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Suborbits and Suborbital Graphs of the Symmetric Group S_n acting on Ordered *r*-element Subsets

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Abstract

The ranks and subdegrees of the symmetric group S_n acting on $X^{[r]}$, the set of all ordered r-element subsets from X have been studied (See Rimberia [4]). In this paper, we examine some properties of suborbits and suborbital graphs of S_n acting on $X^{[r]}$.

Keyword: Suborbital graphs, Suborbits, Symmetric Group, Group action

1. Introduction

Suppose G is a group acting transitively on the set X and let G_X be the stabilizer in G of a point $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \dots, \Delta_{k-1}$ of G_x on X are known as suborbits of G.

Now, let (G, X) be a transitive permutation group. Then G acts on $X \times X$ by

$$g(x, y) = (g(x), g(y)),\ g \in G,\ x, y \in X$$

The orbits of this action are called suborbitals of G. The orbit containing (x, y) is denoted by $\mathrm{O}(x, y)$. Now, if $\mathrm{O} \subseteq X \times X$ is a G-orbit, then for any $x \in X,\ \Delta = \{y \in X \mid (x, y) \in \mathrm{O}\}$ is a G_x-orbit on X. Conversely, if $\Delta \subseteq X$ is a G_x-orbit, then $\mathrm{O} = \{(gx, gy) \mid g \in G,\ y \in \Delta\}$ is G-orbit on $X \times X$ (Neumann [3]).

From $\mathrm{O}(x, y)$ we can form a suborbital graph $G(x, y)$: its vertices are elements of X, and there is a directed edge from x to y if and only if $(x, y) \in \mathrm{O}(x, y)$. Clearly $\mathrm{O}(y, x)$ is also a suborbital, and it is either equal to or disjoint from $\mathrm{O}(x, y)$. In the former case, $G(x, y) = G(y, x)$ and the graph consists of pairs of oppositely directed edges. It is convenient to replace each such pair by a single undirected edge, so that we have an undirected graph which we call self-paired. In the latter case, $G(x, y)$ is just $G(y, x)$ with arrows reversed, and we call $G(x, y)$ and $G(y, x)$ paired suborbital graphs.

The above ideas were first introduced by Sims [5], and are also described in a paper by Neumann [3] and in books by Tsuzuku [6] and Biggs and White [1], the emphasis being on applications to finite groups.

If $x = y$, then $\mathrm{O}(x, x) = \{(x, x) \mid x \in X\}$ is the diagonal of $X \times X$. The corresponding suborbital graph $G(x, x)$, called the trivial suborbital graph, is self-paired and consists of a loop based at each vertex $x \in X$.

2. Notations and Preliminary Results

Notation 2.1

Throughout this paper, G is the symmetric group S_n while $X^{[r]}$ and $\mathscr{G}$ denotes the set of all ordered r-element subsets from $X = \{1, 2, \dots, n\}$ and a suborbital graph respectively.

Definition 2.1
If $\sigma \in S_n$ has α_1 cycles of length 1, α_2 cycles of length 2, …, α_n cycles of length *n*, then the cycle type of σ is the *n*-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n)$.
Definition 2.2
Let *G* act transitively on a set *X* and let Δ be an orbit of G_x on *X*. Define $\Delta^* = \{gx \mid g \in G,\ x \in g\Delta\}$, then Δ^* is also an orbit of G_x and is called the G_x-orbit (or *G*-suborbit) paired with Δ (Wielandt [7]). Clearly $\Delta^{**} = \Delta$ and $|\Delta| = |\Delta^*|$. If $\Delta = \Delta^*$, then Δ is said to be self-paired.
Definition 2.3
Let *G* act on a set *X*, then the character π of permutation representation of *G* on *X* is defined by

$$\pi(g) = |Fix(g)|, \text{ for all g} \in \text{G}.$$

Theorem 2.1 (Cameron [2])
Let *G* act transitively on a set *X*, and let $g \in G$. Suppose π is the character of the permutation representation of *G* on *X*, then the number of self-paired suborbits of *G* is given by

$$n_\pi = \frac{1}{|G|}\sum_{g \in G} \pi(g^2).$$

Theorem 2.2 (Sims [5])
Let *G* be transitive on *X* and let G_x be the stabilizer of the point $x \in X$. Suppose $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \dots, \Delta_{k-1}$ are the orbits of G_x on X and let $\text{O}_i \subseteq X \times X,\ i = 0, 1, \dots, k-1$ be the suborbital corresponding to $\Delta_i,\ i = 0,\ 1, \dots, k-1$. Then *G* is primitive if and only if each suborbital graph $G_i,\ i = 1, 2, \dots, k-1$ is connected.

3. Properties of Suborbits of *G* acting on $X^{[r]}$

Theorem 3.1
Let *G* act on $X^{[r]}$. Suppose $\Delta = [a_1, a_2, \dots, a_r]$ is an orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ of length 1, where $a_i \in \{1, 2, \dots, r\}$, $i = 1, 2, \dots, r$. Then Δ is self-paired if and only if the permutation $\sigma = \begin{pmatrix} 1 & 2 & \dots & r \\ a_1 & a_2 & \dots & a_r \end{pmatrix}$ is such that $\sigma^2 = 1$.
Proof
Let Δ be self-paired. Then there exists $g \in G$ such that
$g[a_1, a_2, \dots, a_r] = [1, 2, \dots, r]$, that is
$[g(a_1), g(a_2), \dots, g(a_r)] = [1, 2, \dots, r]$.
$\Rightarrow g(a_1) = 1,\ g(a_2) = 2, \dots, g(a_r) = r$.
Since Δ is self-paired, then by Definition 2.2
$g(1) = a_1,\ g(2) = a_2, \dots, g(r) = a_r$.
$\Rightarrow$ g exchanges a_i and i if $a_i \neq i$ or fixes i. Thus the permutation
$\sigma = \begin{pmatrix} 1 & 2 & \dots & r \\ a_1 & a_2 & \dots & a_r \end{pmatrix}$ is such that $\sigma^2 = 1$. Conversely, let $\sigma^2 = 1$, then $\sigma = \sigma^{-1}$. Now, $g \in G$ such that $g = \begin{pmatrix} 1 & 2 & \dots & r & \dots & n \\ a_1 & a_2 & \dots & a_r & \dots & a_n \end{pmatrix}$ takes $[a_1, a_2, \dots, a_r]$ to $[1, 2, \dots, r]$ and $[1, 2, \dots, r]$ to $[a_1, a_2, \dots, a_r]$. Therefore Δ is self-paired. □
Theorem 3.2
Let *G* act on $X^{[r]}$ and suppose $\Delta_i = [a_1, a_2, \dots, a_r]$ and $\Delta_j = [b_1, b_2, \dots, b_r]$, where $a_i, b_i \in \{1, 2, \dots, r\}$,

$i=1,2,\ldots,r$ are orbits of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ of length 1. Then Δ_i and Δ_j are paired if and only if the permutations

$\sigma_i=\begin{pmatrix}1 & 2 \ldots r\\ b_1 & b_2 \ldots b_r\end{pmatrix}$ and $\sigma_j=\begin{pmatrix}1 & 2 \ldots r\\ a_1 & a_2 \ldots a_r\end{pmatrix}$ are inverses of each other.

Proof

Suppose Δ_i and Δ_j are paired. Then there exist $g_i, g_j \in G$ such that

$g_i[a_1,a_2,\ldots,a_r]=[1,2,\ldots,r]$ and $g_j[b_1,b_2,\ldots,b_r]=[1,2,\ldots,r]$.

That is,

$[g_i(a_1),g_i(a_2),\ldots,g_i(a_r)]=[1,2,\ldots,r]$ and $[g_j(b_1),g_j(b_2),\ldots,g_j(b_r)]=[1,2,\ldots,r]$.

$\Rightarrow g_i(a_1)=1, g_i(a_2)=2,\ldots,g_i(a_r)=r$ and $g_j(b_1)=1, g_j(b_2)=2,\ldots,g_j(b_r)=r$.

Since Δ_i and Δ_j are paired, then by Definition 2.2

$g_i(1)=b_1, g_i(2)=b_2,\ldots,g_i(r)=b_r$ and $g_j(1)=a_1, g_j(2)=a_2,\ldots,g_j(r)=a_r$.

$\Rightarrow (g_ig_j)(1)=1,(g_ig_j)(2)=2,\ldots,(g_ig_j)(r)=r$.

Similarly,

$(g_jg_i)(1)=1,(g_jg_i)(2)=2,\ldots,(g_jg_i)(r)=r$.

Hence the permutations $\sigma_i=\begin{pmatrix}1 & 2 \ldots r\\ b_1 & b_2 \ldots b_r\end{pmatrix}$ and $\sigma_j=\begin{pmatrix}1 & 2 \ldots r\\ a_1 & a_2 \ldots a_r\end{pmatrix}$ are inverses of each other. Conversely, suppose $\sigma_i=\begin{pmatrix}1 & 2 \ldots r\\ b_1 & b_2 \ldots b_r\end{pmatrix}$ and $\sigma_j=\begin{pmatrix}1 & 2 \ldots r\\ a_1 & a_2 \ldots a_r\end{pmatrix}$ are inverses of each other. Now, if $g_i, g_j \in G$ where $g_i=\begin{pmatrix}1 & 2 \ldots r \ldots n\\ b_1 & b_2 \ldots b_r \ldots b_n\end{pmatrix}$ and $g_j=\begin{pmatrix}1 & 2 \ldots r \ldots n\\ a_1 & a_2 \ldots a_r \ldots a_n\end{pmatrix}$, then g_i takes $[a_1,a_2,\ldots,a_r]$ to $[1,2,\ldots,r]$ and $[1,2,\ldots,r]$ to $[b_1,b_2,\ldots,b_r]$. Similarly, g_j takes $[b_1,b_2,\ldots,b_r]$ to $[1,2,\ldots,r]$ and $[1,2,\ldots,r]$ to $[a_1,a_2,\ldots,a_r]$. Hence Δ_i and Δ_j are paired. □

Lemma 3.1

Let the cycle type of $g\in G$ be $(\alpha_1,\alpha_2,\ldots,\alpha_n)$. If $\alpha_1 \geq r$, then the number of elements in $X^{[r]}$ fixed by *g* is given by

$$|Fix(g)|=r!\binom{\alpha_1}{r}. \qquad 3.1)$$

Proof

Let $[a_1,a_2,\ldots,a_r]\in X^{[r]}$ and $g\in G$. Then *g* fixes $[a_1,a_2,\ldots,a_r]\in X^{[r]}$ if and only if $a_1,a_2,..,a_r$ are mapped onto themselves by *g*. That is

$g[a_1,a_2,\ldots,a_r]=[g(a_1),g(a_2),\ldots,g(a_r)]=[a_1,a_2,\ldots,a_r]$ implying that $g(a_1)=a_1,\ g(a_2)=a_2,\ldots,g(a_r)=a_r$. Therefore each of the elements $a_1,a_2,\ldots,a_r$ comes from a 1-cycle in *g*. Hence the number of unordered *r*-element subsets fixed by $g\in S_n$ is $\binom{\alpha_1}{r}$. But an unordered *r*-element subset say, $\{a_1,a_2,\ldots,a_r\}$ can be rearranged to give $r!$ distinct ordered *r*-element subsets. Hence

$$|Fix(g)|=r!\binom{\alpha_1}{r}.$$

□

Theorem 3.3

Let *G* act on $X^{[r]}$ and suppose $g\in G$ has cycle type $(\alpha_1,\alpha_2,\ldots,\alpha_n)$, then the number of self-paired suborbits of *G* on $X^{[r]}$ is given by

$$n_\pi = \frac{r!}{n!}\sum_g \binom{\alpha_1 + 2\alpha_2}{r}. \qquad (3.2)$$

Proof

Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \ldots, \alpha_n)$,, then g^2 has $(\alpha_1 + 2\alpha_2)$ cycles of length 1. Hence by Lemma 3.1, the number of elements in $X^{[r]}$ fixed by g^2 is given by

$$\left|Fix\left(g^2\right)\right| = r!\binom{\alpha_1 + 2\alpha_2}{r}.$$

By using this together with Theorem 2.1 we see that the number of self-paired suborbits of *G* on $X^{[r]}$ is equal to

$$n_\pi = \frac{1}{|G|}\sum_{g\in G}\pi\left(g^2\right) = \frac{1}{n!}\sum_g r!\binom{\alpha_1 + 2\alpha_2}{r} = \frac{r!}{n!}\sum_g \binom{\alpha_1 + 2\alpha_2}{r}.$$

□

4. Suborbital Graphs of *G* acting on $X^{[r]}$

4.1 Construction of Suborbital Graphs of *G* acting on $X^{[r]}$

Let *G* act on $X^{[r]}$ and let Δ be an orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$. Suppose $[a_1, a_2, \ldots, a_r] \in \Delta$, where $a_i \in \{1, 2, \ldots, n\}$, $i = 1, 2, \ldots, r$. Then the suborbital O corresponding to Δ is given by

$$\text{O} = \left\{\left(g[1,2,\ldots,r],\ g[a_1, a_2, \ldots, a_r]\right) \mid\ g \in G,\ [a_1, a_2, \ldots, a_r] \in \Delta\right\}.$$

We form the suborbital graph $\mathscr{G}$ corresponding to suborbital O by taking $X^{[r]}$ as the vertex set and by including an edge from $[b_1, b_2, \ldots, b_r]$ to $[c_1, c_2, \ldots, c_r]$ if and only if $([b_1, b_2, \ldots, b_r],\ [c_1, c_2, \ldots, c_r]) \in \text{O}$. Now, if the coordinates of $[1, 2, \ldots, r]$ in positions $i, j, k, \ldots$ are respectively identical to the coordinates of $[a_1, a_2, \ldots, a_r]$ in positions $x, y, z, \ldots$., then $([b_1, b_2, \ldots, b_r], [c_1, c_2, \ldots, c_r]) \in \text{O}$ if and only if the coordinates of $[b_1, b_2, \ldots, b_r]$ in positions $i, j, k, \ldots$ are respectively identical to the coordinates of $[c_1, c_2, \ldots, c_r]$ in positions $x, y, z, \ldots$. Consequently we have an edge from $[b_1, b_2, \ldots, b_r]$ to $[c_1, c_2, \ldots, c_r]$ in $\mathscr{G}$.

4.2 Properties of Suborbital Graphs of *G* acting on $X^{[r]}$

Lemma 4.2.1 (Rimberia [4])

The action of *G* on $X^{[r]}$ is imprimitive if $n > r+1$.

Theorem 4.2.1

If $n > r+1$, then all the suborbital graphs corresponding to the action of *G* on $X^{[r]}$ are disconnected.

Proof

By Lemma 4.2.1, *G* acts imprimitively on $X^{[r]}$ if $n > r+1$ hence by Theorem 2.2 all the corresponding suborbital graphs are disconnected provided $n > r+1$. □

Next, we consider the other two cases:

Case 1

If $n = r$, then each orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ is of length 1. Thus the suborbital graphs corresponding to the self-paired suborbits are regular of degree 1 and so must be disconnected. On other hand, the suborbital graphs corresponding to the paired suborbits have vertices each of which has indegree 1 and outdegree 1. Furthermore, any two consecutive vertices *S* and *T* in these graphs need not be adjacent since for there to be a directed edge, say from *S* to *T*, the coordinates of *S* and *T* must satisfy the rule defining the corresponding suborbital. Therefore such a graph cannot have a directed cycle containing every vertex, and so must be disconnected.

Case 2

If $n = r + 1$, then $G_{[1,2,\dots,r]}$ has orbits with exactly r and $(r-1)$ elements from $A = \{1, 2, \dots, r\}$. Now, the former orbits have length 1 while the latter have length $n - r = (r+1) - r = 1$ (Rimberia [4]). Thus the corresponding suborbital graphs have vertices each of which has degree 1 or indegree 1 and outdegree 1. Similarly, these graphs must be disconnected.

Theorem 4.2.2

Let G act on $X^{[r]}$. Then the number of connected components in the suborbital graph $\mathcal{G}_i$ corresponding to a self-paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$ is equal to

$$n(\mathcal{G}_i) = \frac{n!}{2(n-r)!}. \qquad 4.1)$$

Proof

Let $\mathcal{G}_i$ be the suborbital graph corresponding to a self-paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$. Since each vertex of $\mathcal{G}_i$ has degree 1, then the connected components in $\mathcal{G}_i$ are trees with two vertices and one edge. Hence the number of connected components in $\mathcal{G}_i$ is equal to

$$n(\mathcal{G}_i) = \frac{\text{Number of vertices in } \mathcal{G}_i}{2} = |X^{[r]}| \big/ 2 = \binom{n}{r} r! \big/ 2 = \frac{n!}{2(n-r)!}. \qquad \square$$

Theorem 4.2.3

Let G act on $X^{[r]}$. Then the number of connected components in the suborbital graph $\mathcal{G}_j$ corresponding to a paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$ is equal to

$$n(\mathcal{G}_j) = \frac{n!}{3(n-r)!}. \qquad 4.2)$$

Proof

Let $\mathcal{G}_j$ be the suborbital graph corresponding to a paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$. Then each vertex of $\mathcal{G}_j$ has indegree 1 and outdegree 1. Moreover, construction shows that the connected components of $\mathcal{G}_j$ are directed triangles. Hence the number of connected components in $\mathcal{G}_j$ is equal to

$$n(\mathcal{G}_j) = \frac{\text{Number of vertices in } \mathcal{G}_j}{3} = |X^{[r]}| \big/ 3 = \binom{n}{r} r! \big/ 3 = \frac{n!}{3(n-r)!}. \qquad \square$$

Corollary 4.2.1

Let G act on $X^{[r]}$ and let $\mathcal{G}_i$ be the suborbital graph corresponding to a self-paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$. Then $\mathcal{G}_i$ has girth equal to zero.

Proof

By Theorem 4.2.2, the connected components in $\mathcal{G}_i$ are trees with two vertices and one edge. Hence $\mathcal{G}_i$ cannot have a cycle which implies that its girth is equal to zero. $\square$

Corollary 4.2.2

Let G act on $X^{[r]}$ and let $\mathcal{G}_j$ be the suborbital graph corresponding to a paired orbit of $G_{[1,2,\dots,r]}$ on $X^{[r]}$ with exactly r elements from $A = \{1, 2, \dots, r\}$. Then $\mathcal{G}_j$ has girth 3.

Proof

By Theorem 4.2.3, the connected components in $\mathcal{G}_j$ are directed triangles, that is, directed cycles of length 3. Hence the girth of $\mathcal{G}_j$ is equal to 3. $\square$

Theorem 4.2.4

Let G act on $X^{[r]}$ and let $\mathscr{G}$ be the suborbital graph corresponding to the orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with no element from $A=\{1,2,...,r\}$. Then $\mathscr{G}$ has girth 3 provided $n \geq 3r$.

Proof

Let Δ be the orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ with no element from $A=\{1,2,\ldots,r\}$ and suppose $[d_1,d_2,\ldots,d_r] \in \Delta$. Then the suborbital O corresponding to Δ is given by

$$\mathrm{O}=\{(g[1,2,\ldots,r], g[d_1,d_2,\ldots,d_r]) \mid g \in G,\ [d_1,d_2,\ldots,d_r] \in \Delta\}.$$

Therefore the corresponding suborbital graph $\mathscr{G}$ has $X^{[r]}$ as the vertex set and has an edge from $[e_1,e_2,\ldots,e_r]$ to $[f,f,\ldots,f]$ if and only if $\{e_1,e_2,\ldots,e_r\} \cap \{f_1,f_2,\ldots,f_r\} = \phi$. Hence the cycle in Figure 4.1 below exists in $\mathscr{G}$ if and only if the sets $\{d_1,d_2,\ldots,d_r\}$, $\{e_1,e_2,\ldots,e_r\}$ and $\{f_1,f_2,\ldots,f_r\}$ are mutually disjoint. But clearly this is possible if $n \geq 3r$.

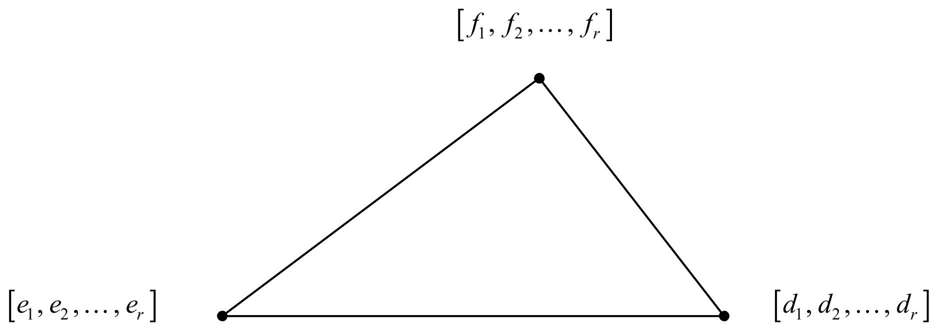

Figure 4.1: A cycle in $\mathscr{G}$

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