(\hat{\beta},\hat{\zeta}) \sim N\left((\beta,\zeta), I_0^{-1}(\hat{\beta},\hat{\zeta})\right), \tag{9}$$

where $I_0(\beta,\zeta)$ is observed information matrix given by

$$I_0(\hat{\beta},\hat{\zeta}) = \begin{bmatrix} -\dfrac{\partial^2\ell(\beta,\xi|\underline{x})}{\partial\beta^2} & -\dfrac{\partial^2\ell(\beta,\xi|\underline{x})}{\partial\beta\partial\zeta} \\ -\dfrac{\partial^2\ell(\beta,\xi|\underline{x})}{\partial\beta\partial\zeta} & -\dfrac{\partial^2\ell(\beta,\xi|\underline{x})}{\partial\zeta^2} \end{bmatrix}_{(\hat{\beta},\hat{\zeta})} \tag{10}$$

where the elements of the Fisher information matrix are given by

$$\frac{\partial^2 \ell(\beta,\xi\,|\,\underline{x})}{\partial\beta^2} = \frac{m}{\beta^2} + \frac{2T_m}{\beta^2}(1+\xi T_m)^{-\frac{1}{\xi}-1} + \frac{(\xi+1)T_m{}^2}{\beta}(1+\xi T_m)^{-\frac{1}{\xi}-2}$$
$$-\frac{(1+\xi)}{\beta^2}\sum_{i=1}^{m}\frac{T_i}{1+\xi T_i} - \frac{\xi(1+\xi)}{\beta^2}\sum_{i=1}^{m}\frac{T_i(1+\xi(T_i-1))}{(1+\xi T_i)^2}, \tag{11}$$

$$\frac{\partial^2 \ell(\beta,\xi\,|\,\underline{x})}{\partial\xi^2} = -\left\{\frac{1}{\zeta^4}\log(1+\xi T_m) - \frac{1}{\xi^2}\left(1+\frac{1}{\xi}\right)\frac{T_m}{(1+\xi T_m)} - \frac{2}{\zeta^3}\right\}(1+\xi T_m)^{-\frac{1}{\xi}}$$
$$-\left\{\frac{T_m}{\zeta^2} + \frac{1}{\zeta}\left(1+\frac{1}{\xi}\right)\frac{T_m^2}{1+\xi T_m} - \frac{T_m}{\zeta^3}\log(1+\xi T_m)\right\}(1+\xi T_m)^{-\frac{1}{\xi}-1} \tag{12}$$
$$-\frac{2}{\xi^3}\sum_{i=1}^{m}\log(1+\xi T_i) + \frac{2}{\xi^2}\sum_{i=1}^{m}\frac{T_i}{1+\xi T_i} + \left(1+\frac{1}{\xi}\right)\sum_{i=1}^{m}\frac{T_i^2}{1+\xi T_i},$$

and

$$\frac{\partial^2 \ell(\beta,\xi\,|\,\underline{x})}{\partial\xi\partial\beta} = \frac{T_m}{\beta\zeta}\left\{\frac{1}{\xi}\log(1+\xi T_m)+1\right\}(1+\xi T_m)^{-\frac{1}{\xi}-1}.$$
$$+\frac{T_m}{\beta}\left\{\frac{-1}{\xi} + \left(1+\frac{1}{\xi}\right)\frac{T_i}{1+\xi T_i}\right\}(1+\xi T_m)^{-\frac{1}{\xi}-1} \tag{13}$$
$$-\frac{1}{\beta}\sum_{i=1}^{m}\frac{T_i}{1+\xi T_i} + \frac{\left(1+\frac{1}{\xi}\right)}{\beta}\sum_{i=1}^{m}\frac{T_i}{(1+\xi T_i)^2}.$$

Approximate confidence intervals for β and ζ can be found by to be bivariately normal distributed with mean (β,ζ) and covariance matrix $I_0^{-1}(\hat{\beta},\hat{\zeta})$. Thus, the $100(1-\alpha)\%$ approximate confidence intervals for β and ζ are:

$$(\hat{\beta}-z_{\frac{\alpha}{2}}\sqrt{v_{11}},\hat{\beta}+z_{\frac{\alpha}{2}}\sqrt{v_{11}}) \text{ and } (\hat{\zeta}-z_{\frac{\alpha}{2}}\sqrt{v_{22}},\hat{\zeta}+z_{\frac{\alpha}{2}}\sqrt{v_{22}}) \tag{14}$$

respectively, where v_{11} and v_{22} are the elements on the main diagonal of the covariance matrix $I_0^{-1}(\hat{\beta},\hat{\zeta})$ and $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\alpha}{2}$.

2.2 Bootstrap confidence intervals

In this section, we propose to use percentile bootstrap method based on the original idea of Efron [20]. The algorithm for estimating the confidence intervals of β and ζ using this method are illustrated below.

1- From the original sample of lower records $\underline{x}$, compute ML estimates $\hat{\beta}$ and $\hat{\zeta}$.

2- Use $\hat{\beta}$ and $\hat{\zeta}$ to generate bootstrap records sample $\{x^*_{L(1)}, x^*_{L(2)},\ldots, x^*_{L(n)}\}$. Use these data to compute the bootstrap estimate $\hat{\beta}^*$ and $\hat{\zeta}^*$.

3- Repeat step 2 , N boot times.

4- Bootstrap estimates $\hat{\beta}_{\text{Boot}} = \frac{1}{N}\sum_{i=1}^{N}\hat{\beta}^{*(i)}$ and $\hat{\zeta}_{\text{Boot}} = \frac{1}{N}\sum_{i=1}^{N}\hat{\zeta}^{*(i)}$.

5- Let $G(x) = P(\hat{\beta}^* \le x)$, be the cumulative distribution of $\hat{\beta}^*$. Define $\hat{\beta}_{Boot} = G^{-1}(x)$ for a given x. The approximate $100(1-2\alpha)\%$ confidence interval of β is given by

$$(\hat{\beta}_{\text{Boot}}(\gamma), \hat{\beta}_{\text{Boot}}(1-\gamma)). \tag{15}$$

Similarly,

$$(\hat{\zeta}_{\text{Boot}}(\gamma), \hat{\zeta}_{\text{Boot}}(1-\gamma)). \tag{16}$$

3. Bayesian Estimation

In this section, we are in a position to consider the Bayesian estimation of the parameters β and ζ for record data, under the assumption that the parameter θ is known. We may consider the joint prior density as a product of independent gamma distribution $\pi_1^*(\beta)$ and $\pi_2^*(\zeta)$, given by

$$\pi_1^*(\beta) \propto \beta^{a-1} \exp(-b\beta), \quad a, b > 0, \tag{17}$$

and

$$\pi_2^*(\zeta) \propto \zeta^{c-1} \exp(-d\zeta), \quad a, b > 0. \tag{18}$$

By using the joint prior distribution of β and ζ and likelihood function, the joint posterior density function of β and ζ given the data, denoted by $\pi(\beta, \zeta \mid \underline{x})$, can be written as

$$\pi(\beta, \zeta \mid \underline{x}) \propto \beta^{a-m-1} \zeta^{c-1} \prod_{i=1}^{m} \left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]^{-1-\frac{1}{\zeta}} \exp\left(-b\beta - d\zeta - \left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]^{\frac{-1}{\zeta}}\right). \tag{19}$$

As expected in this case, the Bayes estimators can not be obtained in closed form. We propose to use the Gibbs sampling procedure to generate MCMC samples, we obtain the Bayes estimates and the corresponding credible intervals of the unknown parameters. A wide variety of MCMC schemes are available, and it can be difficult to choose among them. An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis-within-Gibbs samplers.

It is clear that the posterior density function of β given ζ is

$$\pi_1(\beta \mid \zeta) \propto \beta^{a-m-1} \exp\left(-b\beta - \left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]^{\frac{-1}{\zeta}} - (1 + \frac{1}{\zeta}) \sum_{i=1}^{m} \log\left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]\right), \tag{20}$$

and the posterior density function of ζ given β can be written as

$$\pi_2(\zeta \mid \beta) \propto \zeta^{c-1} \exp\left(-d\zeta - \left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]^{\frac{-1}{\zeta}} - (1 + \frac{1}{\zeta}) \sum_{i=1}^{m} \log\left[1 + \zeta \frac{(x_{L(m)} - \theta)}{\beta}\right]\right). \tag{21}$$

The plots of them show that they are similar to normal distribution. So to generate random numbers from these distributions, we use the Metropolis-Hastings method with normal proposal distribution. Therefore the algorithm of Gibbs sampling procedure as the following algorithm:

algorithm 3.1

1. Set $\beta^{(0)} = \hat{\beta}$ and $\zeta^{(0)} = \hat{\zeta}$ and $M =$ burn-in.
2. Set $t = 1$.
3. Generate $\beta^{(t)}$ from $\pi_1(\beta \mid \zeta^{(t-1)})$ using MH algorithm with the $N(\beta^{(t-1)}, \sigma_1)$ proposal distribution.
4. Generate $\zeta^{(t)}$ from $\pi_2(\zeta \mid \beta^{(t)})$ using MH algorithm with the $N(\zeta^{(t-1)}, \sigma_2)$ proposal distribution.

5. Set $t = t + 1$.
6. Repeat 2-5 and obtain ($\beta^{(1)}, \zeta^{(1)}$), ($\beta^{(2)}, \zeta^{(2)}$), ..., ($\beta^{(N)}, \zeta^{(N)}$).
7. An approximate Bayes estimate of any function $g(\beta, \lambda)$ under a SE loss function can be obtained as

$$\tilde{g} = \frac{1}{N-M} \sum_{i=M+1}^{N} g(\beta^{(i)}, \zeta^{(i)}). \tag{22}$$

8. To compute the credible intervals of β and ζ , order $\beta_{M+1}, \ldots, \beta_N$ and $\zeta_{M+1}, \ldots, \zeta_N$ as $\beta_{(1)}, \ldots, \beta_{(N-M)}$ and $\zeta_{(1)}, \ldots, \zeta_{(N-M)}$. Then the $100(1-2\alpha)\%$ symmetric credible intervals $(\beta_{(\alpha(N-M))}, \beta_{((1-\alpha)(N-M))})$ and $(\zeta_{(\alpha(N-M))}, \zeta_{((1-\alpha)(N-M))})$.

4. Data Analysis

Now, we describe choosing the true values of parameters β and ζ with known prior. For given $(a = 4, b = 2)$ generate random sample of size 100, from gamma distribution, then the mean of the random sample $\beta \cong \frac{1}{100} \sum_{i=1}^{100} \beta_j$, can be computed and considered as the actual population value of $\beta = 1.9$. That is, the prior parameters are selected to satisfy $E(X) = \frac{a}{b} \cong \beta$ is approximately the mean of gamma distribution. Also for given values $(c = 3, d = 2)$, generate according the last $\zeta = 1.4,$ from gamma distribution. The prior parameters are selected to satisfy $E(X) = \frac{c}{d} \cong \zeta$ is approximately the mean of gamma distribution. By using ($\beta = 1.9,$ and $\zeta = \theta = 1.4$), we generate lower record value data from generalized extreme lower bound distribution the simulate data set with $m = 7$, given by: 29.7646, 4.9186, 3.8447, 2.5929, 2.3330, 2.2460, 2.2348

Under this data we compute the approximate MLEs, bootstrap and Bayes estimates of β and ζ using MCMC method, the MCMC samples of size 10000 with 1000 as ' burn-in'. The results of point estimation are displayed in Table 1 and results of interval estimation given in Table 2.

Table 1. The point estimates of parameters β and ζ with $\theta = 3.5$.

Method	β	ζ
MLE	1.95996	1.64485
Boot	2.01135	1.85243
Bayes	1.68060	1.07709

Table 2: Two-sided 95% confidence intervals and it is length of β and ζ

Method	β	Length	ζ	Length
MLE	(-0.81593, 8.04907)	8.8650	(-0.717938, 6.39967)	7.1176
Boot	(1.32546, 4.54325)	3.2178	(0.12548, 3.45781)	3.3323
Bayes	(1.19938, 2.26063)	1.0613	(0.582031, 1.58285)	1.0008

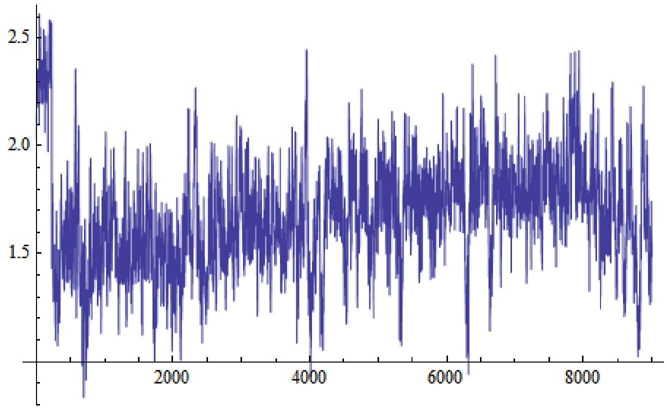

Figure 1. Simulation number of β generated by MCMC method

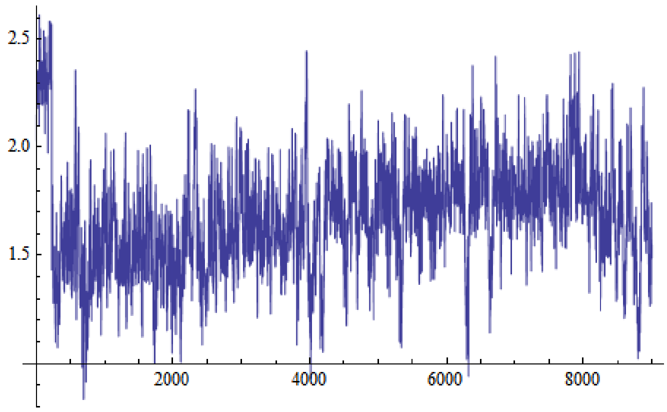

Figure 2. Simulation number of ζ generated by MCMC method

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