

Oscillatory Flow and Particle Suspension in a Fluid Through an Elastic Tube

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Abstract
Womersley gave a solution for the case of a thin-walled elastic tube, it being assumed that the effect of the inertia term in the equations of viscous fluid motion can be neglected. He did not consider the presence of particles, to account for the blood cells in the blood, within the viscous flow through the tube (artery).
In this paper, the corresponding solution for an oscillatory flow and particle suspension in a fluid (blood), to account for blood cells, through an elastic tube is obtained. This solution is the frequency equation as it was obtained by Womersley but it has a different structure. If the volume fraction particle density φ, is removed from this solution it collapses to give the same equation as Womersley's case, without particles.
Keywords: Oscillatory flow, Particle suspension, Elastic tube, Periodic function

1. Introduction:

The problem of blood flow and wave propagation in the arterial system has stimulated the interest of physiologists and mathematicians for years. Its fundamental importance for present day research can be traced to Witzig and later to the works of Womersley [10]. The contributions of Womersley [10] represent the best attempt to date in developing a complete, practical, unified theory of arterial blood flow and pressure propagation [6].
A survey of the published literature on the propagation of waves in the arterial system will show a large variation of assumptions on the nature of flow conditions, fluid properties, and types of vessel walls. The analytical solutions for pulsatile flow in rigid tubes are therefore based on a combination of different assumptions, just the same as those of oscillatory flow through elastic tube.
The model used in this paper consists of three phases as against two phases by Womersley. The three phases are the fluid phase, particulate phase and the equations of motion of the tube.
According to Womersley, the simple solution for the oscillatory motion of viscous liquid in a rigid tube, under a simple-harmonic pressure gradient, was given by Lambossay who gave the formula for velocity and viscous drag solely concerned with the effect of the viscous drag on the frequency response of pressure recording instruments. Womersley obtained the same result independently, in a different form and derived the expression for the rate of flow.
For the equations of the elastic tube, we adopt Oslen J.W et al [7] and Womersley [10]. All the assumptions by Womersley are adopted and applied in this paper alongside his method of solution. The only variation between Womersley and this approach is the introduction of the particulate phase, with φ, as the volume fraction particle density, so that we now simulate blood properly, with proper consideration to the existence of blood cells, as the particles.

2. Methods

2.1 Assumptions and Nomenclature

(u_f, w_f) denote fluid phase velocities, (u_p, w_p) denote particulate phase velocities, ρ_f and, ρ_p are the actual densities of the materials constituting fluid and particulate phase respectively, $(1-\varphi)\rho_f$ is the fluid phase density, $\varphi\rho_p$ the particulate phase density, P denotes the pressure, φ denotes the volume fraction density of the particles, $\mu\varphi$ is the particle fluid mixture viscosity and S is the drag coefficient of interaction for the force exerted by one phase on the other. Inertia terms of the equations of motion are neglected, diffusivity terms are also neglected. φ is chosen as constant, and we assume that the pressure wave is harmonic in time having frequency n and wave velocity c.

2.2 Formulation of the Model

A porous tapered elastic tube filled with a viscous fluid is considered as a model of a vascular bed in which successive branching of the blood cells leads to a rapid decrease in diameter with distance. The porosity of the tapered tube is adjusted to simulate the effect of branching
The motion of the fluid is assumed to be cycisymetric and governed by the Navier-Stokes equations for an incompressible fluid, now modified to simulate the effect of particle suspension as follows:

$$\rho_f(1-\varphi)[\frac{\partial u_f}{\partial t} + u_f\frac{\partial u_f}{\partial z} + w_f\frac{\partial w_f}{\partial r}] = -(1-\varphi)\frac{\partial p}{\partial z} + (1-\varphi)\mu[\frac{\partial^2 u_f}{\partial r^2} + \frac{1}{r}\frac{\partial u_f}{\partial r} + \frac{\partial^2 u_f}{\partial z^2}] + \varphi s(u_p - u_f) \quad (1)$$

$$\rho_f(1-\varphi)[\frac{\partial w_f}{\partial z} + w_f\frac{\partial w_f}{\partial r}] = -(1-\varphi)\frac{\partial p}{\partial r} + (1-\varphi)\mu[\frac{\partial^2 w_f}{\partial r^2} + \frac{1}{r}\frac{\partial w_f}{\partial r} - \frac{w_f}{r^2} + \frac{\partial^2 w_f}{\partial z^2}] + \varphi s(w_p - u_f) \quad (2)$$

With equation of continuity as,

$$\frac{\partial}{\partial r}[(1-\varphi)u_f] + \frac{\partial}{\partial z}[(1-\varphi)w_f] + \frac{(1-\varphi)u_f}{r} = 0, \quad (3)$$

where u_f *and* w_f are the axial and radial components of velocity, P is the pressure and ρ_f and μ are the density and viscosity of the fluid respectively.

The problem of determining the motion of a liquid in an elastic tube when subjected to a pressure – gradient which is a periodic function of the time arises in connection with the flow of blood in the larger arteries. The equations of motion of the tube from [6] are:

$$\frac{\partial^2\alpha}{\partial t^2} = \frac{-\rho_0}{\rho_f}\frac{\alpha}{hR}[\frac{\partial w_f}{\partial y} + \frac{R\partial u_f}{\partial z}]_{y=1} + \frac{B}{\rho_f}[\frac{\partial^2\alpha}{\partial z^2} + \frac{\sigma}{R}\frac{\partial\lambda}{\partial z}] \quad (4)$$

$$\frac{\partial^2\lambda}{\partial t^2} = \frac{P}{h\rho_f} - \frac{\beta}{\rho_f}\left[\frac{\sigma}{R}\frac{\partial\alpha}{\partial Z} + \frac{\lambda}{R^2}\right] \quad (5)$$

Together with the boundary conditions for the motion of the fluid,

$$\left.\begin{aligned} u_f &= \frac{\partial\lambda}{\partial t} \ at\ y = 1 \\ w_f &= \frac{\partial\alpha}{\partial t} \ at\ y = 1 \end{aligned}\right\} \quad (6)$$

The particulate phase: The equations of motion of the particles are:

$$\rho_p\varphi[\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial r} + w_p\frac{\partial u_p}{\partial z}] = -\varphi\frac{\partial p}{\partial r} + \varphi s(u_f - u_p) \quad (7)$$

$$\rho_p[\frac{\partial w_p}{\partial t} + u_p\frac{\partial w_p}{\partial r} + w_p\frac{\partial w_p}{\partial z}] = -\varphi\frac{\partial p}{\partial z} + \varphi s(w_f - w_p) \quad (8)$$

and their equation of continuity is:

$$\frac{\partial}{\partial r}[\varphi u_p] + \frac{\partial}{\partial t}[\varphi w_p] + \frac{\varphi u_p}{r} = 0 \quad (9)$$

With boundary conditions

$$\left.\begin{aligned} u_f &= \frac{\partial\lambda}{\partial t} \ at\ y = 1 \\ w_f &= \frac{\partial\alpha}{\partial t} \ at\ y = 1 \end{aligned}\right\} \quad (10)$$

3. Solution: The expression for the drag coefficient for the present study is selected as,

$s = \frac{q}{2}\frac{\mu_0}{a^2}\lambda'(\psi)$, where $\lambda'(\psi) = \frac{4+3[84-3\varphi^2]^{\frac{1}{2}}}{[2-3\varphi]^2} + 3\varphi$

Where μ_0 is the fluid viscosity, and 'a' is the radius of the particle. Relation where $\lambda'(\psi)$ represents the classical Stokes' drag for small particle Reynolds number, modified to account for the finite particulate fractional volume through the function $\lambda'(\psi)$, obtained by [9].

In order to investigate the Pulsatile flow along the axis of the tube, we assume that the pressure wave is harmonic in time having frequency n and wave velocity c. We therefore assume that,

$$(P, u_f, w_f, u_p, w_p) = [p_1(r), u_1(r), w_1(r), u_2(r), w_2(r)] \exp[in(t - \frac{z}{c})] \quad (11)$$

We shall be concerned with the motion in which

$\frac{u_f}{c}, \frac{w_f}{c}, \frac{u_p}{c}, \frac{w_p}{c}, \frac{nR}{c}$ are small, i.e when the wave velocity, c is very large, as compared to $u_f, w_f, u_p, w_p,$ and nR.

From (11) we can write

$$\left.\begin{aligned} P &= p_1(r)\exp[in(t - \tfrac{z}{c})] \\ U_f &= u_1(r)\exp[in(t - \tfrac{z}{c})] \\ W_f &= w_1(r)\exp[in(t - \tfrac{z}{c})] \\ U_p &= u_2(r)\exp[in(t - \tfrac{z}{c})] \\ W_p &= w_2(r)\exp[in(t - \tfrac{z}{c})] \end{aligned}\right\} \quad (12)$$

$$\left.\begin{aligned} \frac{\partial u_f}{\partial t} &= u_1 in \exp[in(t - \tfrac{z}{c})] \\ \frac{\partial u_f}{\partial x} &= 0 \\ \frac{\partial u_f}{\partial z} &= -\frac{u_1}{c}\exp[in(t - \tfrac{z}{c})] \\ \frac{\partial p}{\partial r} &= \frac{\partial p_1}{\partial r}\exp[in(t - \tfrac{z}{c})] \\ \frac{\partial u_f}{\partial r} &= \frac{\partial u_1}{\partial r}\exp[in(t - \tfrac{z}{c})] \end{aligned}\right\} \quad (13)$$

$$\left.\begin{aligned} \frac{\partial^2 u_f}{\partial r^2} &= \frac{\partial^2 u_1}{\partial r^2}\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial w_f}{\partial t} &= w_1 in\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial p}{\partial z} &= -\frac{p_1 in}{c}\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial w_f}{\partial r} &= \frac{\partial w_1}{\partial r}\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial^2 w_f}{\partial r^2} &= \frac{\partial^2 w_1}{\partial r^2}\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial^2 u_f}{\partial z^2} &= \frac{u_1}{c^2}\exp[in(t-\tfrac{z}{c})] \\ \frac{\partial w_f}{\partial z} &= -\frac{1}{c}w_1 in\exp[in(t-\tfrac{z}{c})] \end{aligned}\right\} \quad (13 \text{ cont.})$$

Neglecting the inertia terms in (1), we can write equation (1) as

$$\rho_f(1-\varphi)\frac{\partial u_f}{\partial z} = -(1-\varphi)\frac{\partial p}{\partial z} + (1-\varphi)\mu[\frac{\partial^2 u_f}{\partial r^2} + \frac{1}{r}\frac{\partial u_f}{\partial r} + \frac{\partial^2 u_f}{\partial z^2}] + \varphi s(u_p - u_f) \quad (14)$$

Now substituting equations (12) and (13) into equation (14) and neglecting the diffusivity terms $\frac{\partial^2 u_f}{\partial z^2}$ we have, Equation (14) simplifies to

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - \frac{inR^2}{\alpha}u_1 = -\frac{R^2 P_1 in}{\mu c} - \frac{R^2\varphi S}{\mu(1-\varphi)}[u_2 - u_1] \quad (15)$$

$$\text{Let } \alpha = R(\frac{n}{\alpha})^{1/2} \Rightarrow \alpha^2 = \frac{R^2 n}{\alpha} \quad (16)$$

Substituting (16) in (15), we get

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - i\alpha^2 u_1 = -\frac{R^2 P_1 in}{\mu c} - \frac{R^2\varphi S}{\mu(1-\varphi)}[u_2 - u_1] \quad (17)$$

Similarly in equation (2), neglecting inertia terms, can be written as

$$\rho_f(1-\varphi)\frac{\partial w_f}{\partial z} = -(1-\varphi)\frac{\partial p}{\partial r} + (1-\varphi)\mu[\frac{\partial^2 w_f}{\partial r^2} + \frac{1}{r}\frac{\partial w_f}{\partial r} - \frac{w_f}{r^2} + \frac{\partial^2 w_f}{\partial z^2}] + \varphi s(w_p - u_f) \quad (18)$$

Now substituting equations (12) and (13) into (18), we get

$$w_1 in = -\frac{1}{\rho_f}\frac{\partial P_1}{\partial r} + \frac{\alpha\partial^2 w_1}{\partial r^2} + \frac{\alpha}{r}\frac{\partial w_1}{\partial r} - \alpha w_1 + \frac{\varphi S}{\rho_f(1-\varphi)}[w_2 - w_1] \quad (19)$$

Let $y = \frac{r}{R} \Rightarrow r = yR$, therefore equation (18) can be written as

$$\Rightarrow \frac{\partial^2 w_1}{\partial y^2} + \frac{1}{y}\frac{\partial w_1}{\partial y} - i\alpha_1^2 w_1 - \frac{w_1}{y^2} = \frac{R}{\mu}\frac{\partial P_1}{\partial y} - \frac{R^2\varphi s}{\mu(1-\varphi)}[w_2 - w_1] \quad (20)$$

Where $\alpha_1^2 = \frac{R^2 n}{\alpha}$

For equations (19) and (20), the conservation of mass equation can be written using (12) and (13) and simplifying, we get

$$\frac{1}{y}\frac{\partial(w_1 y)}{\partial y} = \frac{inRu_1}{c} \quad (21)$$

Equations (19) and (20) are now the equations of motion of the fluid and equation (21) is the conservation of mass equation.

3.1 Evaluating the particulate phase

Now neglecting the inertia terms in equation (7), we can write equation (7) as

$$\rho_p[\frac{\partial u_p}{\partial t}] = -\varphi\frac{\partial p}{\partial r} + \frac{\partial p}{\partial r} + \varphi s(u_f - u_p) \quad (22)$$

Now substituting (12) and (13) into (22), we have

$$u_1 = \frac{su_1}{\rho_p in + s} - \frac{1}{R[\rho_p in + s]}\frac{\partial p_1}{\partial y} \quad (23)$$

Also equation (8) can similarly be written as

$$\rho_p\frac{\partial w_p}{\partial t} = -\varphi\frac{\partial p}{\partial z} + \varphi s(w_f - w_p) \quad (24)$$

Substituting equations (12) and (13) into (24), we get

$$w_1 = \frac{p_1 in}{c[\rho_p in + s]} + \frac{sw_1}{\rho_p in + s} \quad (25)$$

The equation of conservation of mass of the particles can be written using (12) and (13) as

$$\frac{1}{y}\frac{\partial(u_2 y)}{\partial y} = \frac{inRw_2}{c} \quad (26)$$

We can now re-arrange the equations of motion of the particles to be written from (23), (25) and (26) to have:

$$\left.\begin{aligned} u_1 &= \frac{su_1}{\rho_p in + s} - \frac{1}{R[\rho_p in + s]}\frac{\partial p_1}{\partial y} \\ w_1 &= \frac{sw_1}{c[\rho_p in + s]} + \frac{p_1 in}{\rho_p in + s} \end{aligned}\right\} \quad (27)$$

with the equation of continuity as

$\frac{1}{y}\frac{\partial(u_2 y)}{\partial y} = \frac{inRw_2}{c}$

Now to solve equations (19) and (20) [i.e. equations of motion of the fluid]

From equation (23), equation (19) can be written as

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - \frac{u_1}{y^2} + \left[i^3\alpha^2 + \frac{R^2\varphi S}{\mu(1-\varphi)(\rho_p in+s)} - \frac{R\varphi s}{\mu(1-\varphi)}\right]u_1 = \frac{R^2}{\mu R}[1 + \frac{\varphi s}{\mu(1-\varphi)} - \frac{1}{\rho_p in+s}]\frac{\partial p_1}{\partial y} \quad (28)$$

Equation (28), can be written as

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - \frac{u_1}{y^2} + i^3\alpha^2\beta^2 u_1 = \frac{R^2}{\mu R}[1 + \frac{\varphi s}{\mu(1-\varphi)(\rho_p in+s)}]\frac{\partial p_1}{\partial y} \quad (29)$$

where $\beta^2 = 1 + \frac{R^2\varphi S}{i^3\alpha_1^2(1-\varphi)(\rho_p in+s)} - \frac{R\varphi s}{i^3\alpha_1^2(1-\varphi)}$

Equation (29) can be written as

$$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - \frac{u_1}{y^2} + i^3\theta^2 u_1 = \frac{R^2 M}{\mu R}\frac{\partial p_1}{\partial y} \quad (30)$$

Where $\theta = \alpha_1\beta \text{ and } M = 1 + \frac{\varphi s}{(1-\varphi)(\rho_p in+s)}$

Using Bessel function, the solution of equation (30), expressing $p_1 = A_1 J_0(Ky)$, where k is to be determined. Then $\frac{\partial p_1}{\partial y} = A_1 k J_1(ky)$, therefore equation (29) can be written as

$\frac{\partial^2 u_1}{\partial y^2} + \frac{1}{y}\frac{\partial u_1}{\partial y} - \frac{u_1}{y^2} + i^3\theta^2 u_1 = \frac{R^2}{\mu R} M A_1 k J_1(ky)$ With the solution $u_1 = u_c + u_r$, where u_c and u_r are the complementary and particular solutions respectively, giving,

$$u_1 = \frac{C_1 J_0(i^3\theta^2 y)}{J_0(i^3\theta^2)} - \frac{1}{R}\frac{R^2}{\mu}\frac{kA_1 J_1(ky)}{i^3\theta^2 - k^2} \quad (31)$$

Similarly equation (20) has the solution,

$$w_1 = \frac{C_2 J_0(i^3\theta^2 y)}{J_0(i^3\theta^2)} - \frac{inR^2 A_1 J_0(ky)}{i^3\theta^2 - k^2} \quad (32)$$

Now using the approximation, Womersley [10], to equations (31) and (32), where from the equation of continuity of the fluid we get the identities,

$\frac{c_1}{c_2} = \frac{inR}{\theta^2 i^2 c}$

and $\frac{i^2 n^2 R^3 N A_1}{c^2\mu} = \frac{Rk^2 A_1}{\mu}$

$\Rightarrow k = \frac{inRN^{1/2}}{c}$, so that $J_0(ky)$ becomes $J_0\left(\frac{inRN^{1/2}}{c}y\right) = 1$ and $J_1(ky)$ becomes $iJ_1\left(\frac{nRN^{1/2}}{c}y\right) = \frac{inRyN^{1/2}}{2c}$

Also, we use the approximation

$\frac{nRN^{1/2}}{c} = J_0\left(\frac{nRN^{1/2}}{c}\right) = 1$

And $\frac{nRyN^{1/2}}{2c} = J_1\left(\frac{nRN^{1/2}}{c}\right)$.

Inserting these approximations in equations (31) and (32) we get

$u_1 = \frac{C_1 J_1(i^3\theta^2 y)}{J_0(i^3\theta^2)} - \frac{1}{R}\frac{R^2}{\mu}\frac{kA_1 J_1(ky)}{i^3\theta^2 - k^2}$. On further simplification we get

$$u_1 = \frac{C_2 J_0(i^3\theta^2 y)}{J_0(i^3\theta^2)} - \frac{iR^2 nNA_1}{\mu c i^3\theta^2} \quad (33)$$

Since $J_0(ky) = J_0\left(\frac{nRyN^{1/2}}{c}\right) = 1$ and $i^3\theta^2 - k^2 = i^3\theta^2$ similarly,

$$w_1 = \frac{C_2 J_0(i^3\theta^2 y)}{J_0(i^3\theta^2)} + \frac{NA_1}{\rho_f c\beta^2}, \text{ after simplification} \quad (34)$$

At the inner surface of the tube, i.e. when y=1, equation (33) and (34) become respectively,

$$u_1 = \frac{1}{2}\frac{inR}{c}c_2 f_{10}(\theta) + \frac{1}{2}\frac{inR}{c}\frac{NA_1}{\rho_f c\beta^2} \quad (35)$$

Where, $f_{10}(\theta) = \frac{2J_1(i^3\theta^2)}{\theta^2 i^3 J_0(i^3\theta^2)}$

$$\text{And } w_1 = c_2 + \frac{NA_1}{\rho_f c\beta^2} \quad (36)$$

The equations of motion of the tube are: -

$$\frac{\partial^2\alpha}{\partial t^2} = \frac{-\rho_0}{\rho_f}\frac{\alpha}{hR}\left[\frac{\partial w_f}{\partial y} + \frac{R\partial u_f}{\partial z}\right]_{y=1} + \frac{B}{\rho_f}\left[\frac{\partial^2\alpha}{\partial z^2} + \frac{\sigma}{R}\frac{\partial\lambda}{\partial z}\right] \quad (37)$$

$$\frac{\partial^2\lambda}{\partial t^2} = \frac{p}{h\rho_f} - \frac{B}{\rho_f}\left[\frac{\sigma}{R}\frac{\partial\alpha}{\partial z} + \frac{\lambda}{R^2}\right] \quad (38)$$

Together with the boundary conditions for the motion of the liquid,

$u_f = \frac{\partial\lambda}{\partial t}$ $at\ y = 1$

$w_f = \frac{\partial \alpha}{\partial t}\ \ at\ y = 1$

If it is now assumed that from (12) and (13)

$\lambda = D_1 \exp\,[in(t - {}^{z}/_{c})]$ and $\alpha = E_1 \exp\,[in(t - {}^{z}/_{c})]$, where D_1 and E_1 are arbitrary constants, the boundary conditions for u_1 and w_1, using the boundary conditions for the motion of the liquid we have:

$$\frac{\partial \lambda}{\partial t} = D_1\, \text{inexp}[in(t - {}^{z}/_{c})] = u_f \quad (39)$$

But $u_f = D_1 u\exp\,[in(t - {}^{z}/_{c})]$ from (39)

Therefore $u_1 = D_1 in$ (40)

From (36), we can write (37) as

$$inD_1 = \frac{1}{2}\frac{inR}{c}[c_2 f_{10}(\theta) + \frac{NA_1}{\rho_f c\beta^2}] \quad (41)$$

Also $\frac{\partial \alpha}{\partial t} = E_1\, \text{inexp}[in(t - {}^{z}/_{c})] = w_f$ (42)

But from (12), $w_f = w_1 \exp\,[in(t - {}^{z}/_{c})]$, this implies

$$inE_1 = w_1 = c_2 + \frac{NA_1}{\rho_f c\beta^2}]$$

Therefore $inE_1 = c_2 + \frac{NA_1}{\rho_f c\beta^2}]$ (43)

From (37) and (40), the equations of the tube become

$$D_1 n^2 = \frac{A_1}{h\rho} - \frac{B}{\rho}[\frac{J}{R}\left(-\frac{inE_1}{c}\right) + \frac{D_1}{R^2} \quad (44)$$

$$-E_1 n^2 = \frac{\rho_f \alpha}{h\rho R}\left[-\frac{1}{2}i^3\theta^2 c_2 f_{10}(\theta) + \frac{n^2R^2N}{c^2\rho_f\beta^2}\right] + \frac{B}{\rho}\left[-\frac{n^2E_1}{c^2} + \frac{J}{R}\left(-\frac{inD_1}{c}\right)\right], \quad (45)$$

Equations (41), (43), (44) and (45) are four homogeneous equations in the arbitrary constants C, A_1, c_2, D_1 and E_1. Eliminating them will give a frequency equation, which will determine the wave velocity c, in terms of the elastic properties of the tube and the non-dimensional parameter θ. The result of the elimination is:

$$\begin{vmatrix} \frac{N}{\rho_f c\beta^2} & 1 & 0 & -in \\ \frac{inRN}{2c^2\rho_f\beta^2} & \frac{inRf_{10}}{2c} & -in & 0 \\ \frac{1}{h\rho} & 0 & n^2 - \frac{B}{\rho R^2} & \frac{inJ\boldsymbol{B}}{\rho Rc} \\ -[\frac{\rho_f\alpha n^2R^2N + n^2\rho_f\alpha N^2R}{2c^3\rho_f\beta^2 hR}] & \frac{i\rho_f Rn\beta^2 f_{10}}{2h\rho} & -\frac{inJ\boldsymbol{B}}{\rho Rc} & n^2(1 - \frac{B}{\rho c^2}) \end{vmatrix} = 0 \quad (46)$$

Operating on the rows and columns of equation (46) and neglecting $\frac{n^2R^2}{c^2\theta^2}$ and continuing operating on rows and columns until we get

$$\begin{vmatrix} N & 1 & 0 & 1 \\ \frac{N}{2} & \frac{f_{10}}{2} & -1 & 0 \\ \frac{\beta^2}{k} & 0 & \frac{-x}{k} & \frac{-\sigma x}{k} \\ 0 & -\frac{1}{2}f_{10}\beta^2 & \frac{-\sigma x}{k} & 1 - \frac{x}{k} \end{vmatrix} = 0 \quad (47)$$

where $k = \frac{h\rho}{\rho_f R}, and\ x = \frac{kB}{\rho_f c^2}$

Equation (47) can be written as

$$\begin{vmatrix} N & 1 & 0 & 1 \\ N & f_{10} & -2 & 0 \\ \beta^2 & 0 & -x & -\sigma x \\ 0 & \frac{-f_{10}\beta^2 k}{2} & -\sigma x & (k - x) \end{vmatrix} = 0 \quad (48)$$

Whose solution is

$$[N(1-\sigma^2)(1-f_{10})]x^2 - \left[2\beta^2 + kN(1-f_{10}) + f_{10}\beta^2\left(\frac{N}{2} - \sigma - \sigma N\right)\right]x + 2k\beta^2 + f_{10}\beta^4 = 0 \quad (49)$$

This reduces to

$$(1-\sigma^2)(1-f_{10})x^2-\left[2+k(1-f_{10})+f_{10}\left(\frac{1}{2}-2\sigma\right)\right]x+2k+f_{10}=0 \quad (50)$$

I.e. when $N=1$ and $\beta^2=1$ in (48), i.e., when $\varphi=0$, since

$N=\frac{1+\varphi s}{(1-\varphi)(\rho_p in+s)}$ and $\beta^2=\frac{1-R^2\varphi s}{c^3\alpha_1^2(1-\varphi)(\rho_p in+s)}-\frac{R\varphi s}{(1-\varphi)i^3\alpha_1^2}$

Using Womersley [10], the roots of equation (49) are given by

$$(1-\sigma^2)x=G\pm[G^2-(1-\sigma^2)H]^{1/2} \quad (51)$$

where $G=\frac{\left[\frac{\beta^2}{N}+\frac{1}{4}-\sigma\right]}{(1-f_{10})}+\frac{k}{2}+\frac{\sigma\left[1-\frac{f_{10}\beta^2}{2}\left(1+\frac{1}{N}\right)\right]-\frac{1}{4}(1-f_{10}\beta^2)}{(1-f_{10})}$ (52)

When $\beta^2=1$ and $N=1$, i.e. $\varphi=0$ equation (51) reduces to $G=\frac{\left[1+\frac{1}{4}-\sigma\right]}{(1-f_{10})}+\frac{k}{2}+\sigma-\frac{1}{4}$, which is Womersley's [10] solution for the case without suspension and the same as [6].

4. Conclusion

In terms of the notation used in the case of the solid case, $\frac{1}{1-F_{10}}=\exp(-ie)/M'_{10}$ so that the quantities required to compute the roots of equation (49) are already available. When $F_{10}(\theta)$ is complex, x is always complex and the motion is either damped or unstable.

If we write $((1-\sigma^2)\,{}^{x}/_{2})^{1/2}=X-iY$ and denote by C_0 the velocity for the perfect fluid, then if C_1 is the wave-velocity, ${}^{C_0}/_{C_1}=X$ and over a distance of one wavelength the amplitude will be reduced to the ratio $\exp\left(-2\pi\,{}^{Y}/_{X}\right)$ [6] [10]. Since Womersley's case without suspension is considered as the first best description of blood flow in arteries, we have now further proved that our model is in line with this best description of blood flow and the propagation of waves in the arterial system.

Acknowledgement

We wish to acknowledge the contribution of Dr. L. M. Srivastava towards his guide in the M.Sc thesis (1986), "Flow of Particle-Fluid Suspension in Elastic Tube", at the Department of Mathematics, A.B.U, Zaria, Nigeria. This contribution is one of the two major sources, which made this publication possible.

Also, to Associate Professor M. E. Nja of the Mathematics Department, Federal University Lafia, Nigeria. For his great motivation and encouragement for this research. We thank you all.

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