Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.12, 2013

Characterized Proximity L-spaces, Characterized Compact L-spaces and Characterized Uniform L-spaces

A. S. Abd-Allah
Department of Mathematics, Faculty of Science, El-Mansoura University, El-Mansoura, Egypt

Abstract
In this research work, three new spaces are proposed and investigated. The spaces are named characterized proximity L-space, characterized compact L-space and characterized uniform L-space. The properties of such spaces are deeply studied. Some sort of relationship were introduced among such spaces and other published spaces resented by the author. The published spaces are named characterized FT_s -spaces and characterized FR_k -spaces ([2,3,4]), for $s \in \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$ and $k \in \{0, 1, 2, 3\}$. New properties were already added to the characterized FT_s -spaces and to the characterized FR_k -spaces. Moreover, the characterized proximity L-space, characterized compact L-space and characterized uniform L-space are classified according to the characterized FT_s -spaces and the characterized FR_k -spaces for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$.

Keywords: L-filter, topological L-space, operations, isotone and idempotent, characterized L-space, $\varphi_{1,2}$ L-neighborhood filters, uniform L-structure, L-proximity, characterized proximity L-space, characterized compact L-space, characterized uniform L-space, characterized FT_s -space, $\mathrm{F}\varphi_{1,2}$ - T_s space, characterized FR_k -space and $\mathrm{F}\varphi_{1,2}$ - R_k space for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$.

1. Introduction

The notion of fuzzy filter which is named here L-filter has been introduced by Eklund et al. [13]. By means of this notion a point-based approach to L- topology related to usual points has been developed. The more general concept for L-filter introduced by Gä hler in [15] and L-filters are classified by types. Because of the specific type of L-filter however the approach of Eklund is related only to L-topologies which are stratified, that is, all constant L-sets are open. The more specific L-filters considered in the former papers are called now homogeneous.

On the ordinary topological space (X, T), the operation has been defined by Kasahara ([22]) as a mapping φ from T into 2^X such that $A \subseteq A^{\varphi}$, for all $A \in T$. Abd El-Monsef it al. ([7]) extend Kasahara operation to the power set $P(X)$ of a set X. Kandil et al.([21]) extended Kasahars's and Abd El-Monsef's operations by introducing an operation on the class of all L-sets endowed with an L-topology τ as a mapping $\varphi : L^X \to L^X$ such that $\operatorname{int} \mu \le \mu^{\varphi}$ for all $\mu \in L^X$, where μ^{φ} denotes the value of φ at μ.

The notions of the L-filters and the operations on the class of all L-sets on X endowed with an L-topology τ are applied in [1,5,6] to introduce a more general theory including all the weaker and stronger forms of the L-topology. By means of these notions the notion of $\varphi_{1,2}$-interior of L-set, $\varphi_{1,2}$ L-convergence and $\varphi_{1,2}$ L-neighborhood filters are defined and applied to introduced many special classes of separation axioms. The notion of $\varphi_{1,2}$-interior operator for L-sets is defined as a mapping $\varphi_{1,2}.\operatorname{int} : L^X \to L^X$ which fulfill (I1) to (I5) in [1]. There is a one-to-one correspondence between the class of all $\varphi_{1,2}$-open L-subsets of X and these operators, that is, the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$ -open L-subsets of X can be characterized by these operators. Then the triple $(X, \varphi_{1,2}.\operatorname{int})$ as will as the triple $(X, \varphi_{1,2}OF(X))$ will be called the characterized

L-space of $\varphi_{1,2}$-open L-subsets. The characterized L-spaces are characterized by many of characterizing notions in [1,5], for example by: $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}$ L-interior of the L-filters and by the set of $\varphi_{1,2}$-inner points of the L-filters. Moreover, the notions of closeness and compactness in characterized L-spaces are introduced and studied in [6]. The notions of characterized FT_s-spaces, $F\varphi_{1,2}$-T_s spaces, characterized FR_k-spaces and FR_k-spaces spaces are introduced and studied in [2,3,4] for all $s \in \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$ and $k \in \{0, 1, 2, 3\}$. This paper is devoted to introduce and study three new spaces named characterized proximity L-space, characterized compact L-space and characterized uniform L-space. Many relations between these spaces and our spaces characterized FT_s-spaces, $F\varphi_{1,2}$-T_s spaces, characterized FR_k-spaces and $F\varphi_{1,2}$-R_k spaces are investigated for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$.

In section 2, some definitions and notions related to L-sets, L-topologies, L-filters, L-proximity, operations on L-sets, $\varphi_{1,2}$ L- neighborhood filters, characterized L-spaces, characterized FT_s-spaces, $F\varphi_{1,2}$-T_s spaces, characterized FR_k-spaces and $F\varphi_{1,2}$-spaces spaces are given for $s \in \{0, 1, 2, 4\}$ and $k \in \{2, 3\}$. **Section 3**, is devoted to introduce and study the relation between characterized proximity L-spaces and our classes of characterized FT_s-spaces and characterized FR_k-spaces. It will be shown that in the characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$, the L-proximity δ will be identified with the finer relation on the $\varphi_{1,2}$ L-neighborhood filters. Also, we will show that any L-proximity is separated if and only if the associated characterized proximity L-space is characterized FT_0 and to each L-proximity is associated a characterized FR_2-space in our sense. Generally, it will be shown that the associated characterized proximity L-space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is characterized FR_2-space if the related topological L-space (X, τ) is $F\varphi_{1,2}$-R_2 space. Moreover, for each characterized FR_3-space the binary relation on L^X defined by means the $\varphi_{1,2}$-closure operator $\varphi_{1,2}.\mathrm{cl}$ of τ in Eq. (3.6), is L-proximity on X and conversely, to each L-proximity X, which has a $\varphi_{1,2}$-closure operator fulfills the binary relation given in (3.6), is associated characterized FR_3-space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$.

There is a good notion of $\varphi_{1,2}$-compactness of the L-filters and of topological L-spaces introduced and studied by Abd-Allah et all. in [6]. This notion fulfills main properties, for example, it fulfills the Tychonoff Theorem. **In section 4**, we used this notion to introduce and study the notion of characterized compact L-spaces. It will be shown that every $\varphi_{1,2}$-closed subset of a characterized compact L-space is $\varphi_{1,2}$-compact and each $\varphi_{1,2}$-compact subset of a characterized FT_2-space is $\varphi_{1,2}$-closed. Also, it will be shown that each characterized compact FT_2-space is characterized FT_4-space. Moreover, if $(X, \psi_{1,2}.\mathrm{int})$ is characterized compact L-space finer than the characterized FT_2-space $(X, \varphi_{1,2}.\mathrm{int})$, then $(X, \varphi_{1,2}.\mathrm{int})$ is isomorphic to $(X, \psi_{1,2}.\mathrm{int})$. The notion of fuzzy uniform structure which is named here uniform L-structure had been introduced and studied by G$\ddot{a}$hler et all. in [17]. We used in the least section this notion to introduced the notion of characterized uniform L-spaces and the notion of associated characterized L-spaces. Finally, we show that the uniform L-space (X, U) is separated if and only if the associated characterized stratified L-space

$(X, \varphi_{1,2}.\mathrm{int}_U)$ is characterized FT_0 -space and specially, (X, U) is separated if and only if the associated stratified topological L-space (X, τ_U) is $\mathrm{F}\varphi_{1,2}$ - T_0 space.

2. Preliminaries

We begin by recalling some facts on the L-filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$. Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set X, let L^X be the set of all L-subsets of X, that is, of all mappings $f : X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each L-set $\mu \in L^X$, let μ' denote the complement of μ and it is defined by: $\mu'(x) = \mu(x)'$ for all $x \in X$. Denote by $\overline{\alpha}$ the constant L-subset of X with value $\alpha \in L$. For all $x \in X$ .and for all $\alpha \in L_0$, the L-subset x_α of X whose value α at x and 0 otherwise is called an L-point in X .

The fuzzy filter on X ([15]) which is named here L-filter is a mapping $\mathcal{M} : L^X \to L$ such that the following conditions are fulfilled:

(F1) $\mathcal{M}(\overline{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\overline{1}) = 1$.

(F2) $\mathcal{M}(\mu \wedge \rho) = \mathcal{M}(\mu) \wedge \mathcal{M}(\rho)$ for all $\mu, \rho \in L^X$.

The L-filter $\mathcal{M}$ is called homogeneous ([13]) if $\mathcal{M}(\overline{\alpha}) = \alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(\mu) = \mu(x)$ for all $\mu \in L^X$ is a homogeneous L-filter on X . For each $\mu \in L^X$, the mapping $\dot{\mu} : L^X \to L$ defined by $\dot{\mu}(\eta) = \bigwedge_{0<\eta(x)} \eta(x)$ for all $\eta \in L^X$ is also homogeneous L-filter on X, called homogenous L- filter at the L-subset $\mu \in L^X$. Let $\mathcal{F}_L X$ and $F_L X$ will be denote the sets of all L-filters and of all homogeneous L- filters on a set X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are L- filters on a set X, $\mathcal{M}$ is said to be finer than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\mu) \geq \mathcal{N}(\mu)$ holds for all $\mu \in L^X$. Noting that if L is a complete chain then $\mathcal{M}$ is not finer than $\mathcal{N}$, denoted by $\mathcal{M} \nleq \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds.

Proposition 2.1 [11] For all $\mu, \rho \in L^X$, we have

$$\mu \leq \rho \text{ if and only if } \dot{\mu} \leq \dot{\rho}$$

For each non-empty set $\mathcal{A}$ of the L- filters on X the supremum $\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists ([15]) and given by:

$$(\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)$$

for all $\mu \in L^X$. Whereas the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ does not exists in general as an L-filter. If the infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ exists, then we have:

$$(\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M})(\mu) = \bigvee_{\substack{\mu_1 \wedge \ldots \wedge \mu_n \leq \mu, \\ \mathcal{M}_1, \ldots, \mathcal{M}_n \in \mathcal{A}}} (\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n))$$

for all $\mu \in L^X$, where n is an positive integer, $\mu_1, \ldots, \mu_n$ is a collection such that $\mu_1 \wedge \ldots \wedge \mu_n \leq \mu$ and $\mathcal{M}_1, \ldots, \mathcal{M}_n$ are L- filters from $\mathcal{A}$. Let X be a set and $\mu \in L^X$, then the homogeneous L- filter $\dot{\mu}$ at $\mu \in L^X$ is the L-filter on X given by:

$$\dot{\mu} = \bigvee_{0 < \mu(x)} \dot{x} \tag{2.1}$$

L- filter bases. A family $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ of non-empty subsets of L^X is called a valued L- filter base ([15]) if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \mu$.

(V2) For all $\alpha, \beta \in L_0$ with $\alpha \wedge \beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\rho \in \mathcal{B}_\beta$ there are $\gamma \geq \alpha \wedge \beta$ and $\eta \geq \mu \wedge \sigma$ such that $\eta \in \mathcal{B}_\gamma$.

Proposition 2.2 [15] Each valued base $(\mathcal{B}_\alpha)_{\alpha \in L_0}$ defines the L-filter $\mathcal{M}$ on X by $\mathcal{M}(\mu) = \bigvee_{\rho \in \mathcal{B}_\alpha, \rho \leq \mu} \alpha$ for all $\mu \in L^X$. Conversely, each L- filter $\mathcal{M}$ can be generated by a valued base, e.g. by $(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ with $\alpha\text{-pr}\,\mathcal{M} = \{\mu \in L^X \mid \alpha \leq \mathcal{M}(\mu)\}$.

$(\alpha\text{-pr}\,\mathcal{M})_{\alpha \in L_0}$ is a family of prefilters on X and is called the large valued base of $\mathcal{M}$. Recall that a prefilter on X ([25]) is a non-empty proper subset $\mathcal{F}$ of L^X such that:

(1) $\mu, \rho \in \mathcal{F}$ implies $\mu \wedge \rho \in \mathcal{F}$ and (2) from $\mu \in \mathcal{F}$ and $\mu \leq \rho$ it follows $\rho \in \mathcal{F}$.

Proposition 2.3 [15] Let $\mathcal{A}$ be a set of L- filters on a set X. Then the following are equivalent:

(1) The infimum $\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}$ of $\mathcal{A}$ with respect to the finer relation for L- filters exists.

(2) $\mathcal{M}_1(\mu_1) \wedge \ldots \wedge \mathcal{M}_n(\mu_n) \leq \sup(\mu_1 \wedge \ldots \wedge \mu_n)$ for all finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of $\mathcal{A}$ and $\mu_1, \ldots, \mu_n \in L^X$.

(3) $\alpha \leq \sup(\mu_1 \wedge \ldots \wedge \mu_n)$ holds for all non-empty finite subset $\{\mu_1, \ldots, \mu_n\}$ of $\bigcup_{\mathcal{M} \in \mathcal{A}} \alpha\text{-pr}\,\mathcal{M}$ and $\alpha \in L_0$

L-topology. By an L-topology on a set X ([12,20]), we mean a subset of $\mu \in L^X$ which is closed with respect to all suprema and all finite infima and contains the constant
L-sets $\overline{0}$ and $\overline{1}$. A set X equipped with an L-topology τ on X is called topological L-space. For each topological L-space (X, τ), the elements of τ are called open L-subsets of this space. If τ_1 and τ_2 are L-topologies on a set X, τ_2 is said to be finer than τ_1 and τ_1 is said to be coarser than τ_2 provided $\tau_1 \subseteq \tau_2$ holds. The topological L-space (X, τ) and also τ are said to be stratified provided $\overline{\alpha} \in \tau$ holds for all $\alpha \in L$, that is, all constant L-
sets are open ([24]).

Proximity L-space. A binary relation δ on L^X is called L-proximity on X ([23]), provided it fulfill the following conditions:

(P1) $\mu \,\overline{\delta}\, \rho$ implies $\rho \,\overline{\delta}\, \mu$ for all $\mu, \rho \in L^X$, where $\overline{\delta}$ is the negation of δ.

(P2) $(\mu \vee \rho) \,\overline{\delta}\, \eta$ if and only if $\mu \,\overline{\delta}\, \eta$ and $\rho \,\overline{\delta}\, \eta$ for all $\mu, \rho, \eta \in L^X$.

(P3) $\mu = \overline{0}$ or $\rho = \overline{0}$ implies $\mu \,\overline{\delta}\, \rho$ for all $\mu, \rho \in L^X$.

(P4) $\mu \,\overline{\delta}\, \rho$ implies $\mu \leq \rho'$ for all $\mu, \rho \in L^X$.

(P5) If $\mu \,\overline{\delta}\, \rho$, then there is an $\eta \in L^X$ such that $\mu \,\overline{\delta}\, \eta$ and $\eta \,\overline{\delta}\, \rho$.

A set X equipped with an L-proximity δ on X is called a proximity L-space and will be denoted by (X, δ). Every L- proximity δ on a set X is associated an L-topology on X denoted by τ_δ. The L- proximity δ on a set X is said to be separated if and only if for all $x, y \in X$ such that $x \neq y$ we have $x_\alpha \,\overline{\delta}\, y_\beta$ for all $\alpha, \beta \in L_0$.

Operation on L-sets. In the sequel, let a topological L-space (X, τ) be fixed. By the operation ([21]) on a set X we mean a mapping $\varphi: L^X \to L^X$ such that $\operatorname{int} \mu \leq \mu^\varphi$ holds, for all $\mu \in L^X$, where, μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X, \tau)}$. By identity operation on $O_{(L^X, \tau)}$, we mean the operation $1_{L^X}: L^X \to L^X$ such that $1_{L^X}(\mu) = \mu$, for all $\mu \in L^X$. Also by constant operation on $O_{(L^X,\tau)}$ we mean the operation $c_{L^X}: L^X \to L^X$ such that $c_{L^X}(\mu) = \overline{1}$, for all $\mu \in L^X$. If $\leq$ is a partially ordered relation on $O_{(L^X,\tau)}$ defined as follows: $\varphi_1 \leq \varphi_2 \Leftrightarrow \mu^{\varphi_1} \leq \mu^{\varphi_2}$ for all $\mu \in L^X$, then obviously, $O_{(L^X,\tau)}$ is a completely distributive lattice. As an application on this partially ordered relation, the operation $\varphi: L^X \to L^X$ will be called:

(i) Isotone if $\mu \leq \rho$ implies $\mu^{\varphi} \leq \rho^{\varphi}$, for all $\mu, \rho \in L^X$.

(ii) Weakly finite intersection preserving (wfip, for short) with respect to $\mathcal{A} \subseteq L^X$ if $\rho \wedge \mu^{\varphi} \leq (\rho \wedge \mu)^{\varphi}$ holds, for all $\rho \in \mathcal{A}$ and $\mu \in L^X$.

(iii) Idempotent if $\mu^{\varphi} = (\mu^{\varphi})^{\varphi}$, for all $\mu \in L^X$.

The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are said to be dual if $\mu^{\psi} = co((co\,\mu)^{\varphi}$ or equivalently $\mu^{\varphi} = co((co\,\mu)^{\psi}$ for all $\mu \in L^X$, where $co\,\mu$ denotes the complementation of μ. The dual operation of $\varphi: L^X \to L^X$ will be denoted by $\tilde{\varphi}: L^X \to L^X$. In the classical case of $L = \{0,1\}$, by the operation on a set X we mean the mapping $\varphi: P(X) \to P(X)$ such that $\operatorname{int} A \subseteq A^{\varphi}$ for all A in the power set $P(X)$. The identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$, where $i_{P(X)}(A) = A$ for all $A \in P(X)$.

The φ-open L- sets. Let a topological L-space (X, τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. The L-set $\mu: X \to L$ is called φ-open L- set if $\mu \leq \mu^{\varphi}$ holds. We will denote the class of all φ-open L- sets on X by $\varphi OF(X)$. The L- set μ is called φ-closed if its complement $co\,\mu$ is φ-open. The two operations $\varphi, \psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$ if $\varphi OF(X) = \psi OF(X)$.

The $\varphi_{1,2}$-interiors of L- sets. Let a topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the L-set $\mu: X \to L$ is the mapping $\varphi_{1,2}.\operatorname{int}\mu: X \to L$ defined by:

$$\varphi_{1,2}.\operatorname{int}\mu = \bigvee_{\rho \in \varphi_1 OF(X),\, \rho^{\varphi_2} \leq \mu} \rho \tag{2.2}$$

$\varphi_{1,2}.\operatorname{int}\mu$ is the greatest φ_1-open L-set ρ such that ρ^{φ_2} less than or equal to μ ([1]). The L- set μ is said to be $\varphi_{1,2}$-open if $\mu \leq \varphi_{1,2}.\operatorname{int}\mu$. The class of all $\varphi_{1,2}$-open L- sets on X will be denoted by $\varphi_{1,2}OF(X)$. The complement $co\,\mu$ of a $\varphi_{1,2}$-open L-subset μ will be called $\varphi_{1,2}$-closed, the class of all $\varphi_{1,2}$-closed L-subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of $L = \{0,1\}$, the topological L-space (X, τ) is up to an identification by the ordinary topological space (X, T) and $\varphi_{1,2}.\operatorname{int}\mu$ is the classical one. Hence, in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $A \subseteq \varphi_{1,2}.\operatorname{int} A$. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-open and the class of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}O(X)$ and $\varphi_{1,2}C(X)$, respectively. Clearly, F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.\mathrm{cl}_T\, F = F$.

Proposition 2.4 [1] If (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then, the mapping $\varphi_{1,2}.\operatorname{int}\mu: X \to L$ fulfills the following axioms:

(i) If $\varphi_2 \geq 1_{L^X}$, then $\varphi_{1,2}.\operatorname{int}\mu \leq \mu$ holds.

(ii) $\varphi_{1,2}.\operatorname{int}\mu$ is isotone, i.e, if $\mu \leq \rho$ then $\varphi_{1,2}.\operatorname{int}\mu \leq \varphi_{1,2}.\operatorname{int}\rho$ holds for all $\mu, \rho \in L^X$.

(iii) $\varphi_{1,2}.\mathrm{int}\,\overline{1}=\overline{1}$.

(iv) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\mathrm{int}(\mu \wedge \rho)=\varphi_{1,2}.\mathrm{int}\,\mu \wedge \varphi_{1,2}.\mathrm{int}\,\rho$ for all $\mu, \rho \in L^X$.

(v) If φ_2 is isotone and idempotent operation, then $\varphi_{1,2}.\mathrm{int}\,\mu \leq \varphi_{1,2}.\mathrm{int}(\varphi_{1,2}.\mathrm{int}\,\mu)$ holds.

(vi) $\varphi_{1,2}.\mathrm{int}(\bigvee_{i\in I} \mu_i)=\bigvee_{i\in I} \varphi_{1,2}.\mathrm{int}\,\mu_i$ for all $\mu_i \in \varphi_{1,2}OF(X)$.

Proposition 2.5 [1] Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the following are fulfilled:

(i) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-sets on X forms an extended L- topology on X, denoted by $\tau^{\varphi_{1,2}}$ ([18]).

(ii) If $\varphi_2 \geq 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-sets on X forms a supra L- topology on X, denoted by $\overline{\tau}^{\varphi_{1,2}}$ ([18]).

(iii) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ is a pre L-topology on X, denoted by $\tau^{\wedge}_{\varphi_{1,2}}$ ([18]).

(iv) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}OF(X)$ forms an L- topology on X, denoted by $\tau_{\varphi_{1,2}}$ ([12,20]).

From Propositions 2.4 and 2.5, if the topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then

$$\varphi_{1,2}OF(X)=\{\mu \in L^X \mid \mu \leq \varphi_{1,2}.\mathrm{int}\,\mu\} \tag{2.3}$$

and the following conditions are fulfilled:

(I1) If $\varphi_2 \geq 1_{L^X}$, then $\varphi_{1,2}.\mathrm{int}\,\mu \leq \mu$ holds for all $\mu \in L^X$.

(I2) If $\mu \leq \rho$ then $\varphi_{1,2}.\mathrm{int}\,\mu \leq \varphi_{1,2}.\mathrm{int}\,\rho$ holds for all $\mu, \rho \in L^X$.

(I3) $\varphi_{1,2}.\mathrm{int}\,\overline{1}=\overline{1}$.

(I4) If $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\mathrm{int}(\mu \wedge \rho)=\varphi_{1,2}.\mathrm{int}\,\mu \wedge \varphi_{1,2}.\mathrm{int}\,\rho$ for all $\mu, \rho \in L^X$.

(I5) If $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent operation, then $\varphi_{1,2}.\mathrm{int}(\varphi_{1,2}.\mathrm{int}\,\mu)=\varphi_{1,2}.\mathrm{int}\,\mu$ for all $\mu \in L^X$.

Independently on the L- topologies, the notion of $\varphi_{1,2}$-interior operator for L- sets can be defined as a mapping $\varphi_{1,2}.\mathrm{int}: L^X \to L^X$ which fulfill (I1) to (I5). It is well-known that (2.2) and (2.3) give a one-to-one correspondence between the class of all $\varphi_{1,2}$-open L- sets and these operators, that is, $\varphi_{1,2}OF(X)$ can be characterized by $\varphi_{1,2}$-interior operators. In this case $(X, \varphi_{1,2}.\mathrm{int})$ as will as $(X, \varphi_{1,2}OF(X))$ will be called characterized L- space ([1]) of $\varphi_{1,2}$-open L- subsets of X. If $(X, \varphi_{1,2}.\mathrm{int})$ and $(X, \psi_{1,2}.\mathrm{int})$ are two characterized L-spaces, then $(X, \varphi_{1,2}.\mathrm{int})$ is said to be finer than $(X, \psi_{1,2}.\mathrm{int})$ and denoted by $\varphi_{1,2}.\mathrm{int} \leq \psi_{1,2}.\mathrm{int}$ provided $\varphi_{1,2}.\mathrm{int}\,\mu \geq \psi_{1,2}.\mathrm{int}\,\mu$ holds for all $\mu \in L^X$. The characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$ of all $\varphi_{1,2}$-open L-sets is said to be stratified if and only if $\varphi_{1,2}.\mathrm{int}\,\overline{\alpha}=\overline{\alpha}$ for all $\alpha \in L$. As shown in [1], the characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$ is stratified if the related L- topology is stratified. Moreover, the characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$ is said to have the weak infimum property ([18]) provided for all $\mu \in L^X$ and $\alpha \in L$. The characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to be strongly stratified ([18]) provided $\varphi_{1,2}.\text{int}$ is stratified and have the weak infimum property.

If $\varphi_1 = \text{int}$ and $\varphi_2 = 1_{L^X}$, then the class $\varphi_{1,2}OF(X)$ of all $\varphi_{1,2}$-open L-set of X coincide with τ which is defined in [12,20] and hence the characterized L- space $(X, \varphi_{1,2}.\text{int})$ coincide with the topological L-space (X, τ). Another special choices for the operations φ_1 and φ_2 obtained in Table(1).

The $\varphi_{1,2}$ L- neighborhood filters. An important notion in the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is that of a $\varphi_{1,2}$ L-neighborhood filter at the point and at the ordinary subset in this space. Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. As follows by (I1) to (I5) for each $x \in X$, the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$ which is defined by:

$$\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\varphi_{1,2}.\text{int}\,\mu)(x) \tag{2.4}$$

for all $\mu \in L^X$ is L-filter, called $\varphi_{1,2}$ L-neighborhood filter at x ([1]). If $\varphi \neq F \subseteq P(X)$, then the $\varphi_{1,2}$ L-neighborhood filter at F will be denoted by $\mathcal{N}_{\varphi_{1,2}}(F)$ and it will be defined by:

$$\mathcal{N}_{\varphi_{1,2}}(F) = \bigvee_{x \in F} \mathcal{N}_{\varphi_{1,2}}(x).$$

Since $\mathcal{N}_{\varphi_{1,2}}(x)$ is L-filter for all $x \in X$, then $\mathcal{N}_{\varphi_{1,2}}(F)$ is also L-filter on X. Moreover, because of $[\chi_F] = \bigvee_{x \in F} \dot{x}$, then we have $\mathcal{N}_{\varphi_{1,2}}(F) \geq [\chi_F]$ holds.

If the related $\varphi_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, which is defined by (2.4) is an L-stack ([18]), called $\varphi_{1,2}$ L- neighborhood stack at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of $\rho \in L^X$ we take $\overline{\alpha}$, then the mapping $\mathcal{N}_{\varphi_{1,2}}(x): L^X \to L$, is an L-stack with the cutting property, called here $\varphi_{1,2}$ L-neighborhood stack with the cutting property at x. Obviously, the $\varphi_{1,2}$ L-neighborhood filters fulfill the following axioms:

(N1) $\dot{x} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}_{\varphi_{1,2}}(x)(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(x)(\rho)$ holds for all $\mu, \rho \in L^X$ and $\mu \leq \rho$.

(N3) $\mathcal{N}_{\varphi_{1,2}}(x)(y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)) = \mathcal{N}_{\varphi_{1,2}}(x)(\mu)$, for all $x \in X$ and $\mu \in L^X$.

Clearly, $y \mapsto \mathcal{N}_{\varphi_{1,2}}(y)(\mu)$ is the L-set $\varphi_{1,2}.\text{int}\,\mu$.

The characterized L-space $(X, \varphi_{1,2}.\text{int})$ of all $\varphi_{1,2}$-open L-subsets of a set X is characterized as a filter pre L-topology ([1]), that is, as a mapping $\mathcal{N}_{\varphi_{1,2}}(x): X \to \mathcal{F}_L X$ such that the axioms (N1) to (N3) are fulfilled.

The $\varphi_{1,2}$ L-convergence. Let a topological L-spaces (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. If x is a point in the characterized L-space $(X, \varphi_{1,2}.\text{int})$, $F \subseteq X$ and $\mathcal{M}$ is L-filter on X. Then $\mathcal{M}$ is said to be $\varphi_{1,2}$ L-convergence ([1)] to x and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$, provided $\mathcal{M}$ is finer than the $\varphi_{1,2}$ - neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$. Moreover, $\mathcal{M}$ is said to be $\varphi_{1,2}$-convergence to F and written $\mathcal{M} \xrightarrow[\varphi_{1,2}.\text{int}]{} F$, provided $\mathcal{M}$

is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ for all $x \in F$, that is, $\mathcal{M}$ is finer than the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(F)$.

Internal $\varphi_{1,2}$-closure of L-sets and $\varphi_{1,2}$-closure operators. Let a topological L-spaces (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. The internal $\varphi_{1,2}$-closure of the L-set $\mu : X \to L$ is the mapping $\varphi_{1,2}.\text{cl}\,\mu : X \to L$ defined by:

$$(\varphi_{1,2}.\text{cl}\,\mu)(x) = \bigvee_{\mathcal{M} \le \mathcal{N}_{\varphi_{1,2}}(x)} \mathcal{M}(\mu) \tag{2.5}$$

for all $x \in X$. In (2.5) the L-filter $\mathcal{M}$ my have additional properties, e.g, we my assume that is homogeneous or even that is ultra. Obviously, $\varphi_{1,2}.\text{cl}\,\mu \ge \mu$ holds for all $\mu \in L^X$ .The mapping $\varphi_{1,2}.\text{cl} : \mathcal{F}_L X \to \mathcal{F}_L X$ which assigns $\varphi_{1,2}.\text{cl}\,\mathcal{M}$ to each L-filter $\mathcal{M}$ on X, that is,

$$(\varphi_{1,2}.\text{cl}\,\mathcal{M})(\mu) = \bigvee_{\varphi_{1,2}.\text{cl}\,\rho \le \mu} \mathcal{M}(\rho) \tag{2.6}$$

is called $\varphi_{1,2}$-closure operator ([6]) of the characterized L-space $(X, \varphi_{1,2}.\text{int})$ with respect to the related L-topology τ. Obviously, the $\varphi_{1,2}$-closure operator $\varphi_{1,2}.\text{cl}$ is isotone hull operator, that is, for all $\mathcal{M}, \mathcal{N} \in \mathcal{F}_L X$ we have

$$\mathcal{M} \le \mathcal{N} \quad \text{implies} \quad \varphi_{1,2}.\text{cl}\,\mathcal{M} \le \varphi_{1,2}.\text{cl}\,\mathcal{N}$$

and that $\mathcal{M} \le \varphi_{1,2}.\text{cl}\,\mathcal{M}$.

Characterized FT_s and $\text{F}\varphi_{1,2}$-T_s spaces. The notions of characterized FT_s and $\text{F}\varphi_{1,2}$-T_s spaces are investigated and studied in [2,3] for all $s \in \{0, 1, 2, 2\frac{1}{2}, 3, 4\}$. These spaces depend only on the usual points and the operation defined on the class of all L-subsets of X endowed with a topological L-space (X, τ).

Let a topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the characterized L- space $(X, \varphi_{1,2}.\text{int})$ is said to be:

(1) FT_0-space (resp. FT_1-space) if for all $x, y \in X$ such that $x \ne y$ there exists $\mu \in L^X$ and $\alpha \in L_0$ such that $\mu(x) < \alpha \le (\varphi_{1,2}.\text{int}\,\mu)(y)$ holds or (resp. and) there exist $\rho \in L^X$ and $\beta \in L_0$ such that $\rho(y) < \beta \le (\varphi_{1,2}.\text{int}\,\mu)(x)$ holds. The related topological L-space (X, τ) is said to be $\text{F}\varphi_{1,2}$-T_0 (resp. $\text{F}\varphi_{1,2}$-T_1) if for all $x, y \in X$ such that $x \ne y$ we have $\dot{x} \not\le \mathcal{N}_{\varphi_{1,2}}(y)$ or (resp. and) $\dot{y} \not\le \mathcal{N}_{\varphi_{1,2}}(x)$.

(2) FT_2-space if for all $x, y \in X$ such that $x \ne y$, the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(y)$ does not exists. The related topological L-space (X, τ) is said to be $\text{F}\varphi_{1,2}$-T_2 if $\mathcal{M} \xrightarrow[,\varphi_{1,2}.\text{int}]{} x, y$ implies $x = y$ for all $\mathcal{M} \in \mathcal{F}_L X$ and for all $x, y \in X$

Characterized FR_k and $\text{F}\varphi_{1,2}$-R_k spaces. The notions of characterized FR_k and $\text{F}\varphi_{1,2}$-R_k spaces are introduced and studied in [3] for all $k \in \{0, 1\}$. Moreover, the notion of $\varphi_{1,2}$ L-neighborhood filter at a point and at the ordinary subset of the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is applied in [4], to introduced and studied the notions of FR_k-spaces for $k \in \{2, 3\}$. However, the notions of $\text{F}\varphi_{1,2}$-R_k spaces are also given by means of the $\varphi_{1,2}$ L-convergence at a point and at the ordinary set in the space.

Let a topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized L- space $(X, \varphi_{1,2}.\text{int})$ is said to be:

(1) FR_2 -space (resp. FR_3 -space), if for all $x \in X$, $F \in \varphi_{1,2}C(X)$ such that $x \notin F$ (resp. $F_1, F_2 \in \varphi_{1,2}C(X)$ such that $F_1 \cap F_2 = \varphi$), the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(F)$ (resp. $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$) does not exists. The related topological L-space (X,τ) is said to be $\text{F}\varphi_{1,2}\text{-R}_2$ (resp. $\text{F}\varphi_{1,2}\text{-R}_3$) if for all $x \in X$, (resp. $F \in \varphi_{1,2}C(X)$ and $\mathcal{M} \in \mathcal{F}_L X$ such that $\mathcal{M} \xrightarrow[,\varphi_{1,2}.\text{int}]{} x$ (resp. $\mathcal{M} \xrightarrow[,\varphi_{1,2}.\text{int}]{} F$) we have $\varphi_{1,2}.\text{cl}\,\mathcal{M} \xrightarrow[,\varphi_{1,2}.\text{int}]{} x$ (resp. $\varphi_{1,2}.\text{cl}\,\mathcal{M} \xrightarrow[,\varphi_{1,2}.\text{int}]{} F\ \varphi_{1,2}$).

(2) FT_s -space if and only if it is FR_k and FT_1. The related topological L-space (X,τ) is said to be $\text{F}\varphi_{1,2}\text{-T}_s$ if and only if it is $\text{F}\varphi_{1,2}\text{-R}_k$ and $\text{F}\varphi_{1,2}\text{-T}_1$ for $k \in \{2,3\}$ and $s \in \{3,4\}$.

3. Characterized Proximity L-spaces

In this section we are going to introduce and study the notion of characterized proximity L-spaces. We make at first the relation between the farness on L-sets and the finer relation on L-filters. So, we define a $\varphi_{1,2}$ L\$-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(\mu)$ at the L-subset $\mu \in L^X$ and we show some results for this notion. The notion of homogeneous L-filter $\dot{\mu}$ which is defined in (2.1) and the notion of $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ at the L-subset $\mu \in L^X$ are applied to study the relation between the L-proximity δ defined in [23] and our L-separation axioms in [2,3,4]. Moreover, the relation between characterized proximity L-spaces and the characterized FT_s -spaces, characterized FR_2 -spaces and characterized FR_3 -spaces are introduce

Proposition 3.1 Let a topological L-space (X,τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and idempotent and φ_1 is wfip with respect to $\varphi_1 OF(X)$. Then the supremum of the $\varphi_{1,2}$ L-neighborhood filters $\mathcal{N}_{\varphi_{1,2}}(x)$ at $x \in X$ which is given by:

$$\mathcal{N}_{\varphi_{1,2}}(\mu) = \bigvee_{0<\mu(x)} \mathcal{N}_{\varphi_{1,2}}(x) \tag{3.1}$$

for all $\mu \in L^X$ is L-filter on X called a $\varphi_{1,2}$ L-neighborhood filter at $\mu \in L^X$.

Proof. Fix an $\alpha \in L_0$, then because of (2.4) and the condition $\varphi_2 \geq 1_{L^X}$, we have

$$\mathcal{N}_{\varphi_{1,2}}(\overline{\alpha}) = \bigwedge_{0<\mu(y)} \mathcal{N}_{\varphi_{1,2}}(y)(\overline{\alpha}) = \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\overline{\alpha})(y) \leq \bigwedge_{0<\mu(y)} \overline{\alpha}(y) = \alpha$$

and

$$\mathcal{N}_{\varphi_{1,2}}(\overline{1}) = \bigwedge_{0<\mu(y)} \mathcal{N}_{\varphi_{1,2}}(y)(\overline{1}) = \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\overline{1})(y) = \bigwedge_{0<\mu(y)} \overline{1}(y) = 1.$$

Thus, condition F1) is fulfilled. To prove condition F2, let $\rho, \eta \in L^X$, then because of Proposition 2.4 and (2.4) we have

$$\begin{aligned}\mathcal{N}_{\varphi_{1,2}}(\mu)(\rho \wedge \eta) &= \bigwedge_{0<\mu(y)} \varphi_{1,2}.\text{int}(\rho \wedge \eta)(y)\\ &= \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\rho)(y) \wedge \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\eta)(y)\\ &= \mathcal{N}_{\varphi_{1,2}}(\mu)(\rho) \wedge \mathcal{N}_{\varphi_{1,2}}(\mu)(\eta).\end{aligned}$$

Hence, $\mathcal{N}_{\varphi_{1,2}}(\mu)$ is L- filter on X . Since $(\varphi_{1,2}.\text{int})(x) \le \rho(x)$ holds for all $x \in X$ and $\rho \in L^X$, then $\mathcal{N}_{\varphi_{1,2}}(\mu)(\rho) \le \dot{\mu}(\rho)$ holds for all $\rho \in L^X$. Thus, $\dot{\mu} \le \mathcal{N}_{\varphi_{1,2}}(\mu)$ and therefore $\mathcal{N}_{\varphi_{1,2}}(\mu)$ fulfills condition (N1). For condition (N2), let $\rho, \eta \in L^X$ such that $\rho \le \eta$. Because of Proposition 2.4, we have $\varphi_{1,2}.\text{int}\,\rho \le \varphi_{1,2}.\text{int}\,\eta$ which implies $\bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\rho)(y) \le \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\eta)(y)$ holds for all $y \in X$. Hence $\mathcal{N}_{\varphi_{1,2}}(\mu)(\rho) \le \mathcal{N}_{\varphi_{1,2}}(\mu)(\eta)$ and therefore (N2) is fulfilled. Since for any $y \in X$ we have $\bigwedge_{0<\mu(y)} y \mapsto \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\rho)(y)$ represents the mapping $\varphi_{1,2}.\text{int}\,\rho$. Then from Proposition 2.4 we have

$$\mathcal{N}_{\varphi_{1,2}}(\mu)(\varphi_{1,2}.\text{int}\,\rho) = \bigwedge_{0<\mu(x)} (\varphi_{1,2}.\text{int}\ \varphi_{1,2}.\text{int}\,\rho)(x) = \bigwedge_{0<\mu(x)} (\varphi_{1,2}.\text{int}\,\rho)(x),$$

and then $\mathcal{N}_{\varphi_{1,2}}(\mu)(\bigwedge_{0<\mu(y)} y \mapsto \bigwedge_{0<\mu(y)} (\varphi_{1,2}.\text{int}\,\rho)(y)) = \mathcal{N}_{\varphi_{1,2}}(\mu)(\rho)$ for all $y \in X$ and $\rho \in L^X$. Thus, condition (N3) is also fulfilled and therefore $\mathcal{N}_{\varphi_{1,2}}(\mu)$ fulfilled the conditions (N1) to (N3) of the $\varphi_{1,2}$ - neighborhood filters. □

Not that in [11] the supremum of the empty set of the L-filters is the finest L-filter. This means $\mathcal{N}_{\varphi_{1,2}}(\overline{0}) \le \dot{\mu}'$ for all $\mu \in L^X$. Because of (2.4) the equations (2.1) and (2.2) can be written as in the following:

$$\dot{\mu}(\rho) = \bigwedge_{0<\mu(x)} \rho)(x) \tag{3.2}$$

$$\mathcal{N}_{\varphi_{1,2}}(\mu)(\rho) = \bigwedge_{0<\mu(x)} \mathcal{N}_{\varphi_{1,2}}(x)(\rho) = \bigwedge_{0<\mu(x)} (\varphi_{1,2}.\text{int}\,\rho)(x) \tag{3.3}$$

for all $\rho \in L^X$. Here a useful remark is given

Remark 3.1. The homogeneous L-filter $\dot{x}$ at the ordinary point x is nothing that a homogeneous L-filter $\dot{x}_\alpha$ at the L- point x_α, that is, $\dot{x}_\alpha = \dot{x}$ for all $x \in X$ and $\alpha \in L_0$. Moreover, the $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x)$ at $x \in X$ is itself a $\varphi_{1,2}$ -neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(x_\alpha)$ at x_α.

The $\varphi_{1,2}$ L-neighborhood filter $\mathcal{N}_{\varphi_{1,2}}(\mu)$ at the L-subset $\mu \in L^X$ and the homogeneous L-filter $\dot{\mu}$ fulfill the following properties.

Lemma 3.1 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then for all $\mu, \rho \in L^X$ the following properties are fulfilled:

(1) $\dot{\mu} \le \dot{\rho}$ implies $\mathcal{N}_{\varphi_{1,2}}(\rho') \le \dot{\mu}'$ and $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\rho}$ implies $\mathcal{N}_{\varphi_{1,2}}(\rho') \le \dot{\mu}'$.

(2) $\mu \le \rho$ implies $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \mathcal{N}_{\varphi_{1,2}}(\rho)$.

(3) $\mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho) = \mathcal{N}_{\varphi_{1,2}}(\mu) \vee \mathcal{N}_{\varphi_{1,2}}(\rho)$.

(4) $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\rho}$ implies $\mu \le \rho$.

(5) $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\rho}$ implies there is an $\eta \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\eta}$ and $\mathcal{N}_{\varphi_{1,2}}(\eta) \le \dot{\rho}$.

Proof. Let $\dot{\mu} \le \dot{\rho}$. From (N1) we have $\dot{\mu} \le \mathcal{N}_{\varphi_{1,2}}(\rho)$ and therefore for all $\eta \in L^X$ we have $\bigwedge_{0<\mu(x)} \eta(x) \ge \bigwedge_{0<\rho(y)} (\varphi_{1,2}.\text{int}\,\eta)(y)$. Hence, $\bigwedge_{0<\mu'(x)} \eta(x) \le \bigwedge_{0<\rho'(y)} (\varphi_{1,2}.\text{int}\,\eta)(y)$. Thus, $\mathcal{N}_{\varphi_{1,2}}(\rho')(\eta) \ge \dot{\mu}'(\eta)$ and therefore, $\mathcal{N}_{\varphi_{1,2}}(\rho') \le \dot{\mu}'$. Similarly, if $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\rho}$, then from (N1) we have $\dot{\mu} \le \dot{\rho}$ which implies $\mathcal{N}_{\varphi_{1,2}}(\rho') \le \dot{\mu}'$. Thus, (1) is fulfilled. Since $\mu \le \rho$ implies $\mu(x) \le \rho(x)$ for all $x \in X$, then

$$\bigwedge_{0<\mu(x)} (\varphi_{1,2}.\text{int}\,\eta)(x) \ge \bigwedge_{0<\rho(x)} (\varphi_{1,2}.\text{int}\,\eta)(x).$$

Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.12, 2013

Hence, $\mathcal{N}_{\varphi_{1,2}}(\mu)(\eta) \geq \mathcal{N}_{\varphi_{1,2}}(\rho)(\eta)$ for all $\eta \in L^X$ and therefore $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(\rho)$. Hence, (2) is fulfilled.

Since $\mu,\ \rho \leq \mu \vee \rho$, then from (2) we have $\mathcal{N}_{\varphi_{1,2}}(\mu) \wedge \mathcal{N}_{\varphi_{1,2}}(\rho) \leq \mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho)$. Now, let $\eta \in L^X$ then

$$\begin{aligned}(\mathcal{N}_{\varphi_{1,2}}(\mu) \wedge \mathcal{N}_{\varphi_{1,2}}(\rho))(\eta) &= \bigvee_{k_1 \wedge k_2 \leq \eta} (\mathcal{N}_{\varphi_{1,2}}(\mu)(k_1) \wedge \mathcal{N}_{\varphi_{1,2}}(\rho)(k_2)) \\ &= \bigvee_{k_1 \wedge k_2 \leq \eta} (\bigwedge_{0<\mu(x)} \varphi_{1,2}.\mathrm{int}\, k_1(x) \wedge \bigwedge_{0<\rho(y)} \varphi_{1,2}.\mathrm{int}\, k_2(y)) \\ &\leq \bigvee_{k_1 \wedge k_2 \leq \eta} \bigwedge_{0<(\mu\vee\rho)(z)} \varphi_{1,2}.\mathrm{int}\,(k_1 \wedge k_2)(z) \\ &\leq \bigwedge_{0<(\mu\vee\rho)(z)} \varphi_{1,2}.\mathrm{int}\,\eta(z) = \mathcal{N}_{\varphi_{1,2}}(\mu \wedge \rho)(\eta)\end{aligned}$$

Hence, $\mathcal{N}_{\varphi_{1,2}}(\mu) \wedge \mathcal{N}_{\varphi_{1,2}}(\rho) \geq \mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho)$ and therefore (3) is fulfilled. To prove (4), let $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\rho}$ holds. Because of (2.1), (3.1) and (N1) we have $\dot{\mu} \leq \mathcal{N}_{\varphi_{1,2}}(\mu)$ and then $\dot{\mu} \leq \dot{\rho}$. Hence, Proposition 2.1 implies $\mu \leq \rho$. Thus, (4) is fulfilled. Finally, let $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\rho}$. Then $\bigwedge_{0<\mu(x)} (\varphi_{1,2}.\mathrm{int}\,\lambda)(x) \geq \bigwedge_{0<\rho(y)} \lambda(y)$ for all $\lambda \in L^X$. Hence there is $\eta \in L^X$ such that

$$\bigwedge_{0<\mu(x)} (\varphi_{1,2}.\mathrm{int}\,\lambda)(x) \geq \bigwedge_{0<\eta(z)} \lambda(z) \geq \bigwedge_{0<\eta(z)} (\varphi_{1,2}.\mathrm{int}\,\lambda)(z) \geq \bigwedge_{0<\rho(y)} \lambda(y).$$

This means there is $\eta \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(\mu)(\lambda) \geq \dot{\eta}(\lambda)$ and $\mathcal{N}_{\varphi_{1,2}}(\mu)(\lambda) \geq \dot{\rho}(\lambda)$ are hold for all $\lambda \in L^X$. Thus, $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\eta}$ and $\mathcal{N}_{\varphi_{1,2}}(\eta) \leq \dot{\rho}$ are also hold. Consequently, (5) is fulfilled. □

In the characterized L-space $(X, \varphi_{1,2}.\mathrm{int})$, the L-proximity will be identified with the finer relation on the L-filters, specially with the finer relation on the $\varphi_{1,2}$ L- neighborhood filters. This shown in the following proposition.

Proposition 3.2 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the binary relation δ on L^X which is defined by:

$$\mu \,\bar{\delta}\, \rho \text{ if and only if } \mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\mu}'$$

for all $\mu, \rho \in L^X$ is L-proximity on X.

Proof. Let $\mu, \rho \in L^X$ such that $\mu \,\bar{\delta}\, \rho$, then $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\mu}'$. Because of (1) in Lemma 3.1, we have $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\rho}'$ and therefore $\rho \,\bar{\delta}\, \mu$. Hence, condition (P1) is fulfilled.

Since $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho)$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho)$ are hold for all $\mu, \rho \in L^X$, then $\mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho) \leq \dot{\eta}'$ implies $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\eta}'$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\eta}'$ are hold for all $\eta \in L^X$. This means $\eta \,\bar{\delta}(\mu \vee \rho)$ implies $\eta \,\bar{\delta}\, \mu$ and $\eta \,\bar{\delta}\, \rho$. Conversely, let $\eta \,\bar{\delta}\, \mu$ and $\eta \,\bar{\delta}\, \rho$ for all $\mu, \rho, \eta \in L^X$, then $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\eta}'$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\eta}'$ are hold. Hence, (3) in Lemma 3.1 implies $\mathcal{N}_{\varphi_{1,2}}(\mu \vee \rho) = \mathcal{N}_{\varphi_{1,2}}(\mu) \vee \mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\eta}'$ holds and therefore $\eta \,\bar{\delta}(\mu \vee \rho)$. Consequently, (P2) is fulfilled. To prove (P3), since $\mathcal{N}_{\varphi_{1,2}}(\bar{0}) \leq \dot{\mu}'$ holds for all $\mu \in L^X$. Then, $\mu \,\bar{\delta}\, \bar{0}$ for all $\mu \in L^X$. Hence, $\mu = \bar{0}$ or $\rho = \bar{0}$ implies $\mu \,\bar{\delta}\, \rho$ for all $\mu, \rho \in L^X$. Thus, (P3) is fulfilled.

Let $\mu, \rho \in L^X$ such that $\mu \,\bar{\delta}\, \rho$, then $\mathcal{N}_{\varphi_{1,2}}(\rho) \leq \dot{\mu}'$. Because of (1) and (4) in Lemma 3.1, we have $\mathcal{N}_{\varphi_{1,2}}(\mu) \leq \dot{\rho}'$ and therefore $\mu \leq \rho'$, that is, (P4) is fulfilled. Finally, let $\mu, \rho \in L^X$ such that $\mu \,\bar{\delta}\, \rho$, then

$\mathcal{N}_{\varphi_{1,2}}(\rho) \le \dot{\mu}'$ which implies $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\rho}'$. Because of (5) in Lemma 3.1, there is an $\eta \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(\mu) \le \dot{\eta}$ and $\mathcal{N}_{\varphi_{1,2}}(\eta) \le \dot{\rho}'$ are hold. Hence, $\mathcal{N}_{\varphi_{1,2}}(\eta') \le \dot{\mu}'$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \le \dot{\eta}'$ are also hold, that is, $\mu \bar{\delta} \eta'$ and $\eta \ \bar{\delta} \ \rho$. Thus, (P5) holds and consequently, δ is L-proximity on X . □

If the topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then each L-proximity δ on X is associated a set of all $\varphi_{1,2}$ -open L-subsets of X with respect to δ denoted by $\varphi_{1,2}OF(X)_\delta$. In this case the triple $(X, \varphi_{1,2}OF(X)_\delta)$ as will as the triple $(X, \varphi_{1,2}.\text{int}_\delta)$ is said to be characterized proximity L-space. The related $\varphi_{1,2}$ -interior and $\varphi_{1,2}$ -closure operators $\varphi_{1,2}.\text{int}_\delta$ and $\varphi_{1,2}.\text{cl}_\delta$ are given by:

$$\varphi_{1,2}.\text{int}_\delta \mu = \bigvee_{\mu' \bar{\delta} \rho} \rho \tag{3.4}$$

and

$$\varphi_{1,2}.\text{cl}_\delta \mu = \bigwedge_{\rho' \bar{\delta} \mu} \rho \tag{3.5}$$

respectively, for all $\mu \in L^X$. In the following we will show that the characterized proximity L-space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FT_0 -space as in sense of ([2]) if and only if δ is separated.

Proposition 3.3 Let (X, τ) be a topological L-space, $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ and δ is an L-proximity on X . Then the characterized proximity L-space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FT_0 -space if and only if δ is separated.

Proof. Let $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FT_0 -space and let $x, y \in X$ such that $x \ne y$. Then $\dot{x} \not\le \mathcal{N}^\delta_{\varphi_{1,2}}(y)$ and therefore there is $\mu \in L^X$ such that $\varphi_{1,2}.\text{int}_\delta \mu(y) > \mu(x)$. Because of (3.4), we have $\bigvee_{\mu' \bar{\delta} \rho} \rho(y) > \mu(x)$ and hence $\mu(x) < \rho(y)$ holds for all $\rho \in L^X$ with $\mu' \bar{\delta} \rho$, that is, $\mu(x) < \rho(y)$ holds for all $\rho \in L^X$ with $\mathcal{N}^\delta_{\varphi_{1,2}}(\rho) \le \dot{\mu}$. Choice $\dot{\mu} = x_1'$ and $\rho = y_1$, then because of Remark 3.1, we get $\mathcal{N}^\delta_{\varphi_{1,2}}(y_1) \le x_1'$. Using Proposition 3.2 we get $x_1 \bar{\delta} y_1$ and therefore $x_\alpha \bar{\delta} y_\beta$ holds for all $\alpha, \beta \in L_0$. Thus, δ is separated.

Conversely, let δ is separated L-proximity and let $x, y \in X$ such that $x \ne y$. Then, $x_1 \bar{\delta} y_1$ and because of Proposition 3.2 and Remark 3.1, we have $\mathcal{N}^\delta_{\varphi_{1,2}}(y) \le \dot{x}'$. Therefore, $\varphi_{1,2}.\text{int}_\delta \mu(y) > \bigwedge_{z \ne x} \mu(z)$ holds for all $\mu \in L^X$. Consider, $\mu = x_1'$ we get $\varphi_{1,2}.\text{int}_\delta x_1'(y) = 1$ and $x_1'(x) = 0$. Hence, there exists $\mu = x_1' \in L^X$ such that $\varphi_{1,2}.\text{int}_\delta \mu(y) = 1 > \mu(x)$, that is, $\dot{x} \not\le \mathcal{N}^\delta_{\varphi_{1,2}}(y)$ and therefore $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FT_0 -space. □ $\Box$

In the following proposition, the $\varphi_{1,2}$ -closure of the L-subsets in the characterized proximity L-space $(X, \varphi_{1,2}.\text{int}_\delta)$ are equivalent with the L-subsets by the L-proximity δ on X .

Proposition 3.4 Let (X, τ) be a topological L-space, $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ such that $\varphi_2 \ge 1_{L^X}$ and δ is an L-proximity on X . Then, $\mu \bar{\delta} \rho$ if and only if $\varphi_{1,2}.\text{cl}_\delta \mu \bar{\delta} \varphi_{1,2}.\text{cl}_\delta \rho$ for all $\mu, \rho \in L^X$

Proof. Let $\mu, \rho \in L^X$ such that $\varphi_{1,2}.\text{cl}_\delta \mu \bar{\delta} \varphi_{1,2}.\text{cl}_\delta \rho$, then Proposition 3.2 implies $\mathcal{N}^\delta_{\varphi_{1,2}}(\varphi_{1,2}.\text{cl}_\delta \rho) \le (\varphi_{1,2}.\text{cl}_\delta \mu)'$. Since $\varphi_2 \ge 1_{L^X}$ and $\mathcal{N}^\delta_{\varphi_{1,2}}(\eta)$ is isotone operator, then $\mu \le \varphi_{1,2}.\text{cl}_\delta \mu$

and $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho) \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho)$ are hold for all $\mu, \rho \in L^X$. Hence, $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho) \le \dot{\mu}'$ and therefore $\mu \bar{\delta} \rho$.

Conversely, Let $\mu, \rho \in L^X$ such that $\mu \bar{\delta} \rho$. Because of Proposition 3.2 we have $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho) \le \dot{\mu}$. Since $\varphi_2 \ge 1_{L^X}$ and $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\eta)$ is isotone operator, then $\mu' \le \varphi_{1,2}.\mathrm{cl}_{\delta}\mu'$ holds for all $\mu' \in L^X$ and therefore $\dot{\rho} \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\mu') \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\mu')$. From Lemma 3.1, we have $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\mu) \le \dot{\rho}'$ and then $\rho \bar{\delta} \varphi_{1,2}.\mathrm{cl}_{\delta}\mu$. Therefore, $\varphi_{1,2}.\mathrm{cl}_{\delta}\mu \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho')$ holds. Using Lemma 3.1 we get $\varphi_{1,2}.\mathrm{cl}_{\delta}\mu \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho') \le \mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho')$. Thus, $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho) \le (\varphi_{1,2}.\dot{\mathrm{cl}}_{\delta}\mu)'$ and therefore $\varphi_{1,2}.\mathrm{cl}_{\delta}\mu \bar{\delta} \varphi_{1,2}.\mathrm{cl}_{\delta}\rho$ for all $\mu, \rho \in L^X$. □

In the following theorem we give new description for the characterized FR_2-spaces and its related $\mathrm{F}\varphi_{1,2}$-R_2 spaces.

Theorem 3.1 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the following statements are equivalent:

(1) (X, τ) is $\mathrm{F}\varphi_{1,2}$-R_2 space.

(2) $(X, \varphi_{1,2}.\mathrm{int})$ is characterized FR_2-space.

(3) For all $x \in X$ and $\mu \in L^X$ with $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\mu}$, there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho) \le \dot{\mu}$ are hold.

Proof. The equivalent between (1) and (2) is already proved in [4, Theorem 2.1]. Now, let $(X, \varphi_{1,2}.\mathrm{int})$ is characterized FR_2-space and let $\rho \in L^X$ holds for all $x \in X$ and $\mu \in L^X$. Then because of part (5) in Lemma 3.1, there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \le \dot{\mu}$ are hold. By using (2.1) and (3.1) we have also $\bigwedge_{0<\rho(z)} (\varphi_{1,2}.\mathrm{int}\,\eta)(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$ holds for all $\eta \in L^X$. Since $(X, \varphi_{1,2}.\mathrm{int})$ is characterized FR_2-space, then we have $\varphi_{1,2}.\mathrm{cl}(\mathcal{N}_{\varphi_{1,2}}(x)) = \mathcal{N}_{\varphi_{1,2}}(x)$ for all $x \in X$ and then $\varphi_{1,2}.\mathrm{int}\,\eta(z) = \bigwedge_{0<\rho(z)} \bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma \le \eta} \varphi_{1,2}.\mathrm{int}\,\sigma(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$ holds for all $\eta \in L^X$. Thus, $\bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma \le \eta} \bigwedge_{0<\rho(z)} \varphi_{1,2}.\mathrm{int}\,\sigma(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$ holds. Since $\varphi_{1,2}.\mathrm{cl}$ is hull operator, then for all $\rho \in L^X$ we have

$$\bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma \le \eta} \bigwedge_{0 \le \varphi_{1,2}.\mathrm{cl}\rho(z)} \varphi_{1,2}.\mathrm{int}\,\sigma(z) = \bigwedge_{0 \le \varphi_{1,2}.\mathrm{cl}\rho(z)} \bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma \le \eta} \varphi_{1,2}.\mathrm{int}\,\sigma(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$$

holds and therefore (2.4) and (2.6)imply $\bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma \le \eta} \varphi_{1,2}.\mathrm{int}\,\sigma(z) = \mathcal{N}_{\varphi_{1,2}}(z)(\eta) = \varphi_{1,2}.\mathrm{int}\,\eta(z)$ for all $z \in X$ and $\eta \in L^X$. Then $\bigwedge_{0 \le \varphi_{1,2}.\mathrm{cl}\rho(z)} \varphi_{1,2}.\mathrm{int}\,\eta(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$ holds and therefore $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho)(\eta) \ge \dot{\mu}(\eta)$ is also holds. Thus, $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho) \le \dot{\mu}$, that is, $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\mu}$ implies that, there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho) \le \dot{\mu}$ are hold. Hence, (3) is fulfilled. Finally, let (3) is fulfilled and let $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Then, $x \in F'$ and therefore $\dot{x} \le \dot{\chi}_{F'}$. Because of Proposition 2.1 and Lemma 3.1 we have $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\chi}_{F'}$ holds and then there is $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(x) \le \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\rho) \le \dot{\chi}_{F'}$ are hold. By using part (1) from

Lemma 3.1, we get $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\rho}'$ and hence $\mathcal{N}_{\varphi_{1,2}}(x)(\eta) \wedge \mathcal{N}_{\varphi_{1,2}}(F)(\sigma) \ge \dot{\rho}(\eta) \wedge \dot{\rho}'(\sigma) = \bigwedge_{0<\rho(y)} \eta(y) \wedge \bigwedge_{0<\rho'(z)} \sigma(z)$ holds for all $\eta, \sigma \in L^X$.

Consider, $\rho = x_1 \vee y_1$ for all $x \ne y \in F'$ we get

$$\mathcal{N}_{\varphi_{1,2}}(x)(\eta) \wedge \mathcal{N}_{\varphi_{1,2}}(F)(\sigma) \ge \bigwedge_{0<(x_1 \vee y_1)(y)} \eta(y) \wedge \bigwedge_{0<(x_1 \vee y_1)'(z)} \sigma(z)$$

holds for all $\eta, \sigma \in L^X$. Since for choice $\eta = x_1 \vee y_1$ and $\sigma = (x_1 \vee y_1)'$ we have $\sup(\eta \wedge \sigma) = 0$ and $\mathcal{N}_{\varphi_{1,2}}(x)(\eta) \wedge \mathcal{N}_{\varphi_{1,2}}(F)(\sigma) > 0$ are fulfilled, then the infimum $\mathcal{N}_{\varphi_{1,2}}(x) \wedge \mathcal{N}_{\varphi_{1,2}}(F)$ does not exists. Hence, $(X, \varphi_{1,2}.\text{int})$ is characterized FR_2-space and therefore (3) and (2) are equivalent. □

In the following proposition we show that the associated characterized proximity L-space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FR_2-space if the related topological L-space (X, τ) is $\text{F}\varphi_{1,2}\text{-R}_2$ space.

Proposition 3.5 Let (X, τ) be a topological L-space, $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ and δ is an L-proximity on X. Then the associated characterized proximity L-space $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FR_2-space if (X, τ) is $\text{F}\varphi_{1,2}\text{-R}_2$ space.

Proof. Let $x \in X$ and $\mu \in L^X$ with $\mathcal{N}_{\varphi_{1,2}}(x) \le \dot{\mu}$. Because of Proposition 3.2, we have $\mu' \overline{\delta} x_1$ and from (P5), there is $\rho \in L^X$ such that $\mu' \overline{\delta} \rho$ and $\rho' \overline{\delta} x_1$. Therefore Proposition 3.4 implies $\varphi_{1,2}.\text{cl}_\delta \mu' \overline{\delta}\, \varphi_{1,2}.\text{cl}_\delta \rho$ and hence $\mathcal{N}^\delta_{\varphi_{1,2}}(\varphi_{1,2}.\text{cl}_\delta \rho) \le (\varphi_{1,2}.\dot{\text{cl}}_\delta \mu')'$ and $\mathcal{N}^\delta_{\varphi_{1,2}}(x) \le \dot{\rho}$ are hold. Hence, $\mathcal{N}^\delta_{\varphi_{1,2}}(x) \le \dot{\mu}$ implies there is $\rho \in L^X$ such that $\mathcal{N}^\delta_{\varphi_{1,2}}(x) \le \dot{\rho}$ and $\mathcal{N}^\delta_{\varphi_{1,2}}(\varphi_{1,2}.\text{cl}_\delta \rho) \le \dot{\mu}$ are hold. Since (X, τ) is $\text{F}\varphi_{1,2}\text{-R}_2$ space, then from Theorem 3.1 we have $(X, \varphi_{1,2}.\text{int}_\delta)$ is characterized FR_2-space. □

In the following theorem we give a new description for the characterized FR_3-spaces and its related $\text{F}\varphi_{1,2}\text{-R}_3$ spaces.

Theorem 3.2 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the following statements are equivalent:

(1) (X, τ) is $\text{F}\varphi_{1,2}\text{-R}_3$ space.

(2) $(X, \varphi_{1,2}.\text{int})$ is characterized FR_3-space.

(3) For all $F \in \varphi_{1,2}C(X)$ and $\mu \in L^X$ with $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\mu}$, there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\text{cl}_\delta \rho) \le \dot{\mu}$ are hold.

Proof. The equivalent between (1) and (2) is already proved in [4, Theorem 3.1]. Now, let $(X, \varphi_{1,2}.\text{int})$ is characterized FR_3-space and let $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\mu}$ holds for all $F \in \varphi_{1,2}C(X)$ and $\mu \in L^X$. Then because of part (5) in Lemma 3.2, there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\rho) \le \dot{\mu}$. By using (2.1) and (3.1) we also have $\bigwedge_{0<\rho(z)} (\varphi_{1,2}.\text{int}\,\eta)(z) \ge \bigwedge_{0<\mu(y)} \eta(y)$ holds for all $\eta \in L^X$. Since $(X, \varphi_{1,2}.\text{int})$ is characterized FR_3-space, then $\varphi_{1,2}.\text{cl}(\mathcal{N}_{\varphi_{1,2}}(F)) = \mathcal{N}_{\varphi_{1,2}}(F)$ and therefore $\bigwedge_{x \in F} \varphi_{1,2}.\text{int}\,\eta(x) = \bigvee_{\varphi_{1,2}.\text{cl}\sigma \le \eta} \bigwedge_{x \in F} \varphi_{1,2}.\text{int}\,\sigma(x)$ holds for all $\eta \in L^X$. Since $\mathcal{N}_{\varphi_{1,2}}(F) \le \dot{\rho}$, then Proposition 2.1 and Lemma 3.1

we have $\dot{F} \leq \dot{\rho}$ and therefore $F \subseteq S_0(\rho)$, that is, $\bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\eta(x) \geq \bigwedge_{0<\rho(x)} \varphi_{1,2}.\mathrm{int}\,\eta(x)$ holds for all $\eta \in L^X$. Hence,

$$\bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\eta(x) \geq \bigwedge_{0<\rho(x)} \bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\eta(x)$$
$$= \bigwedge_{0<\rho(x)} \bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma\leq\eta} \bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\sigma(x)$$
$$\geq \bigwedge_{0<\rho(z)} \varphi_{1,2}.\mathrm{int}\,\eta(z) \geq \bigwedge_{0<\rho(y)} \eta(y).$$

Since $\varphi_{1,2}.\mathrm{cl}$ is hull operator, then we have $\bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma\leq\eta} \bigwedge_{0\leq\varphi_{1,2}.\mathrm{cl}\rho(x)} \bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\sigma(x) \geq \bigwedge_{0<\mu(y)} \eta(y)$ holds for all $\rho \in L^X$ and therefore from the distributivity of L we get

$$\bigwedge_{0\leq\varphi_{1,2}.\mathrm{cl}\rho(x)} \bigvee_{\varphi_{1,2}.\mathrm{cl}\sigma\leq\eta} \bigwedge_{x\in F} \varphi_{1,2}.\mathrm{int}\,\sigma(x) = \bigwedge_{0\leq\varphi_{1,2}.\mathrm{cl}\rho(x)} \bigwedge_{x\in F} \varphi_{1,2}.\mathrm{cl}\,\eta(x) \geq \bigwedge_{0<\mu(y)} \eta(y).$$

Hence, $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho)(\eta) \geq \dot{\mu}(\eta)$ holds for all $\eta \in L^X$ and therefore $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq \dot{\mu}$, that is, $\mathcal{N}_{\varphi_{1,2}}(F) \leq \dot{\mu}$, implies that there exists $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F) \leq \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_\delta \rho) \leq \dot{\mu}$ are hold. Hence, (3) is fulfilled. Finally, let (3) is fulfilled and let $F_1, F_2 \in \varphi_{1,2}C(X)$ such that $F_1 \cap F_2 = \varphi$. Because of Proposition 2.1 and Lemma 3.1 we have $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \dot{\chi}_{F_2}$ holds and then there is $\rho \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq \dot{\chi}_{F_2}$ are hold. Hence, $\mathcal{N}_{\varphi_{1,2}}(F_1) \leq \dot{\rho}$ and $\mathcal{N}_{\varphi_{1,2}}(F_2) \leq (\varphi_{1,2}.\mathrm{c\dot{l}}\,\rho)' \leq \dot{\rho}'$ are hold and hence $\mathcal{N}_{\varphi_{1,2}}(F_1)(\eta) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)(\sigma) \geq \dot{\rho}(\eta) \wedge \dot{\rho}'(\sigma)$ holds for all $\eta, \sigma \in L^X$. Consider, $\rho = \chi_{F_1} \vee x_1$ for all $x \in F_2' \setminus F_1$. Then for $\eta = \rho$ and $\sigma = \rho'$ we have $\sup(\eta \wedge \sigma) = 0$ and $\mathcal{N}_{\varphi_{1,2}}(F_1)(\eta) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)(\sigma) > \sup(\eta \wedge \sigma) = 0$ are fulfilled, then the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$ does not exists. Hence, $(X, \varphi_{1,2}.\mathrm{int})$ is characterized FR_3-space and therefore (3) and (2) are equivalent. □

In the following we are going to show an important relation between the associated characterized proximity L-space and the characterized FR_3-space.

Proposition 3.5 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$ such that $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, where L is a complete chain. If (X, τ) is F$\varphi_{1,2}$-R_3 space, then the binary relation δ on X which is defined by:

$$\mu \bar{\delta} \rho \Leftrightarrow \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\mathrm{c\dot{l}}\,\rho)' \tag{3.6}$$

for all $\mu, \rho \in L^X$ is L-proximity on X and (X, δ) is proximity L-space. On other hand if (X, δ) is proximity L-space for δ is defined by (3.6)}, then the associated characterized proximity L-space $(X, \varphi_{1,2}.\mathrm{int}_\delta)$ is characterized FR_3-space.

Proof. Let (X, τ) is F$\varphi_{1,2}$-R_3-space and δ a binary relation on X defined by (3.6). Then, $\mu \bar{\delta} \rho$ implies $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\mathrm{c\dot{l}}\,\rho)'$ and from Lemma 3.1 part (1) we get $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq (\varphi_{1,2}.\mathrm{c\dot{l}}\,\mu)'$ and then $\rho \bar{\delta} \mu$.. Hence, condition (P1) is fulfilled. For showing condition (P2), let $(\mu \vee \rho) \bar{\delta} \eta$ for a fixed L-subsets $\mu, \rho, \eta \in L^X$. Then, $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}(\mu \vee \rho)) \leq (\varphi_{1,2}.\mathrm{c\dot{l}}\,\eta)'$. Since L is complete chain, $\varphi_2 \geq 1_{L^X}$ is isotone and φ_1 is wfip with respect to $\varphi_1 OF(X)$, then $\varphi_{1,2}.\mathrm{cl}(\mu \vee \rho) = \varphi_{1,2}.\mathrm{cl}\,\mu \vee \varphi_{1,2}.\mathrm{cl}\,\rho$ and therefore

$\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu \vee \varphi_{1,2}.\mathrm{cl}\,\rho) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$. Because of Lemma 3.1 part (3) we have $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$ are hold and therefore $\mu\,\overline{\delta}\,\eta$ and $\rho\,\overline{\delta}\,\eta$. Thus, $(\mu \vee \rho)\,\overline{\delta}\,\eta$ implies $\mu\,\overline{\delta}\,\eta$ and $\rho\,\overline{\delta}\,\eta$. On the other hand let $\mu\,\overline{\delta}\,\eta$ and $\rho\,\overline{\delta}\,\eta$ are hold for all $\mu, \rho, \eta \in L^X$. Then from Lemma 3.1 we have $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$ are hold and therefore $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}(\mu \vee \rho)) = \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \wedge \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\rho) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$, that is, $\mu\,\overline{\delta}\,\eta$ and $\rho\,\overline{\delta}\,\eta$ imply $(\mu \vee \rho)\,\overline{\delta}\,\eta$. Hence, (P2) is fulfilled. Now, let $\mu, \rho \in L^X$ such that $\mu = \overline{0}$ or $\rho = \overline{0}$. Since $\mathcal{N}_{\varphi_{1,2}}(\overline{0})$ is the finest L-filter on X and from the fact $\varphi_{1,2}.\mathrm{cl}\,\overline{0} = \overline{0}$, we get $\mathcal{N}_{\varphi_{1,2}}(\overline{0}) = \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\overline{0}) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$ holds for all $\rho \in L^X$. Thus, $\overline{0}\,\overline{\delta}\rho$ for all $\rho \in L^X$. Since $\mu = \overline{0}$ or $\rho = \overline{0}$, then we have $\mu\,\overline{\delta}\,\rho$, that is, (P3) is also fulfilled. Since $\mu\,\overline{\delta}\,\rho$ implies $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$ which means by the inequality $(\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\mu) \leq \mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu)$ that $(\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$. Because of Proposition 2.1 and the fact that $\varphi_{1,2}.\mathrm{cl}$ is hull operator we get $\mu \leq \varphi_{1,2}.\mathrm{cl}\,\mu \leq (\varphi_{1,2}.\mathrm{cl}\,\rho)' \leq \rho'$. Thus, (P4) is fulfilled. Let $\mu, \rho \in L^X$ such that $\mu\,\overline{\delta}\,\rho$, then $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$. Consider, $F = S_0(\varphi_{1,2}.\mathrm{cl}\,\mu)$, hence $F \in \varphi_{1,2}C(X)$ and therefore $\mathcal{N}_{\varphi_{1,2}}(F) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$ holds. Since (X, τ) is F $\varphi_{1,2}$ - R_3 -space, then from Theorem 3.2 there exists $\eta' \in L^X$ with arbitrary choice such that $\mathcal{N}_{\varphi_{1,2}}(F) \leq \dot{\eta}'$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\eta') \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$ are hold. Therefore, there exists $\eta \in L^X$ such that $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\mu) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\eta)'$ and $\mathcal{N}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}\,\eta) \leq (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}\,\rho)'$, which means that $\mu\,\overline{\delta}\,\eta$ and $\eta'\,\overline{\delta}\,\rho$. Hence, (P5) is also fulfilled. Consequently, δ is L-proximity on X.

Conversely, $F_1, F_2 \in \varphi_{1,2}C(X)$ such that $F_1 \cap F_2 = \varphi$. Then, $F_1 \subseteq F_2'$ and therefore $\dot{\chi}_{F_1} \leq \dot{\chi}'_{F_2} = \dot{\chi}_{F_2}$. Hence because of Lemma 3.1 part (1) we have $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\chi_{F_2}) \leq \dot{\chi}'_{F_1}$. Since $F_1, F_2 \in \varphi_{1,2}C(X)$, then $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\varphi_{1,2}.\mathrm{cl}_{\delta}\chi_{F_2}) = \mathcal{N}^{\delta}_{\varphi_{1,2}}(\chi_{F_2}) \leq \dot{\chi}'_{F_1} = (\varphi_{1,2}.\dot{\mathrm{c}}\mathrm{l}_{\delta}\chi_{F_1})'$ and therefore $\dot{\chi}_{F_1}\,\overline{\delta}\,\dot{\chi}_{F_2}$. From (P5), there exists $\rho \in L^X$ such that $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\chi_{F_2}) = \mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2) \leq \dot{\rho}$ and $\mathcal{N}^{\delta}_{\varphi_{1,2}}(\rho) \leq \dot{\chi}'_{F_1} = F_1'$ are hold. Because of Lemma 3.1 part (1), we have $\mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2) \leq \dot{\rho}'$. Hence, $\mathcal{N}^{\delta}_{\varphi_{1,2}}(F_1)(\mu) \wedge \mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2)(\eta) \geq \dot{\rho}(\mu) \wedge \dot{\rho}'(\eta)$ holds for all $\mu, \eta \in L^X$. Consider $\eta = \chi_{F_2} \vee x_1 = \rho$ and $\mu = (\chi_{F_2} \vee x_1)' = \rho'$ for all $x \in F_1' \setminus F_2$, then we get $\sup(\mu \wedge \eta) = 0$ and $\mathcal{N}^{\delta}_{\varphi_{1,2}}(F_1)(\mu) \wedge \mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2)(\eta) \geq 0$ are fulfilled. Hence, there exist $\mu, \eta \in L^X$ such that $\mathcal{N}^{\delta}_{\varphi_{1,2}}(F_1)(\mu) \wedge \mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2)(\eta) \geq \sup(\mu \wedge \eta)$, that is, the infimum $\mathcal{N}^{\delta}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}^{\delta}_{\varphi_{1,2}}(F_2)$ does not exists. Consequently, $(X, \varphi_{1,2}.\mathrm{int}_{\delta})$ is characterized

FR_3-space. □

4. Characterized Compact L-spaces

The notion of $\varphi_{1,2}$ -compactness of the L-filters and of the topological L-spaces are introduced in [6] by means of the $\varphi_{1,2}$ L-convergence in the characterized L-spaces. Moreover, the compactness in the characterized L-spaces is also introduced by means of the $\varphi_{1,2}$ -compactness of the L-filters and therefore it will be suitable to study here the relation between the characterized compact L-spaces and some of our classes of separation axioms in the characterized L-spaces.

Let (X, τ) be a topological L-space, $F \subseteq X$ and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then $x \in X$ is said to be $\varphi_{1,2}$ -adherence point for the L-filter $\mathcal{M}$ on X ([6]), if the infimum $\mathcal{M} \wedge \mathcal{N}_{\varphi_{1,2}}(x)$ exists for all $\varphi_{1,2}$ L-neighborhood filters $\mathcal{N}_{\varphi_{1,2}}(x)$ at $x \in X$. As shown in [6], the point $x \in X$ is said to be $\varphi_{1,2}$ -adherence point for the L-filter $\mathcal{M}$ on X if and only if there exists an L-filter $\mathcal{K} \in \mathcal{F}_L X$ finer than $\mathcal{M}$ and $\mathcal{K} \xrightarrow[\varphi_{1,2}.\text{int}]{} x$, that is, $\mathcal{K} \leq \mathcal{M}$ and $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ are hold for some $\mathcal{K} \in \mathcal{F}_L X$. The ordinary subset F is said to be $\varphi_{1,2}$ -closed with respect to $\varphi_{1,2}.\text{int}$ if $\mathcal{M} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ implies $x \in F$ for some $\mathcal{M} \in \mathcal{F}_L X$. The subset F is said to be $\varphi_{1,2}$ -compact subset ([6]), if every L-filter on F has a finer $\varphi_{1,2}$ L-converging L-filter, that is, every L-filter on F has a $\varphi_{1,2}$ -adherence point in F. Moreover, the topological L-space (X, τ) is said to be $\varphi_{1,2}$ -compact if X is $\varphi_{1,2}$ -compact. Generally, the characterized L-space $(X, \varphi_{1,2}.\text{int})$ is said to be compact L-space if the related topological L-space (X, τ) is $\varphi_{1,2}$ -compact.

In the following proposition, we show that in the characterized FT_2 -space $(X, \varphi_{1,2}.\text{int})$, every $\varphi_{1,2}$ -compact subset is $\varphi_{1,2}$ -closed with respect to the $\varphi_{1,2}$ -interior operator $\varphi_{1,2}.\text{int}$.

Proposition 4.1 Let a topological L-space (X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then every $\varphi_{1,2}$ -compact subset of a characterized FT_2 -space is $\varphi_{1,2}$ -closed.

Proof. Let $(X, \varphi_{1,2}.\text{int})$ is characterized FT_2 -space and let F is a $\varphi_{1,2}$ -compact subset of X. Then, for all $\mathcal{M} \in \mathcal{F}_L F$ there exists $\mathcal{K} \in \mathcal{F}_L F$ such that $\mathcal{K} \leq \mathcal{M}$ and $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ are hold for some $x \in F$. Since $\mathcal{K} \in \mathcal{F}_L F \leq \mathcal{F}_L X$ and $(X, \varphi_{1,2}.\text{int})$ is characterized FT_2 -space, then Then $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(x)$ and $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(y)$ imply that $x = y$. Therefore, $y \in F$ for some $\mathcal{K} \in \mathcal{F}_L F$. Hence, F is $\varphi_{1,2}$ -closed. □

In the following proposition we give new property for the characterized FT_2 -spaces by using the $\varphi_{1,2}$ L-neighborhood filters for the L-subsets.

Proposition 4.2 Let (X, τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then in the characterized FT_2 -space $(X, \varphi_{1,2}.\text{int})$, every disjoint $\varphi_{1,2}$ -compact subsets F_1 and F_2 of X have the $\varphi_{1,2}$ L-neighborhood filters $\mathcal{N}_{\varphi_{1,2}}(F_1)$ and $\mathcal{N}_{\varphi_{1,2}}(F_2)$ such that the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$ does not exists.

Proof. Let F_1 and F_2 are two $\varphi_{1,2}$ -compact subsets of the characterized FT_2 -space $(X, \varphi_{1,2}.\text{int})$ such that $F_1 \cap F_2 = \varphi$. Then, for all $\mathcal{M}_i \in \mathcal{F}_L F_i$ there exists $\mathcal{K}_i \in \mathcal{F}_L F_i$ such that $\mathcal{K}_i \leq \mathcal{M}_i$ and $\mathcal{K}_i \leq \mathcal{N}_{\varphi_{1,2}}(x_i)$ for some $x_i \in F_i$, where $i \in \{1,2\}$. Since $\mathcal{F}_L F_i \leq \mathcal{F}_L X$ for all $i \in \{1,2\}$, then we can say that $\mathcal{K}_i \leq \mathcal{N}_{\varphi_{1,2}}(x_i) \leq \mathcal{N}_{\varphi_{1,2}}(F_i)$ and therefore there is $\mathcal{K} = (\mathcal{K}_1 \wedge \mathcal{K}_2) \in \mathcal{F}_L X$ such that $\mathcal{K} \leq \mathcal{N}_{\varphi_{1,2}}(x_i)$ for some $x_i \in F_i$. Since $(X, \varphi_{1,2}.\text{int})$ is FT_2 -space, then $x_1 = x_2$ which contradicts $F_1 \cap F_2 = \varphi$. Hence, for every $\mathcal{L} \in \mathcal{F}_L X$ we get $\mathcal{L} \not\leq \mathcal{N}_{\varphi_{1,2}}(F_1)$ or $\mathcal{L} \not\leq \mathcal{N}_{\varphi_{1,2}}(F_2)$ which means that the infimum $\mathcal{N}_{\varphi_{1,2}}(F_1) \wedge \mathcal{N}_{\varphi_{1,2}}(F_2)$ does not exists. Hence, F_1 and F_2 can be separated by two disjoint $\varphi_{1,2}$ L-neighborhood filters. □

The notion of compactness for the characterized L-spaces fulfills the following property which will be used in the prove of the important result given in Proposition 4.3.

Lemma 4.1 Let (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every $\varphi_{1,2}$-closed subset of a characterized compact L-space $(X, \varphi_{1,2}.\text{int})$ is $\varphi_{1,2}$-compact.

Proof. Let F is $\varphi_{1,2}$-closed subset of a characterized compact L-space $(X, \varphi_{1,2}.\text{int})$ and let $\mathcal{M} \in \mathcal{F}_L F$. Then, $\mathcal{M} \le \mathcal{N}_{\varphi_{1,2}}(x)$ implies that $x \in F$.. Since $\mathcal{F}_L F \le \mathcal{F}_L X$, then $\mathcal{M} \in \mathcal{F}_L X$ and hence there exists $\mathcal{K} \in \mathcal{F}_L X$ such that $\mathcal{K} \le \mathcal{M}$ and $\mathcal{K} \le \mathcal{N}_{\varphi_{1,2}}(x)$ are hold. Since $\mathcal{M} \in \mathcal{F}_L F$ and $\mathcal{K} \le \mathcal{M}$, then $\mathcal{K} \in \mathcal{F}_L F$. Thus, for all $\mathcal{M} \in \mathcal{F}_L F$ we get $\mathcal{K} \le \mathcal{M}$ such that $\mathcal{K} \le \mathcal{N}_{\varphi_{1,2}}(x)$. Therefore, $x \in F$ is $\varphi_{1,2}$-adherence point of $\mathcal{M}$, that is, F is $\varphi_{1,2}$-compact. □

The following proposition give an important relation between the characterized compact FT_2-spaces and the characterized FT_4-spaces.

Proposition 4.3 Let (X,τ) be a topological L-space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized compact FT_2-space $(X, \varphi_{1,2}.\text{int})$ is characterized FT_4-space.

Proof. Follows directly from Lemma 4.1 and Proposition 4.2. □

Lemma 4.2 Let (X,τ) and (X,σ) are two topological L-spaces such that τ is finer than σ. If $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and $\psi_1, \psi_2 \in O_{(L^X,\sigma)}$ and $(X, \psi_{1,2}.\text{int})$ is characterized compact L-space, then $(X, \varphi_{1,2}.\text{int})$ is also characterized compact L-space.

Proof. Let $\mathcal{N}_{\varphi_{1,2}}(x)$ and $\mathcal{N}_{\psi_{1,2}}(x)$ are the $\varphi_{1,2}$ L-neighborhood and $\psi_{1,2}$ L-neighborhood at $x \in X$ with respect to $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$ respectively. Since τ is finer than σ, then $\mathcal{N}_{\varphi_{1,2}}(x) \le \mathcal{N}_{\psi_{1,2}}(x)$ for all $x \in X$. Because of $(X, \psi_{1,2}.\text{int})$ is characterized compact L-space, then for all $\mathcal{M} \in \mathcal{F}_L X$, there exists $\mathcal{K} \in \mathcal{F}_L X$ such that $\mathcal{K} \le \mathcal{M}$ and $\mathcal{K} \le \mathcal{N}_{\psi_{1,2}}(x)$ are hold for all $x \in X$, therefore $\mathcal{K} \le \mathcal{N}_{\varphi_{1,2}}(x)$ holds for all $x \in X$. Thus, $(X, \varphi_{1,2}.\text{int})$ is characterized compact L-space. □

Proposition 4.4 [2] If $(X, \varphi_{1,2}.\text{int})$ is characterized FT_2-space and $\varphi_{1,2}.\text{int}$ is finer than $\psi_{1,2}.\text{int}$, then $(X, \psi_{1,2}.\text{int})$ is also characterized FT_2-space.

Proposition 4.5 Let (X,τ) and (X,σ) are two topological L-spaces such that τ is finer than σ, $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ and $\psi_1, \psi_2 \in O_{(L^X,\sigma)}$. If $(X, \psi_{1,2}.\text{int})$ is characterized compact L-space and $(X, \varphi_{1,2}.\text{int})$ is characterized FT_2-space, then $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$ are isomorphic.

Proof. Since τ is finer than σ, then $\psi_{1,2}.\text{int} \le \varphi_{1,2}.\text{int}$. Hence, because of Proposition 4.4, $(X, \psi_{1,2}.\text{int})$ is characterized FT_2-space. From Lemma 4.2, we have $(X, \varphi_{1,2}.\text{int})$ is characterized compact L-space. Hence, we can find the identity mapping $\text{id}_X : (X, \varphi_{1,2}.\text{int}) \to (X, \psi_{1,2}.\text{int})$ which is bijective $\varphi_{1,2}\psi_{1,2}$ L-continuous and its inverse is $\psi_{1,2}\varphi_{1,2}$ L-continuous, that is, $\varphi_{1,2}\psi_{1,2}$ L-isomorphism. Consequently, $\varphi_{1,2}.\text{int}$ and $\psi_{1,2}.\text{int}$ are isomorphic. □

5. Characterized Uniform L-spaces

In this section, we go to investigate the notion of characterized uniform L-space by the notion of uniform L-spaces introduced in [17]. Moreover, the relation between the separated uniform L-spaces, the associated characterized uniform FT_0-spaces and the F$\varphi_{1,2}$-T_0 spaces which introduced in [2] are investigated.

By an L-relation on a set X we mean a mapping $R : X \times X \to L$, that is, an L-subset of $X \times X$. For each L-relation R on X, the inverse R^{-1} of R is the L-relation on X defined by $R^{-1}(x, y) = R(y, x)$ for all $x, y \in X$. Let $\mathcal{U}$ be an L-filer on $X \times X$. The inverse $\mathcal{U}^{-1}$ of $\mathcal{U}$ is an L-filter on $X \times X$ defined by:

$$\mathcal{U}^{-1}(R) = \mathcal{U}(R^{-1})$$

for all $R \in L^{X \times X}$.

The composition $R_1 \circ R_2$ of two L-relations R_1 and R_2 on a set X is the L-relation on X defined by:

$$(R_1 \circ R_2)(x, y) = \bigvee_{x \in X} \big(R_2(x, z) \wedge R_1(z, y)\big)$$

for all $x, y \in X$. For each pair (x, y) of elements x and y of X, the mapping $(x, y)^{\dot{}} : L^{X \times X} \to L$ defined by:

$$(x, y)^{\dot{}}(R) = R(x, y)$$

for all $R \in L^{X \times X}$ is a homogeneous L-filter on $X \times X$.

Let $\mathcal{U}$ and $\mathcal{V}$ are L-filers on $X \times X$ such that $(x, y)^{\dot{}} \le \mathcal{U}$ and $(y, z)^{\dot{}} \le \mathcal{V}$ hold for some $x, y, z \in X$. Then the composition $\mathcal{V} \circ \mathcal{U}$ of $\mathcal{V}$ and $\mathcal{U}$ is an L-filter ([17]) on $X \times X$ defined by:

$$(\mathcal{V} \circ \mathcal{U})(R) = \bigvee_{R_2 \circ R_1 \le R} \big(\mathcal{U}(R_1) \wedge \mathcal{V}(R_2)\big)$$

for all $R \in L^{X \times X}$.

By the uniform L-structure $\mathcal{U}$ on a set X, we mean an L-filter ([17]) on $X \times X$ such that the following axioms are fulfilled:

(U1) $(x, x)^{\dot{}} \le \mathcal{U}$ for all $x \in X$.

(U2) $\mathcal{U} = \mathcal{U}^{-1}$.

(U3) $\mathcal{U} \circ \mathcal{U} \le \mathcal{U}$.

The pair $(X, \mathcal{U})$ is called uniform L-space.

A uniform L-structure $\mathcal{U}$ on a set X is called separated if for all $x, y \in X$ with $x \ne y$ there is $R \in L^{X \times X}$ such that $\mathcal{U}(R) = 1$ and $R(x, y) = 0$. In this case the uniform L-space $(X, \mathcal{U})$ is called separated uniform L-space.

To each uniform L-structure $\mathcal{U}$ on a set X is associated a stratified L-topology $\tau_{\mathcal{U}}$. Consider $\varphi_1, \varphi_2 \in O_{(L^X, \tau_{\mathcal{U}})}$, then the set of all $\varphi_{1,2}$-open L-subsets of X related to $\tau_{\mathcal{U}}$ forms a characterized stratified L-topology on X generated by the $\varphi_{1,2}$-interior operator with respect to $\tau_{\mathcal{U}}$ denoted by $\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}$ and $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ is a characterized stratified L-space. The characterized stratified L-space $(X, \varphi_{1,2}.\mathrm{int}_{\mathcal{U}})$ will be called associated characterized uniform L-space which is stratified. The related $\varphi_{1,2}$-interior operator $\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}$ is given by:

$$\varphi_{1,2}.\mathrm{int}_{\mathcal{U}}(\mu) = \mathcal{U}(\dot{x})(\mu) \qquad (5.1)$$

for all $x \in X$ and $\mu \in L^X$.

Proposition 5.1 Let X be non-empty set, $\mathcal{U}$ is a uniform L-structure on X and $\varphi_1, \varphi_2 \in O_{(L^X, \tau_{\mathcal{U}})}$. Then the uniform L-space $(X, \mathcal{U})$ is separated if and only if the associated characterized uniform L-space $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ is FT_0 -space.

Proof. Let $(X, \mathcal{U})$ is separated and let $x, y \in X$ such that $x \neq y$. Then, there exists $R \in L^{X \times X}$ such that $\mathcal{U}(R) = 1$ and $R(x, y) = 0$. Consider $\mu = R[y_1]$ for which

$$\mu(x) = R[y_1](x) = \bigvee_{x \in X} R(z, x) \wedge y_1(z) = 0$$

and

$$(\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu)(y) = \mathcal{U}[\dot{y}](\mu) = \bigvee_{R(\eta) \leq \mu} \mathcal{U}(R) \wedge \eta(y) = 1$$

for all $\eta \in L^X$. Hence, there exists $\mu \in L^X$ and $\alpha \in L_0$ such that $\mu(x) < \alpha \leq (\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu)(y)$, that is, $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ is FT_0 -space.

Conversely, let $(X, \varphi_{1,2}.\text{int}_{\mathcal{U}})$ is FT_0 -space and let $x \neq y$ in X . Then, there exists $\mu \in L^X$ and $\alpha \in L_0$ such that $\mu(x) < \alpha \leq (\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu)(y)$. This means that $\bigvee_{R(\eta) \leq \mu} \mathcal{U}(R) \wedge \eta(y) > \mu(x)$ holds for all $\eta \in L^X$. Hence, there is $R \in L^{X \times X}$ for which

$$R(x, y) = (\varphi_{1,2}.\text{int}_{\mathcal{U}}\, \mu)(y) \text{ if } x = y \text{ and } R(x, y) = \mu(x) \text{ if } x \neq y$$

such that $R(x, y) = 0$ and $\mathcal{U}(R) = 1$. Thus, $(X, \mathcal{U})$ is separated. □

Corollary 5.1 Let X be non-empty set, $\mathcal{U}$ is a uniform L-structure on X and $\varphi_1, \varphi_2 \in O_{(L^X, \tau_{\mathcal{U}})}$. Then the uniform L-space $(X, \mathcal{U})$ is separated if and only if the associated stratified topological L-space $(X, \tau_{\mathcal{U}})$ is F $\varphi_{1,2}$ - T_0 space.

Proof. Immediate from Proposition 5.1 and Theorem 2.1 in [2]. □

In Table(1), we give some special choices for the operations φ_1 and φ_2 to obtained some special classes of the $\varphi_{1,2}$ -open L-sets, $\varphi_{1,2}$ L-neighborhood filters, characterized proximity L-spaces, characterized compact L-spaces and the characterized uniform L-spaces.

6. Conclusion

In this paper, we introduced and studied three new types of spaces which are named characterized proximity L-spaces, characterized compact L-spaces and characterized uniform L-spaces. The relation between such spaces with the characterized FT_s -spaces F $\varphi_{1,2}$ - T_s spaces, characterized FR_k -spaces and F $\varphi_{1,2}$ -spaces are investigated for $s \in \{0, 2, 4\}$ and $k \in \{2, 3\}$. Some new properties for the characterized FT_s -spaces, F $\varphi_{1,2}$ - T_s spaces, characterized FR_k -spaces and F $\varphi_{1,2}$ -spaces will be added by applied these new spaces. Many new special classes from the $\varphi_{1,2}$ -open L-sets, $\varphi_{1,2}$ L-neighborhood filters, characterized proximity L-spaces, characterized compact L-spaces and the characterized uniform L-spaces are listed in Table (1).

	Operations	$\varphi_{1,2}$ -pen L-sets	$\varphi_{1,2}$ L-nbd filters	Characterized Proximity L-spaces	Characterized Comact L-spaces	Characterized Uniform L-spaces
1	$\varphi_1 = \text{int}$ $\varphi_2 = 1_{L^X}$	τ [13,18]	Fuzzy nbd. filter [16]	Proximity L-space [11]	Compact L-space [11]	Uniform L-space [11]
2	$\varphi_1 = \text{int}$ $\varphi_2 = \text{cl}$	τ_θ [25]	θ-fuzzy nbd.filter	θ-proximity L-space	θ-compact L-space	θ-uniform L-space
3	$\varphi_1 = \text{int}$ $\varphi_2 = \text{int}\circ\text{cl}$	τ_δ [19]	δ -fuzzy nbd. filter	δ-proximity L-space	δ -compact L-space	δ -uniform L-space
4	$\varphi_1 = \text{cl}\circ\text{int}$ $\varphi_2 = 1_{L^X}$	$SOF(X)$ [10]	Semi fuzzy nbd. filter	Semi -proximity L-space	Semi-compact L-space	Semi -uniform L-space
5	$\varphi_1 = \text{cl}\circ\text{int}$ $\varphi_2 = \text{cl}$	$\tau_{(\theta.S)}$	$(\theta.S)$ -fuzzy nbd. filter	$(\theta.S)$ -proximity L-space	$(\theta.S)$ -compact L-space	$(\theta.S)$ -uniform L-space
6	$\varphi_1 = \text{cl}\circ\text{int}$ $\varphi_2 = \text{int}\circ\text{cl}$	$\tau_{(\delta.S)}$	$(\delta.S)$ -fuzzy nbd. filter	$(\delta.S)$ -proximity L-space	$(\delta.S)$ -compact L-space	$(\delta.S)$ -uniform L-space
7	$\varphi_1 = \text{int}\circ\text{cl}$ $\varphi_2 = 1_{L^X}$	$POF(X)$ [14]	Pre fuzzy nbd. filter	Pre proximity L-space	Pre compact L-space	Pre uniform L-space
8	$\varphi_1 = \text{cl}\circ\text{int}$ $\varphi_2 = S.\text{cl}$	$\tau_{(S.\theta)}$	$(S.\theta)$ -fuzzy nbd. filter	$(S.\delta)$ -proximity L-space	$(S.\delta)$ -compact L-space	$(S.\delta)$ -uniform L-space
9	$\varphi_1 = \text{cl}\circ\text{int}$ $\varphi_2 = S.\text{int}\circ S.\text{cl}$	$\tau_{(S.\delta)}$	$(S.\delta)$ -fuzzy nbd. filter	$(S.\delta)$ -proximity L-space	$(S.\delta)$ -compact L-space	$(S.\delta)$ -uniform L-space
10	$\varphi_1 = \text{cl}\circ\text{int}\circ\text{cl}$ $\varphi_2 = 1_{L^X}$	$\beta OF(X)$ [9]	β -fuzzy nbd. filter	β -proximity L-space	β -compact L-space	β -uniform L-space
11	$\varphi_1 = \text{int}\circ\text{cl}\circ\text{int}$ $\varphi_2 = 1_{L^X}$	$\lambda OF(X)$ [14]	λ -fuzzy nbd. filter	λ -proximity L-space	λ -compact L-space	λ -uniform L-space
12	$\varphi_1 = S.\text{cl}\circ\text{int}$ $\varphi_2 = 1_{L^X}$	$fOF(X)$	Feebly-fuzzy nbd. filter	Feebly proximity L-space	Feebly compact L-space	Feebly uniform L-space

Table (1) : Some special classes of $\varphi_{1,2}$ -open L-sets, $\varphi_{1,2}$ L-neighborhood filters, characterized proximity L-spaces, characterized compact L-spaces, characterized uniform L-spaces.

References

[1] Abd-Allah, A. S. (2002). General notions related to fuzzy filters, J. Fuzzy Math., 10(2) , 321 - 358.
[2] Abd-Allah, A. S. (2007). Separation axioms in characterized fuzzy spaces, J. Fuzzy Math., 15(2), 291 - 329.
[3] Abd-Allah, A. S. (2008). Fuzzy regularity axioms and $FT_{2\frac{1}{2}}$ axioms in characterized spaces, J. Egypt. Math. Soc., 16(2), 225 - 253.
[4] Abd-Allah, A. S. (2010). On characterized fuzzy regular and fuzzy normal spaces, J. Egypt. Math. Soc., 18(2), 9 - 38.
[5] Abd-Allah, A. S. and El-Essawy, M. (2003). On characterizing notions of characterized spaces, J. Fuzzy Math., 11(4), 835 - 875.

[6] Abd-Allah, A. S. and El-Essawy, M. (2004). Closedness and compactness in characterized spaces, J. Fuzzy Math., 12(3), 591 - 632.
[7] Abd El-Monsef, M. E., Zeyada, F. M., Mashour A. S. and El-Deeb, S. N. (1983). Operations on the power set P(X) of a topological space (X,T), Colloquium on topology,
Janos Bolyai Math. Soc. Eger, Hungry.
[8] Adàmek, J. Herrlich, H. and Strecker, G. 1990. Abstract and Concrete Categories, John Wiley & Sons, Inc., New York et al.
[9] Andrijevic, D. (1986). Semi-preopen sets, Math. Vesnik, 38, 24 - 32.
[10] Azad, K. K. (1981). On fuzzy semicontinuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl., 82, 14 - 32.
[11] Bayoumi, F. and Ibedou, . (2004). The relation between GT_i-spaces and fuzzy proximity spaces, fuzzy uniform spaces, Chaos, Solitons and Fractals, 20, 955 - 966.
[12] Chang, C. L. (1968). Fuzzy topological spaces, J. Math. Anal. Appl., 24, 182 - 190.
[13] Eklund, P. and Gähler, W. Dordrecht et al. (1992). Fuzzy filter functors and convergence, in: Applications of Category Theory Fuzzy Subsets, Kluwer Academic Publishers, 109 - 136.
[14] Fath Alla, M. A. (1980). On Fuzzy Topological Spaces Ph.D. Thesis, Assiut University, Egypt.
[15] Gähler , W. (1995). The general fuzzy filter approach to fuzzy topology, I, Fuzzy Sets and Systems, 76, 205 - 224.
[16] Gähler , W. (1995). The general fuzzy filter approach to fuzzy topology, II, Fuzzy Sets and Systems, 76, 225 - 246.
[17] Gähler , W., Bayoumi, F., Kandil, A. and Nouh, A. (1998). The theory of global fuzzy neighborhood structures, (III), Fuzzy uniform structures, Fuzzy Sets and Systems, 98, 99 - 175.
[18] Gähler , W., Abd-Allah, A. S. and Kandil, A. (2000). On Extended fuzzy topologies, Fuzzy Sets and Systems, 109, 149 - 172.
[19] Ganguly, S. and Saha, S. (1988). A not on δ-continuity and δ-connected sets in fuzzy set theory, Simon Stevin , 62(2), 127 - 141.
[20] Goguen, J. A. (1967). L-fuzzy sets, J. Math. Anal. Appl., 18, 145 - 174.
[21] Kandil, A., Abd-Allah, A. S. and Nouh, A. A. (1999). Operations and its applications on L-fuzzy bitopological spaces, Part I, Fuzzy Sets and Systems, 106, 255 - 274.
[22] Kasahara, S. (1979). Operation-compact spaces, Math. Japon., 24(1), 97 - 105.
[23] Katsaras, A. K. (1980). Fuzzy proximities and fuzzy completely regular spaces, I, An Stiinto Univ. Iassi Sect Ia Math. (NS), 26, 31 - 41.
[24] Lowen, R. (1976). Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl., 56, 621 - 633.
[25] Lowen, R. (1979). Convergence in fuzzy topological spaces, General Topology and Appl., 10, 147 - 160.