Development of Implicit Rational Runge-Kutta Schemes for Second Order Ordinary Differential Equations

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Abstract
In this paper, the development of One – Stage Implicit Rational Runge – Kutta methods are considered using Taylor and Binomial series expansion for the direct solution of general second order initial value problems of ordinary differential equations with constant step length. The basic properties of the developed method were investigated and found to be consistent and convergent.
Keywords: Implicit Rational Runge Kutta scheme, Second Order Equations, Convergence and Consistent

1. Introduction

Consider the numerical approximation first order initial value problems of the form,

$$y' = f(x, y), \qquad y(x_0) = y_0, \quad a \le x \le b \tag{1.1}$$

A Runge-Kutta method is the most important family of implicit and explicit iterative method of approximation of initial value problems of ordinary differential equations. So far many work and schemes have been developed for solving problem (1). The numerical solution of (1.1) is.

$$y_{n+1} = y_n + h\phi(x_n, y_n, h) \tag{1.2}$$

where

$$\phi(x, y, h) = \sum_{i=1}^{s} c_i k_i$$

$$k_1 = f(x, y), \quad \text{and} \quad k_r = f\left(x + ha_i, y + h\sum_{j=1}^{r} b_{ij}k_j\right), \quad r = 1(1)s \tag{1.3}$$

with constraints

$$a_i = \sum_{j=1}^{i} b_{ij}, \quad i = 1(1)s$$

The derivative of suitable parameters a_{ij}, b_i and c_i of higher order term involves a large amount of tedious algebraic manipulations and functions evaluations which is both time consuming and error prone, Julyan and Oreste (1992). The derivation of the Runge – Kutta methods is extensively discussed by Lambert (1973), Butcher (1987), Fatunla (1987), According to Julyan and Oreste (1992) the minimum number of stages necessary for an explicit method to attain order *p* is still an open problem. Therefore so many new schemes and approximation formula have been derived this includes the work of Ababneh *et al*. (2009a), Ababneh *et al*. (2009b) Faranak and Ismail (2010).

Since the stability function of the implicit Runge-Kutta scheme is a rational function, Butcher (2003); Hong (1982) first proposes rational form of Runge-Kutta method (1.2), then Okunbor (1987) investigate rational form and derived the explicit rational Runge-Kutta scheme:

$$y_{n+1} = \frac{y_n + h\sum_{i=1}^{r} w_i K_i}{1 + hy_n\sum_{i=1}^{r} v_i H_i} \tag{1.4}$$

where

$$k_i = f\left(x_n + c_i h, y_n + h\textstyle\sum_{i=1}^{r} a_{i-1j}\, k_j\right), \; i = 1(1)r \tag{1.5}$$

and

$$H_i = g\left(x_n + d_i h, z_n + h\textstyle\sum_{i=1}^{r} b_{i-1j} H_j\right), \; i = 1(1)r \tag{1.6}$$

in which

$$g(x_n, z_n) = -z_n^2 f(x_n, y_n) \qquad \text{and} \qquad z_n = \frac{1}{y_n} \tag{1.7}$$

where c_i, a_{ij}, b_{ij}, d_i are arbitrary constants to be determined.

$$d_i = \sum_{j=1}^{i} b_{ij} \, , \tag{1.8}$$

is imposed to ensure consistency of the method.

In view of these inadequacies of the explicit schemes and the superior region of absolute stability associated with implicit schemes, Ademuluyi and Babatola (2000) generate implicit rational Runge-Kutta and generates also the parameters so that the resulting numerical approximation method shall be A-stable and will have low bound for local truncation error. Since then many new rational Runge – Kutta schemes have been developed for the solution of first order initial value problems and found out to give better estimates. Among these authors are: Ademuluyi and Babatola (2001), Odekunle (2001), Odekunle *et al.* (2004), Bolarinwa (2005), Babatola *et al.* (2007), Bolarinwa *et al.* (2012) and Abhulimen and Uluko (2012) The schemes are absolutely stable, consistent and convergent and was used to approximate a variety of first order differential equations. However, the methods are presently receiving more attention as efficient schemes for the solutions of various types of first order initial value problems are considered.

2. Derivation of the Scheme

Consider the second order initial value problems

$$y'' = f(x, y, y'), \qquad y(x_0) = y_0, \;\; y'(x_0) = y_0 \, , \; a \le x \le b \tag{2.1}$$

The general $s-$stage Runge-Kutta scheme for general second order initial value problems of ordinary differential equations of the form (1) as defined by Jain (1984) is

$$y_{n+1} = y_n + hy'_n + \sum_{r=1}^{s} c_r k_r \tag{2.2}$$

and

$$y'_{n+1} = y'_n + \frac{1}{h}\sum_{r=1}^{s} c'_r k_r \tag{2.3}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{r} a_{ij} k_j, y'_n + \frac{1}{h}\sum_{j=1}^{r} b_{ij} k_j \right), \;\; i = 1(1)s \tag{2.4}$$

with

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j=1}^{i} b_{ij} \;\; , \;\; i(1)r$$

The rational form of (2.2) and (2.3) can be defined as

$$y_{n+1} = \frac{y_n + hy'_n + \sum_{r=1}^{s} w_r K_r}{1 + y'_n \sum_{r=1}^{s} v_r H_r} \tag{2.5}$$

$$y'_{n+1} = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \tag{2.6}$$

where

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right), \; i = 1(1)s \tag{2.7}$$

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right), \; i = 1(1)s \tag{2.8}$$

with constraints

$$c_i = \sum_{j=1}^{i} a_{ij} = \frac{1}{2}\sum_{j} b_{ij}, \;\; i = 1(1)r$$

$$d_i = \sum_{j=1}^{i} \alpha_{ij} = \frac{1}{2}\sum_{j}^{i} \beta_{ij}, \quad i = 1(1)r \tag{A}$$

in which

$$g(x_n, z_n, z'_n) = -z_n^2 f(x_n, y_n, y'_n) \quad and\ z_n = \frac{1}{y_n} \tag{B}$$

The constraint equations are to ensure consistency of the method, h is the step size and the parameters $a_{ij}, b_{ij}, c_i, d_{i,} \alpha_{ij}, \beta_{ij}$ are constants called the parameters of the method.

Using Bobatola e*tal* (2007), the following procedures are adapted.

i. Obtain the Taylor series expansion of K_r and H_r about the point (x_n, y_n, y'_n) and binomial series expansion of right side of (2.1) and (2.2).

iv. Insert the Taylor series expansion into (2.1) and (2.2) respectively.

v. Compare the final expansion of K_r and H_r about the point (x_n, y_n, y'_n) to the Taylor series expansion of y_{n+1} and y'_{n+1} about (x_n, y_n, y'_n) in the powers of h.

Normally the numbers of parameters exceed the number of equations, these parameters are chosen to ensure that (one or more of the following conditions are satisfied.

iv. Minimum bound of local truncation error exists.

v. The method has maximized interval of absolute stability.

vi. Minimized computer storage facilities are utilized.

To derive a One – stage scheme, we set s = 1 in equations (2.5), (2.6), (2.7) and (2.8) to have

$$y_{n+1} = \frac{y_n + hy'_n + w_1 K_1}{1 + y'_n v_1 H_1} \tag{2.9}$$

and

$$y'_{n+1} = \frac{y_n + \frac{1}{h} w'_1 K_1}{1 + \frac{1}{h} y'_n v'_1 H_1} \tag{2.10}$$

where

$$k_1 = \frac{h^2}{2} f\left(x_n + c_1 h, y_n + hc_1 y'_n + a_{11} K_1, y'_n + \frac{1}{h} b_{11} K_1\right), i = 1(1)s \tag{2.11a}$$

and

$$H_1 = \frac{h^2}{2} g\left(x_n + d_1 h, z_n + hd_1 z'_n + \alpha_{11} H_1, z'_n + \frac{1}{h} \beta_{11} H_1\right), i = 1(1)s \tag{2.11b}$$

with constraints

$$c_1 = a_{11} = \tfrac{1}{2} b_{11} \qquad and \qquad d_1 = \alpha_{11} = \tfrac{1}{2}\beta_{11} \tag{2.12}$$

where $c_1, a_{11}, b_{11}, d_1, \alpha_{11}, \beta_{11}, w_1, w'_1, v_1$ and v'_1 are all constants to be determined.

Equation (2.9) can be written as

$$y_{n+1} = \left(y_n + hy'_n + w_1 k_1\right)\left(1 + y_n v_1 H_1\right)^{-1} \tag{2.13}$$

Expanding the bracket and neglecting 2nd and higher orders gives

$$y_{n+1} = \left(y_n + hy'_n + w_1 k_1\right)\left(1 - y_n v_1 H_1\right) \tag{2.14}$$

Expanding (2.14) and re-arranging, gives

$$y_{n+1} = y_n + hy'_n - \left(y_n^2 v_1 + hy_n y'_n v_1\right) H_1 + \left(w_1 - y_n v_1 H_1 w_1\right) K_1 \tag{2.15}$$

Equation (2.10) can be written as

$$y'_{n+1} = \left(y'_n + \tfrac{1}{h} w'_1 K_1\right)\left(1 + \tfrac{1}{h} y'_n v'_1 H_1\right)^{-1} \tag{2.16}$$

$$y'_{n+1} = \left(y'_n + \frac{1}{h} w'_1 k_1\right)\left(1 - \frac{1}{h} y'_n v'_1 H_1\right)$$

Expanding the binomial and re-arranging also gives

$$y'_{n+1} = y'_n + \tfrac{1}{h} w'_1 K_1 - \left(\tfrac{1}{h} y'^2_n v'_1 + \tfrac{1}{h^2} y'_n w'_1 v'_1 K_1\right) H_1 \tag{2.17}$$

Now, the Taylor's series expansion of y_{n+1} about x_n is given as

$$y_{n+1} = y_n + hy'_n + \frac{h^2 y_n''}{2!} + \frac{h^3 y_n'''}{3!} + \frac{h^4 y_n^{iv}}{4!} + \ \dots \qquad (2.18)$$

and

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2 y_n'''}{2!} + \frac{h^3 y_n^{iv}}{3!} + \ \dots \qquad (2.19)$$

where

$$y_n'' = f(x_n, y_n, y_n') = f_n$$
$$y_n''' = f_x + y' f_y + f_n f_{y'} = \Delta f_n$$
$$y_n'^{v} = f_{xx} + y'^2_n f_{yy} + f^2 f_{y'y'} + 2y' f_n f_{yy'} + 2 f_n f_{xy'} + f_{y'} \Delta f_n$$
(2.20)
$$y_n'^{v} = \Delta^2 f_n + f_{y'} \Delta f_n + f_n f_y$$

Since $\Delta = \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + f_n \frac{\partial}{\partial y'}$ (2.21)

Using the Taylor's series of the function of three variables we have from

$$\frac{2}{h^2} K_1 = f_n + \left(c_1 h f_x + (hc_1 y'_n + a_{11} K_1) f_n + \frac{1}{h} b_{11} f_{y'} \right)$$
$$+ \frac{1}{2!} \Bigg((c_1 h)^2 f_{xx} + 2c_1 h (hc_1 y'_n + a_{11} K_1) f_{xy} + 2 c_1 h \left(\frac{1}{h} b_{11} K_1 \right) f_{xy'} + (hc_1 y'_n + a_{11} K_1)^2 f_{yy}$$
$$+ 2(hc_1 y'_n + a_{11} K_1) \left(\frac{1}{h} b_{11} K_1 \right) f_{yy'} + \left(\frac{1}{h} b_{11} K_1 \right)^2 f_{y'y'} \Bigg) + \ \dots$$

Simplifying further and arranging the equation in powers of h gives,

$$K_1 = \frac{h}{2}\left[b_{11} K_1 f_{y'} + a_{11} b_{11} K_1^2 f_{yy'} \right] + \frac{h^2}{2}\left[f_n + a_{11} K_1 f_y + c_1 b_{11} K_1 f_{xy'} + a_{11}^2 K_1^2 f_{yy} + c_1 y_n' b_{11} K_1 f_{yy'} \right] + \frac{h^3}{2}\left[c_1 f_x + c_1 y_n' f_y + c_1 a_{11} K_1 f_{xy} + c_1 a_{11} y_n' K_1 f_{yy} \right] + \frac{h^4}{4}\left[c_1^2 f_{xx} + c_1^2 y_n' f_{xy} + c_1^2 y'^2_n f_{yy} \right] + 0(h^5) \qquad (2.22)$$

Equation (2.22) is implicit; one cannot proceed by successive substitution. Following *Lambert (1973)*, we can assume that the solution for K_1 may be express in the form

$$K_1 = hA_1 + h^2 B_1 + h^3 C_1 + h^4 D_1 + 0(h^5) \qquad (2.23)$$

Substituting equation (2.23) into (2.22) gives

$$K_1 = \frac{h}{2}\left[b_{11}(hA_1 + h^2 B_1 + h^3 C_1) f_{y'} + a_{11} b_{11} (hA_1 + h^2 B_1)^2 f_{yy'} \right]$$
$$+ \frac{h^2}{2}\Big[f_n + a_{11}(hA_1 + h^2 B_1) f_y + c_1 b_{11} (hA_1 + h^2 B_1) f_{xy'} + a_{11}^2 (hA_1)^2 f_{yy}$$
$$+ c_1 y_n' b_{11} (hA_1 + h^2 B_1) f_{yy'} \Big] + \frac{h^3}{2}\left[c_1 f_x + c_1 y_n' f_y + c_1 a_{11} (hA_1) f_{xy} + c_1 a_{11} y_n' (hA_1) f_{yy} \right]$$
$$+ \frac{h^4}{4}\left[c_1^2 f_{xx} + c_1^2 y_n' f_{xy} + c_1^2 y'^2_n f_{yy} \right] + 0(h^5) \qquad (2.24)$$

On equating powers of h from equation (2.22) and (2.23), gives

$$A_1 = 0, \ B_1 = \tfrac{1}{2} f_n, \quad C_1 = \tfrac{1}{2}\left(c_1 f_x + c_1 y_n' f_y + 1/2 b_{11} f_n f_{y'} \right) = \tfrac{1}{2} c_1 \Delta f_n \text{ , since } c_1 = \tfrac{1}{2} b_{11}$$
$$D_1 = \tfrac{1}{4}\left(c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y \right) \qquad (2.25)$$

Substituting A_1, B_1, C_1 and D_1 into (2.23) gives.

$$K_1 = \tfrac{h^2}{2} f_n + \tfrac{h^3}{2} c_1 \Delta f_n + \tfrac{h^4}{2}\left(c_1^2 \Delta^2 f_n + b_{11} \Delta f_n f_{y'} + a_{11} f_n f_y \right)$$
(2.26)

Similarly, expanding H_1 in Taylor's series about (x_n, z_n, z_n'), from (2.11b), we have

$$H_1 = \frac{h}{2}\left[\beta_{11} H_1 g_{z'} + \alpha_{11} \beta_{11} H_1^2 g_{zz'} \right] + \frac{h^2}{2}\left[g_n + \alpha_{11} H_1 g_z + d_1 \beta_{11} H_1 g_{xz'} + \alpha_{11}^2 H_1^2 g_{zz} + d_1 z_n' \beta_{11} H_1 g_{zz'} \right]$$
$$+ \frac{h^3}{2}\left[d_1 g_x + d_1 z_n' g_z + d_1 \alpha_{11} H_1 g_{xz} + d_1 \alpha_{11} z_n' H_1 g_{zz} \right] + \frac{h^4}{4}\left[c_1^2 g_{xx} + d_1^2 z_n' g_{xz} + d_1^2 z'^2_n g_{zz} \right]$$
$$+ 0(h^5) \qquad (2.27)$$

Equation (2.27) is also implicit which cannot be proceed by successive substitution. Assuming a solution of the equation is of the form

$$H_1 = hL_1 + h^2 M_1 + h^3 N_1 + h^4 R_1 + 0(h^5) \qquad (2.28)$$

Substituting the values of H_1 in (2.28) into equation (2.27) and equating powers of h of the equation, we can get the following after substitutions:

$$L_1 = 0, \quad M_1 = \frac{1}{2} g_{n,} \quad N_1 = \frac{1}{2} d_1 \Delta g_n \quad and\ R_1 = \frac{1}{4}(d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z) \tag{2.29}$$

Substituting equation (2.29) into equation (2.28) gives

$$H_1 = \frac{h^2}{2} g_n + \frac{h^3}{2} d_1 \Delta g_n + \frac{h^4}{2}(d_1^2 \Delta^2 g_n + \beta_{11} \Delta g_n g_{z'} + \alpha_{11} g_n g_z) \tag{2.30}$$

Using equations (2.23) and (2.28) into equations (2.15) and (2.17) respectively gives

$$y_{n+1} = y_n + h y_n' - (y_n^2 v_1 + h y_n y_n' v_1)(h^2 M_1 + h^3 N_1 + h^4 R_1) + [w_1 - y_n v_1 w_1 (h^2 M_1 + h^3 N_1 + h^4 R_1)](h^2 B_1 + h^3 C_1 + h^4 D_1)$$

Expanding the brackets and re-arranging in powers of h gives

$$y_{n+1} = y_n + h y_n' + h^2 (w_1 B_1 - y_n^2 v_1 M_1) + h^3 (w_1 C_1 - y_n^2 v_1 N_1 - y_n y_n' v_1 M_1) + 0(h^4) \tag{2.31}$$

Also for y_{n+1}' gives

$$y_{n+1}' = y_n' + \frac{1}{h} w_1' (h^2 B_1 + h^3 C_1 + h^4 D_1) - \left[\frac{1}{h} y'^2_n v_n' + \frac{1}{h^2} y_n' w_1' v_n' (h^2 B_1 + h^3 C_1 + h^4 D_1)\right](h^2 M_1 + h^3 N_1 + h^4 R_1)$$

Expanding the brackets and re-arrange in powers of h gives

$$y_{n+1}' = y_n' + h(w_1' B_1 - y'^2_n v_n' M_1) + h^2 (w_1' C_1 - y'^2_n v_n' N_1 - y_n' w_1' v_n' B_1 M_1) + h^3 (w_1' D_1 - y'^2_n v_n' R_1 - y_n' w_1' v_n' B_1 N_1 - y_n' w_1' v_n' C_1 M_1) + 0(h^4) \tag{2.32}$$

Comparing the corresponding powers in h of equations (2.31) and (2.32) with equations (2.18) and (2.19) we obtain

$$\left.\begin{aligned}
&\tfrac{1}{2} w_1 f_n - \tfrac{1}{2} y_n' v_1 g_n = \tfrac{1}{2} f_n \\
&\tfrac{1}{2} w_1 w c_1 \Delta f_n - \tfrac{1}{2} y_n'^2 v_1 d_1 \Delta g_n - \tfrac{1}{2} y_n y_n' v_1 g_n = \tfrac{1}{6} \Delta f_n \\
&\tfrac{1}{2} w_1' f_n - \tfrac{1}{2} y_n'^2 v_1' g_n = f_n \\
&\tfrac{1}{2} w_1' c_1 \Delta f_n - \tfrac{1}{2} y_n'^2 v_1' d_1 \Delta g_n - \tfrac{1}{2} y_n' w_1' v_1' f_n (\tfrac{1}{2} g_n) = \tfrac{1}{2} \Delta f_n
\end{aligned}\right\} \tag{2.33}$$

(By using the equations in (2.25) and (2.29))

Since from equation (1.7)

$$\left.\begin{aligned}
&g_n = -\frac{f_n}{y_n^2}, \quad g_x = -\frac{f_x}{y_n^2}, \quad g_z = -2\frac{f_n}{y_n} + f_y, \quad g_{z'} = -2\frac{f_n}{y_n} + f_{y'}, \quad z_n' = -\frac{y_n'}{y_n^2} \\
&\text{and} \\
&\Delta g_n = g_n + z_n' g_z + g_n g_{z'}
\end{aligned}\right\} \tag{2.34}$$

Using those equations into equation (2.33), we get the following simultaneous equations

$$\left.\begin{aligned}
&w_1 + v_1 = 1 \\
&w_1 c_1 + v_1 d_1 = \tfrac{1}{3} \\
&w_1' + v_1' = 2 \\
&w_1' c_1 + v_1' d_1 = 1
\end{aligned}\right\} \tag{2.35}$$

Equation (2.35) has (4) equations with (6) unknowns; there will not be a unique solution for (2.35). There will be a family of one-stage scheme of order four.

i. Choosing the parameters

$$w_1 = \frac{1}{3}, \quad v_1 = \frac{2}{3}, \quad c_1 = a_{11} = b_{11} = 0, \quad w_1' = 0, \quad v_1' = 2, \quad d_1 = \alpha_{11} = \frac{1}{2}, \quad \beta_{11} = 1$$

arbitrarily the following scheme is obtain.

$$y_{n+1} = \frac{h y_n' + \frac{1}{3} K_1}{1 + \frac{2}{3} y_n H_1} \tag{2.36}$$

and

$$y_{n+1}' = \frac{y_n'}{1 + \frac{2}{h} H_1} \tag{2.37}$$

where

$$K_1 = \frac{h^2}{2} f(x_n, y_n, y_n')$$

$$H_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{2}h, z_n + \frac{1}{2}hz_n' + \frac{1}{2}H_1, z_n' + \frac{1}{h}H_1\right)$$, since $d_1 = \alpha_{11} = \frac{1}{2}\beta_{11}$

ii. Choosing the parameters

From (2.35) setting $w_1 = v_1 = \frac{1}{2},\ c_1 = a_{11} = \frac{1}{2},\ d_1 = \alpha_{11} = \frac{1}{6},\ w_1' = 2,\ v_1' = 0,\ b_{11} = 1,\ \beta_{11} = \frac{1}{3}$

Then,

$$y_{n+1} = \frac{y_n + hy_n' + \frac{1}{2}K_1}{1 + \frac{1}{2}y_n H_1} \tag{2.38}$$

and

$$y_{n+1}' = y_n' + \frac{2}{h}K_1 \tag{2.39}$$

where

$$K_1 = \frac{h^2}{2} f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hy_n' + \frac{1}{2}K_1, y_n' + \frac{1}{h}K_1\right)$$, since $c_1 = a_{11} = \frac{1}{2}b_{11}$

and

$$H_1 = \frac{h^2}{2} g\left(x_n + \frac{1}{6}h, z_n + \frac{1}{6}hz_n' + \frac{1}{6}H_1, z_n' + \frac{1}{3h}H_1\right)$$, since $d_1 = \alpha_{11} = \frac{1}{2}\beta_{11}$

3. CONVERGENCE

A numerical method is said to be convergent if the numerical solution approaches the exact solution as the step size tends to zero.

Convergent $= \lim_{h\to 0}|y(x_{n+1}) - y_{n+1}|$

In other words, if the *discretiation* error at x_{n+1} tends to zero as $h \to \infty$, i.e if

$$e_{n+1} = |y(x_{n+1}) - y_{n+1}| \to 0 \quad \text{as } n \to \infty \tag{3.1}$$

From equation (2.5),

$$y_{n+1}' = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w_r' K_r}{1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r} \tag{3.2}$$

while the exact solution $y'(x_{n+1})$ seems to satisfy the equation of the form

$$y'(x_{n+1}) = \frac{y(x_n) + \frac{1}{h}\sum_{r=1}^{s} w_r' K_r}{1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r' H_r} + T_{n+1} \tag{3.3}$$

Where T_{n+1} is a local truncation error.

Subtracting equation (3.3) from (3.2) gives

$$y_{n+1}' - y'(x_{n+1}) = \frac{y_n + \frac{1}{h}\sum_{r=1}^{s} w_r' K_r}{1 + \frac{1}{h} y_n' \sum_{r=1}^{s} v_r' H_r} - \frac{y(x_n) + \frac{1}{h}\sum_{r=1}^{s} w_r' K_r}{1 + \frac{1}{h} y'(x_n) \sum_{r=1}^{s} v_r' H_r} + T_{n+1} \tag{3.4}$$

Adopting equation (3.4) gives

$$e_{n+1} = \frac{\left(1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right)\left(y_n+\frac{1}{h}\sum_{r=1}^{s}w'_rK_r\right)-\left(1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right)\left(y(x_n)+\frac{1}{h}\sum_{r=1}^{s}w'_rK_r\right)}{\left(1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right)\left(1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right)}+T_{n+1} \tag{3.5}$$

Expanding the brackets and re-arranging gives

$$e_{n+1} = \frac{e_n+\frac{1}{h^2}(y'_n-y'(x_n))\left[\left(\sum_{r=1}^{s}w'_rK_r\sum_{r=1}^{s}v'_rH_r\right)\right]}{\left(1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right)\left(1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right)}+T_{n+1}$$

This implies that

$$e_{n+1} = \frac{e_n+e_n\frac{1}{h^2}\left[\left(\sum_{r=1}^{s}w'_rK_r\sum_{r=1}^{s}v'_rH_r\right)\right]}{\left(1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right)\left(1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right)}+T_{n+1} \tag{3.6}$$

$$e_{n+1} = \frac{e_n\left[1+\frac{1}{h^2}\left(\sum_{r=1}^{s}w'_rK_r\sum_{r=1}^{s}v'_rH_r\right)\right]}{\left(1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right)\left(1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right)}+T_{n+1} \tag{3.7}$$

From equations (3.7), setting

$$A_n = \left[1+\frac{1}{h^2}\left(\sum_{r=1}^{s}w'_rK_r\sum_{r=1}^{s}v'_rH_r\right)\right],\quad B_n = \left[1+\frac{1}{h}y_n^{'}\sum_{r=1}^{s}v'_rH_r\right],\quad C_n = \left[1+\frac{1}{h}y'(x_n)\sum_{r=1}^{s}v'_rH_r\right]$$

and $T_{n+1} = T$

Then

$$e_{n+1} = \frac{A_n}{B_nC_n}e_n + T \tag{3.8}$$

Let $B = \max B_n > 0$, $C = \max C_n > 0$ and $A = \max A_n < 0$ then (3.8) becomes,

$$e_{n+1} \le \frac{A}{BC}e_n + T$$

Set $\frac{A}{BC} = K < 1$, then

$$e_{n+1} \le Ke_n + T \tag{3.9}$$

If $n = 0$, then from (3.9),

$$e_1 = Ke_0 + T$$

$e_2 = Ke_1 + T = K^2e_0 + KT + T$ by substituting the value of e_1

$$e_3 = Ke_2 + T = K^3e_0 + K^2T + T$$

Continuing in this manner, we get the following

$$e_{n+1} = K^{n+1}e_0 + \sum_{t=0}^{n+1} K^tT \tag{3.10}$$

Since $\frac{A}{BC} = K < 1$, then one can see that as $n \to \infty$, $e_{n+1} \to 0$. This proves that the scheme converges.

7. CONSISTENCY

A scheme is said to be consistent, if the difference equation of the computation formula exactly approximates the differential equation it intends to solve as the step size ends to zero. To prove if equation (2.5) is consistent, subtract y_n from both side of (2.5), then

$$y_{n+1} - y_n = \frac{y_n + hy'_n + \sum_{r=1}^{s}w_rK_r}{1+y_n\sum_{r=1}^{s}v_rH_r} - y_n \tag{4.1}$$

$$y_{n+1} - y_n = \frac{y_n + hy'_n + \sum_{r=1}^{s}w_rK_r - y_n - y_n^2\sum_{r=1}^{s}v_rH_r}{1+y_n\sum_{r=1}^{s}v_rH_r}$$

$$y_{n+1} - y_n = \frac{hy'_n + \sum_{r=1}^{s} w_r K_r - y_n^2 \sum_{r=1}^{s} v_r H_r}{1 + y_n \sum_{r=1}^{s} v_r H_r} \tag{4.2}$$

but

$$K_r = \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right)$$

and

$$H_r = \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)$$

Then (4.2) becomes

$$y_{n+1} - y_n = \frac{hy'_n + \sum_{r=1}^{s} w_r \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right) - y_n^2 \sum_{r=1}^{s} v_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}{1 + y_n \sum_{r=1}^{s} v_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}$$

Dividing the above equation throughout by h and taking the limit as h tends to zero on both sides gives

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = y'_n \tag{4.3}$$

Again recall that from (2.6), subtracting y'_n on both sides gives

$$y'_{n+1} - y'_n = \frac{y'_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} - y'_n$$

$$y'_{n+1} - y'_n = \frac{y'_n + \frac{1}{h}\sum_{r=1}^{s} w'_r K_r - y'_n - \frac{1}{h} y'^{2}_n \sum_{r=1}^{s} v'_r H_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r}$$

Simplify further gives

$$y'_{n+1} - y'_n = \frac{\frac{1}{h}\sum_{r=1}^{s} w'_r K_r - \frac{1}{h} y'^{2}_n \sum_{r=1}^{s} v'_r H_r}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r H_r} \tag{4.4}$$

Substituting the values of K_r and H_r (4.4) becomes

$$y'_{n+1} - y'_n = \frac{\frac{1}{h}\sum_{r=1}^{s} w'_r K_r \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right) - \frac{1}{h} y'^{2}_n \sum_{r=1}^{s} v'_r H_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}$$

Dividing all through by h and taking the limit as h tends to zero on both sides gives

$$y'_{n+1} - y'_n = \frac{\frac{1}{h}\sum_{r=1}^{s} w'_r K_r \frac{h^2}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right) - \frac{1}{h} y'^{2}_n \sum_{r=1}^{s} v'_r H_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}{1 + \frac{1}{h} y'_n \sum_{r=1}^{s} v'_r \frac{h^2}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}$$

$$\lim_{h\to 0} \frac{y_{n+1} - y_n}{h} = \frac{\frac{1}{2} f\left(x_n + c_i h, y_n + hc_i y'_n + \sum_{j=1}^{s} a_{ij} K_j, y'_n + \frac{1}{h}\sum_{j=1}^{s} b_{ij} K_j \right) - y'^{2}_n \frac{1}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}{1 + y'_n \frac{1}{2} g\left(x_n + d_i h, z_n + hd_i z'_n + \sum_{j=1}^{s} \alpha_{ij} H_j, z'_n + \frac{1}{h}\sum_{j=1}^{s} \beta_{ij} H_j \right)}$$

but by definition

$$f_n = y_n'^2 g(x_n, z_n, z_n')$$

hence the above equation becomes

$$\lim_{h \to 0} \frac{y_{n+1} - y_n}{h} = f_n$$

Hence, the numerical method is consistent.

Conclusion

The new numerical schemes derived follows the techniques of rational form of Runge – Kutta methods proposed by Hong (1982) which was adopted by Okunbor (1987) and Ademiluyi and Babatola (2000) by using Taylor and Binomial expansion in stages evaluation. The order condition obtained in this research is up to five (5) and the stage is up to three (3). This is an improvement on the work of earlier authors.

Due it convergence and consistency of the new schemes, the scheme will be of high accuracy for direct numerical solution of general second order ordinary differential equations. The steps to the derivation of the new schemes are presented in the methodology while the analysis of the schemes proved to be consistent, convergent.

The implementation of the schemes will be highlighted in the forthcoming paper.

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