Unique Fixed Point Theorems for Generalized Weakly Contractive Condition in Ordered Partial Metric Spaces

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Abstract
The aim of this paper to prove some fixed point theorems for generalized weakly contractive condition in ordered partial metric spaces. The result extend the main theorems of Nashine and altun[17] on the class of ordered partial metric ones.
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Introduction and preliminaries

The concept of partial metric space was introduced by Methuews [16] in 1994.In such spaces the distance of a point to itself may not be zero. Specially from the point of sequences, a convergent sequence need not have unique limit. Methuews [16] extended the well known Banach contraction principle to complete partial metric spaces. After that many interesting fixed point results were established in such spaces. In this direction we refer the reader to Velero[21]. Oltra and Velero[23]. Altun et at[4]. Ramaguera[24]. Altun and Erduran[2] and Aydi[6,7,8]. First we recall some definitions and properties of partial metric spaces [see 2, 4, 16, 22,23,24,25 for more details]

Definition 1.1:- A partial metric on non empty set X is a function

$p : X \times X \to \mathbb{R}_+$ such that for all $x, y, z \in X$.

$(p_1)\, x = y \Leftrightarrow p(x,y) = p(x,y) = p(y,y)$

$(p_2)\, p(x,x) \le p(x,y)$

$(p_3)\, p(x,y) = p(y,x)$

$(p_4)\, p(x,y) \le p(x,x) + p(z,y) - p(z,z)$

A partial metric space is a pair (X, p) such that X is a non-empty set and p is a partial metric on X.

Remarks 1.2:- It is clear that if $p(x, y) = 0$, then from (p_1) and (p_2).

$x = y$ *But if* $x = y$, $p(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}_+, p)$ *where* $p(x, y) = max\{x, y\}$ *for all* $x, y \in \mathbb{R}_+$

Each partial metric p on X generates a T_0 topology τ_p on X which has a base the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \varepsilon > 0\}$ Where

$B_p(x\ \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ *for all* $x \in X$ *and* $\varepsilon > 0$.

If p is a partial metric on X then function $P^s : X \times X \to R^+$ given by

$$P^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

Is metric on X.

Definition 1.3:- let (X, p) be a partial metric space and $\{x_n\}$ be a sequence in X. then

$(i)\ \{x_n\}$ *converges to a point* $x \in X$ *if and only if* $p(x,x) = \lim_{p\to+\infty} p(x,x_n)$.

$(ii)\ \{x_n\}$ *is called Cauchy sequence if there exists* (*and is finite*) $\lim_{n,m\to+\infty} p(x_n\ x_m)$.

Definition 1.4:- A partial metric space (X,p) is said to be complete if every Cauchy sequence (x_n) in x converges, with respect to τ_p to a point $x\in$X, such that p(x,x)=lim p(x_n,x_m).

Lemma 1.5:- Let (X,p)be a partial metric space then.

(a) { x_n} is a Cauchy sequence in (X,p)if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) (X,p) is complete if and only if the metric space (X,p^s) is complete. Further more $\lim_{n\to+\infty} p^s(x_n,x)=0$ if and only if

$$p(x,x)=\lim_{n\to+\infty} p(x_n,x)=\lim_{n,m\to+\infty} p(x_n,x_m)$$

Definition 1.6:- (([2]) suppose that (X, p) is a metric space. A mapping $F:(X,p)\to(X,p)$ is said to be continuous at $x\in X$, if for every ε>0 there exists δ>0 such that $F\left(B_p(x,\ \delta)\right)\subseteq B_p(Fx,\varepsilon)$.

The following results are easy to check.

Lemma 1.7 :- let (x,p) be a partial metric space F:X→X is continuous if and only if given a sequence { x_n }$\in\mathbb{N}$ and $x\in$X such that p(x,x)= $\lim_{n\to+\infty}$ p(x, x_n),

hence p(Fx,Fx)= $\lim_{n\to+\infty}$ p(Fx,Fx_n).

Remarks 1.8: ([22]) let (x, p) be a partial metric space and F :(x, p) →(x, p) if F is continuous on (X, p) then F: (X, p^s) →(X, p^s) is continuous.

On the other hand. Fixed point problems of contractive mapping in metric spaces endowed with a partially order have been studied by many authors (see [1,3,5,9,10,11,12,14,15,17,18,19,20,21]. In particular Nashine and altun [17] proved the following.

Theorem 1.9:- Let (X, ≤) be a partially ordered set and (X, d) be a complete metric space. Suppose that T:X→X is a nondecreasing mapping such that for every to comparable elements. x,y$\in$X

$$\Psi\,(d(Tx,Ty))\le\Psi\,(m(x,y))-\varphi\,(m(x,y)) \qquad .(1.2)$$

Where

$$M(x,y)=a_1d(x,y)+a_2d(x,Tx)+a_3d(y,Ty)+a_4[d(y,Tx)+d(x,Ty)]+a_5[d(y,Ty)+d(x,Tx)]$$

With a_1>0 , a_2,a_3,a_4,a_5 >0,$a_1+a_2+a_3+2(a_4+a_5)\le 1$ and Ψ , φ :[(0,+∞] → [0,+∞] φ is a continuous non decreasing φ is a lower semi continues function and Ψ(t)=0= φ (t) If and only if t=0. Also suppose there exists $x_0\in$X with $x_0\le$ Tx_0. Assume that

i. T is continuous or

ii. If a nondcreasing sequence { x_n} converges to x, then $x_n\leftarrow x$ for all n.

Then T has a fixed point

The purpose of this paper is to extend theorem (1.9) on the class of ordered partial metric space. Also a common fixed point result is given.

2. Main results

Theorem 2.1:- (X, ≤) be a partially ordered set and (X, p) be a complete partial metric space. Suppose that T: X→X is a nondecreasing such that for every two comparable element x, y$\in$X.

$$\Psi(p(Tx,Ty))<=\Psi\,(\theta(x,y))-\varphi\,(\theta(x,y)) \qquad (2.1)$$

Where

$$\theta(x,y)=a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Ty)+a_4[p(y,Tx)+p(x,Ty)]$$
$$+a_5[p(y,Ty)+p(x,Tx)] \qquad (2.2)$$

With . (a_1, a_4,a_5)>0 , (a_2,a_3) >=0,($a_1+a_2+a_3+2(a_4+a_5)$)≤ 1 and Ψ , φ :[(0,+∞] → [0,+∞] Ψ is a continuous nondecreasing, φ is a lower semi continuous function and

Ψ(t)=0= φ (t). if and only if t=0 . Also suppose there exists there exists $x_0\in$X with $x_0\le$ Tx_0.

Assume that:

i. T is continuous or

ii. If a non decreasing sequence { x_n } converges to x in (X, p) then $x_n\le x$ for all n.

Then T has a fixed point, say z moreover p(z,z)=0.

Proof: - If $Tx_0=x_0$ then the proof is completed. Suppose that $Tx_0\neq x_0$. Now since $x0<Tx0$ and T is non decreasing we have

$x_0<Tx_0\leq T^2x_0\leq$ ……………………………..$\leq T^n x_0\leq T^{n+1}x_0\leq$………

Put $x_n=T^n x_0$, hence $x_{n+1}=Tx_n$. If there exists $n_0\in\{1,2,\ldots.\}$ such that $\theta\left(x_{n_0},x_{n_0-1}\right)=0$ then by definition (2.2), it is clear that $p\left(x_{n_0-1},x_{n_0}\right)=p\left(x_{n_0},Tx_{n_0-1}\right)=0$,

So $x_{n_0-1}=x_{n_0}=Tx_{n_0-1}$ and so we are finished. Now we can suppose

$$\theta(x_n,x_{n-1})>0 \tag{2.3}$$

For all n≥1, let us check that

$$\lim_{n\to+\infty} p(x_{n+1}, x_n)=0 \tag{2.4}$$

By (2.2), we have using condition (p_4)

$\theta(x_n,x_{n-1})=a_1p(x_nx_{n-1}) + a_2p(x_n,Tx_n) + a_3p(x_{n-1},Tx_{n-1}) + a_4[p(x_{n-1},Tx_n)+p(x_n,Tx_{n-1})$
$+ a5[p(x_{n-1},Tx_{n-1})+p(x_n,Tx_n)]$
$= a_1p(x_n, x_{n-1}) + a_2p(x_n, x_{n+1}) +a_3p(x_{n-1}, x_n)+a4[p(x_{n-1}, x_{n+1})+p(x_n,x_n)]$
$+a_5[p(x_{n-1},x_n)+p(x_n,x_{n+1})]$
$\leq(a_1+a_3+a_4+a_5)p(x_n,x_{n-1})+(a_2+a_4+a_5)p(x_n,x_{n+1})$[by (p4)]

Now we claim that

$$P(x_{n+1},x_n)\leq p(x_n,x_{n-1}) \tag{2.5}$$

For all n≥1. Suppose that is not true, that is there exists $n_0\geq1$ such that $p\left(x_{n_0+1},x_{n_0}\right)>p\left(x_{n_0},x_{n_0-1}\right)$ now since $x_{n_0}\leq x_{n_0+1}$ we can use the inequality(2.1) then we have

$$\Psi\left(p\left(x_{n_0+1},x_{n_0}\right)\right)=\Psi\left(p\left(Tx_{n_0},Tx_{n_0-1}\right)\right)$$
$$\leq\Psi(\theta\left(x_{n_0},x_{n_0-1}\right)-\varphi(\theta\left(x_{n_0},x_{n_0-1}\right))$$
$$\leq\Psi\left(a_1+a_3+a_4+a_5\right)p\left(x_{n_0},x_{n-1}\right)$$
$$+\left(a_2+a_4+a_5\right)p\left(x_{n_0},x_{n_0+1}\right)-\varphi\left(\theta\left(x_{n_0},x_{n_0\ ?\ 1}\right)\right)$$
$$\leq\Psi\left(a_1+a_2+a_3+2a_4+2a_5\right)p\left(x_{n_0},x_{n_0+1}\right)-\varphi\left(\theta\left(x_{n_0},x_{n_o-1}\right)\right)$$
$$\leq\Psi\left(p\left(x_{n_0}x_{n_0+1}\right)-\varphi\left(\theta\left(x_{n_0},x_{n_o-1}\right)\right)\right)$$

Which implies that $\varphi(\theta\left(x_{n_0}, x_{n_0-1}\right))\leq 0$ and by property of φ given that $\theta\left(x_{n_0}, x_{n_0-1}\right)=0$, this contradict (2.3) hence(2.5)holds and so the sequence $p(x_{n+1},x_n)$ is non increasing and bounded below. Thus there exists ρ>0 such that limit

$\lim_{n\to+\infty} p(x_{n+1},x_n)=\rho$. Assume that ρ >0, by (2.2), we have

$$a_1\rho=\lim_{n\to+\infty} a_1p(x_n,x_{n-1})\leq\lim_{n\to+\infty}\sup\theta(x_n,x_{n-1})$$
$$=\lim_{n\to+\infty}\sup[(a_1+a_3)p(x_n,x_{n-1})+a_2p(x_n,x_{n+1})$$
$$+a_4[p(x_{n-1},x_{n+1})+p(x_n,x_n)]+a_5[p(x_{n-1},x_n)+p(x_n,x_{n+1})]$$
$$\leq\lim_{n\to+\infty}\sup[(a_1+a_3+a_4+a_5)p(x_n,x_{n-1})+(a_2+a_4+a_5)p(x_n,x_{n-1})]$$

This implies that

$$0<a_1\rho\leq\lim_{n\to+\infty}\sup\theta(x_n, x_{n-1})\leq(a_1+a_2+a_3+2a_4+2a_5)\rho\leq\rho$$

And so there exists $\rho_1>0$ and subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that

$$\lim_{k\to+\infty}\theta(x_{n(k)},x_{n(k)-1})=\rho_1\leq\rho$$

By the lower semi-continuity of φ we have

$$\varphi(\rho_1) \le \lim_{k\to+\infty} \inf \varphi(\theta(x_{n(k)}, x_{n(k)+1}))$$

From(2.1) we have

$\Psi(p(x_{n(k)+1}, x_{n(k)})) = \Psi(p(Tx_{n(k)}, Tx_{n(k)-1}))$

$\le \Psi(\theta(x_{n(k)}, x_{n(k)-1})) - \varphi(\theta(x_{n(k)}, x_{n(k)-1}))$

And taking upper limit as K→+∞ we have using the properties of Ψ and φ

$$\Psi(\rho) \le \psi(\rho_1) - \lim_{k\to+\infty} \inf \varphi(\theta(x_{n(k)}, x_{n(k)+1}))$$

$$\le \Psi(\rho_1) - \varphi(\rho_1)$$

$$\le \Psi(\rho) - \varphi(\rho_1)$$

That is $\varphi(\rho_1) = 0$ thus by the property of φ we have $\rho_1 = 0$ which is a contradiction. Therefore we have $\rho = 0$ that is (2.4) holds.

Now we show that $\{x_n\}$ is a cauchy sequence in the partial metric space (x,p).

From lemma 1.5 it is sufficient to prove that $\{x_n\}$ is a Cauchy sequence in the metric space (X,p^s)suppose to the contrary. Then there is a $\in>0$ such that for and integer K there exist integer m(k)>n(k)>k such that

$$p^s(x_{n(k)}, x_{m(k)}) > \varepsilon \tag{2.6}$$

For ever integer K let m(k) be the least positive integer exceeding n(k) satisfying (2.6)and such that

$$p^s(x_{n(k)}, x_{m(k)-1}) \le \varepsilon \tag{2.7}$$

Now using (2.4)

$\varepsilon < p^s(x_{n(k)}, x_{m(k)}) \le p^s(x_{n(k)}, x_{m(k)-1}) + p^s(x_{m(k)-1}, x_{m(k)})$

$\le \varepsilon + p^s(x_{m(k)-1}, x_{m(k)})$

Then by (2.4) it follows that

$$\lim_{k\to+\infty} p^s\left(x_{n(k)}, x_{m(k)}\right) = \varepsilon \tag{2.8}$$

Also by the triangle inequality. We have

$$\left| p^s\left(x_{n(k)}, x_{m(k)-1}\right) - p^s\left(x_{n(k)}, x_{m(k)}\right) \right| \le p^s\left(x_{m(k)-1}, x_{m(k)}\right)$$

By using (2.4), (2.8) we get

$$\lim_{k\to+\infty} p^s\left(x_{n(k)}, x_{m(k)-1}\right) = \varepsilon \tag{2.9}$$

On the other hand by definition of p^s .

$$p^s\left(x_{n(k)}, x_{m(k)-1}\right) = 2p\left(x_{n(k)}, x_{m(k)}\right) - p\left(x_{n(k)}, x_{n(k)}\right) - p\left(x_{m(k)}, x_{m(k)}\right)$$

Hence

$$p^s\left(x_{n(k)}, x_{m(k)-1}\right) = 2p\left(x_{n(k)}, x_{m(k)-1}\right) - p\left(x_{n(k)}, x_{n(k)}\right) - p\left(x_{m(k)-1}, x_{m(k)-1}\right)$$

letting k→+∞, we find thanks to (2.8), (2.9) and the condition p_3 in (2.4)

$$\lim_{k\to+\infty} p\left(x_{n(k)}, x_{m(k)}\right) = \frac{\varepsilon}{2} \tag{2.10}$$

$$\lim_{k\to+\infty} p\left(x_{n(k)}, x_{m(k)-1}\right) = \frac{\varepsilon}{2} \tag{2.11}$$

In view of (2.2) we get

$$
\begin{aligned}
a_1 p\left(x_{n(k)}, x_{m(k)-1}\right) \le \theta\left(x_{n(k)}, x_{m(k)-1}\right)
&= a_1 p\left(x_{n(k)}, x_{m(k)-1}\right) + a_2 p\left(x_{n(k)}, \mathrm{T}x_{\mathrm{n}(k)}\right) \\
&+ a_3 p\left(x_{\mathrm{m}(k)-1}, \mathrm{T}x_{m(k)-1}\right) \\
&+ a_4 [p\left(x_{\mathrm{m}(k)-1}, \mathrm{T}x_{\mathrm{n}(k)-1}\right) + p\left(x_{n(k)}, \mathrm{T}x_{m(k)-1}\right)] \\
&+ a_5\left[p\left(x_{\mathrm{m}(k)-1}, \mathrm{T}x_{m(k)-1}\right) + p\left(x_{\mathrm{n}(k)}, \mathrm{T}x_{\mathrm{n}(k)}\right)\right] \\
&= a_1 p\left(x_{n(k)}, x_{m(k)-1}\right) + a_2 p\left(x_{n(k)}, x_{\mathrm{n}(k)+1}\right) \\
&+ a_3 p\left(x_{\mathrm{m}(k)-1}, x_{m(k)}\right) \\
&+ a_4 [p\left(x_{\mathrm{m}(k)-1}, x_{\mathrm{n}(k)}\right) + p\left(x_{n(k)}, x_{\mathrm{n}(k)+1}\right)] \\
&+ a_5\left[p\left(x_{\mathrm{m}(k)-1}, x_{m(k)}\right) + p\left(x_{\mathrm{n}(k)}, x_{\mathrm{n}(k)+1}\right)\right] \\
&\le a_1 p\left(x_{n(k)}, x_{m(k)-1}\right) + a_2 p\left(x_{n(k)}, x_{\mathrm{n}(k)+1}\right) \\
&+ a_3 p\left(x_{\mathrm{m}(k)-1}, x_{m(k)}\right) \\
&+ a_4\left[p\left(x_{\mathrm{m}(k)-1}, x_{\mathrm{n}(k)}\right) + p\left(x_{n(k)}, x_{\mathrm{n}(k)+1}\right) + p\left(x_{n(k)}, x_{m(k)}\right)\right] \\
&+ a_5\left[p\left(x_{\mathrm{m}(k)-1}, x_{m(k)}\right) + p\left(x_{\mathrm{n}(k)}, x_{\mathrm{n}(k)+1}\right)\right]
\end{aligned}
$$

Taking upper limit as K→ +∞ and using (2.4),(2.10) and (2.11) we have

$$
0 < a_1 \frac{\varepsilon}{2} \le \lim_{k\to+\infty} \sup \theta\left(x_{n(k)}, x_{m(k)-1}\right) \le (\mathrm{a}_1 + 2a_4 + 2\,\mathrm{a}_5)\frac{\varepsilon}{2} \le \frac{\varepsilon}{2}. \quad \ldots
$$

this implies that there exists $\varepsilon_1 > 0$ And subsequence *$\{x_{n(k(p))}\}$ of $\{x_{n(k)}\}$* Such that

$$
\lim_{p\to+\infty} \theta\left(x_{n(k(\mathrm{p}))}, x_{m(k(\mathrm{p}))-1}\right) = \varepsilon_1 \le \frac{\varepsilon}{2}
$$

By the lower semi continuity of φ we have

$$
\varphi(\varepsilon_1) \le \lim_{k\to+\infty} \inf \varphi\left(\theta\left(x_{n(k)}, x_{m(k)-1}\right)\right)
$$

Now by (2.1) we get…

$$
\begin{aligned}
\psi\left(\frac{\varepsilon}{2}\right) &= \lim_{p\to+\infty} \sup \psi\left(p\left(x_{n(k(\mathrm{p}))}, x_{m(k(\mathrm{p}))}\right)\right) \\
&\le \lim_{p\to+\infty} \sup \psi\left(p\left(x_{n(k(\mathrm{p}))}, x_{m(k(\mathrm{p}))+1}\right)\right) + \left(p\left(Tx_{n(k(\mathrm{p}))}, \mathrm{T}x_{m(k(\mathrm{p}))-1}\right)\right) \\
&= \lim_{p\to+\infty} \sup \psi(\mathrm{p}(\mathrm{Tx}_{n(\mathrm{k(p))}}), \mathrm{Tx}_{m(\mathrm{k(p)}-1)}))
\end{aligned}
$$

$$\leq \lim_{p\to+\infty} \sup\left[\psi\left(\theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right) - \varphi\left(\theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right)\right]$$
$$= \psi(\varepsilon_1) - \lim_{p\to+\infty} \inf \varphi\left(\theta\left(x_{n(k(p))}, x_{m(k(p))-1}\right)\right)$$
$$\leq \psi(\varepsilon_1) - \varphi(\varepsilon_1)$$
$$\leq \psi\left(\frac{\varepsilon}{2}\right) - \varphi(\varepsilon_1)$$

Which is a contradiction? Therefore $\{x_n\}$ is a Cauchy sequence in the metric space (X, p^s) from lemma (1.5) (X, p^s) is a complete metric space. Then there is $z\in X$ such that $\lim_{n\to+\infty} p^s(x_n, z) = 0$

Again from lemma (1.5), we have thanks to(2.4) and the condition (p_2).

$$p(z,z) = \lim_{n\to+\infty} p(x_n, z) = \lim_{n\to+\infty} p(x_n, x_n) = 0 \qquad (2.12)$$

We will prove that Tz=z

1. Assume that (i) hold, that is T is continuous. By(2.12) the sequence converges in(X,p)to z, and since T is continuous hence the sequence.. converges to Tz that is

$$p(Tz, Tz) = \lim_{n\to+\infty} p(Tx_n, Tz) \qquad (2.13)$$

Again thanks to (2.12)

$$P(z,Tz)= \lim_{n\to+\infty} p(x_n, z) = \lim_{n\to+\infty} p(Tx_{n-1}, Tz) = p(Tz, Tz) \qquad (2.14)$$

On the other hand by(2.1),(2.14)

$\Psi((p(z,Tz))=\Psi (p(Tz,Tz))\leq\Psi (\theta(z,z)- \varphi (\theta(z,z))$

Where from (2.12) and the condition p_2

$\theta(z,z) = a_1p(z,z) +(a_2+a_3+2a_4+2a_5)p(z,Tz)$

$= (a_2+a_3+2a_4+2a_5)p (z,Tz)\leq p(z,Tz)$

Thus, $\Psi (p(z,Tz), \leq\Psi(\theta(z,z))- \varphi (\theta((z,z))$

$\leq\Psi(p(z,Tz))- \varphi (\theta(z,z))$

In follows that $\varphi (\theta(z, Z))=0$ so $\theta (z,z)=(a_2+a_3+2a_4+2a_5)p(z,Tz) =0$ that is $p(z,Tz)=0$, because $\varepsilon> 0$. Hence z=Tz that is z is a fixed point of T

Assume that ii holds than we have $x_n\leq z$ for all n, Therefore all n, we can use the inequality (2.1) for x_n and z since

$\theta(z,x_n)=a_1p(z,x_n)+a_2p(z,Tz)+a_3p(x_n,Tx_n)+a_4[p(x_n,Tz)+p(z,Tx_n)+a_5[p(x_n,Tx_n)+p(z,Tz)]$

$= a_1p(z,x_n)+a_2p(z,Tz)+a_3p(x_n,x_{n+1})+a_4[p(x_n,Tz)+p(z,x_{n+1})+a_5[p(x_n,x_{n+1})+p(z,Tz)]$

Hence from (2.4), (2.12)

$$\lim_{n\to+\infty} \theta(z, x_n) = (a_1 + a_4 + a_5)p(z, Tz)$$

we have,

$$\psi(p(z,Tz)) = \lim_{n\to+\infty} \sup\psi\left(p\left(Tz, x_{n+1}\right)\right)$$
$$= \lim_{n\to+\infty} \sup\psi\left(p\left(Tz, Tx_n\right)\right)$$
$$\leq \lim_{n\to+\infty} \sup\psi\left[\left(\psi\left(z, x_n\right)\right) - \varphi(\theta(z,x_n))\right]$$

$\leq\Psi ((a_1 +a_4+a_5)p(Tz,z))- \varphi (a_2+a_4+a_5)p(Tz,z).$

$\leq\Psi(p(Tz,z))- \varphi ((a_2+a_4+a_5)p(Tz,z)).$

Then $\varphi ((a_2+a_4+a_5)p(Tz,z))=0$ And since $(a_4,a_5) >0$ hence by the property of φ we have $p(Tz,z)=0$ so $Tz=z$, This complete the proof of theorem (2.1)

Remarks 2.2 Theorem 2.1 holds for ordered partial metric spaces, so it is an extension of the result of Noshine and altun (17) given in theorem (1.9) which is verified just for ordered metric ones.

Corollary 2.3 :- Let $(X,\leq)$ be a partially ordered set and (x,p) be a complete partial metric space suppose that $T:X\rightarrow X$ be a non decreasing mapping such that for every two comparable elements $x,y\in X$

$$P(Tx,Ty)\ ,\leq\theta(x,y)-\ \varphi\ (\theta(x,y)) \qquad (2.15)$$

Where

$$\theta(x,y)=a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Ty)$$
$$+a_4[p(y,Tx)+p(x,Ty)+a_5[p(y,Ty)+p(x,Tx)] \qquad (2.16)$$

With $(a_1,a_4,a_5)>0.(a_2,a_3)\geq 0,(a_1+a_2+a_3+2a_4+2a_5)\leq 1$ and φ : $(0,+\infty)\rightarrow(0,+\infty)$. φ is a lower semi continuous function and φ (t)=0 if and only if t=0 also suppose that there exists $x_0\in X$ with $x_o\leq Tx_0$, Assume that

i. T is continuous or φ
ii. If a nondcreasing sequence $\{x_n\}$ converges to x, in (X,p),then $x_n\leq x$ for all n.

Then T has a fixed point, say z moreover p(z,z)=0

Proof: - It is sufficient to take Ψ (t)=t in theorem.

Corollary 2.4:- Let $(X.\leq)$ be a partially ordered set and (X,p) be a complete partial metric space suppose that $T:X\rightarrow X$ be a non decreasing mapping such that for every two comparable elements $x,y\in X$

$$P\ (Tx,\ Ty)\leq k\theta(x,y) \qquad (2.17)$$

Where

$$\theta(x,y)=a_1p(x,y)+a_2p(x,Tx)+a_3p(y,Ty)$$
$$+a_4[p(y,Tx)+p(x,Ty)+a_5[p(y,Ty)+p(x,Tx)] \qquad (2.18)$$

With $k\in[0,1],(a_1,a_4,a_5)>0,(a_2,a_3)\geq 0,(a_1+a_2+a_3+2a_4+2a_5)\leq 1$ also suppose , there exists $x_0\in X$ with $x_o\leq Tx_0$, Assume that

i. T is continuous or
ii. If a nondcreasing sequence $\{x_n\}$ converges to x in(X,p) then $x_n\leq x$ for all n.

Then T has a fixed point, say z moreover p(z,z)=0

Proof: - It sufficient to take $\Psi(t)=(1-k)t$ in corollary (2.3)

We give in the following a sufficient condition for the uniqueness of the fixed point of the mapping T.

Theorem 2.5 :- Let all the conditions of the theorem (2.1) be fulfilled and let the following condition hold for arbitrary two points $x,y\in X$ there exists $z\in X$ which is comparable with both x and y. If $(a_1+2a_2+2a_4+2a_5)\leq 1$ or $(a_1+2a_3+2a_4+2a_5)\leq 1$. Then the fixed point of T is unique.

Proof :- Let u and v be two fixed point of T, i.e Tu=u and Tv=v. we have in mind, p(u,u)=p(v,v)=0. Consider the following two cases.

1. U and v are comparable. Then we can apply condition 2.1 and obtain that

$$\Psi(p(u,v))=\Psi(p(Tu,Tv))<=\Psi(\theta(u,v))-\ \varphi\ (\theta(u,v))$$

Where

$$\begin{aligned}\theta(u,v)&=a_1p(u,v)+a_2p(u,Tu)+a_3p(v,Tv)+a_4[p(u,Tv)+p(v,Tu)]+a5[p(v,Tv)+p(u,Tu)]\\&=a_1p(u,v)+a_2p(u,u)+a_3p(v,v)+a_4[p(u,v)+p(v,u)]+a_5[p(v,v)+p(u,u)]\\&=((a_1+2a_4+2a_5)p(u,v))+a_2p(u,u)+a_3p(v,v)\\&\leq(a_1+a_2+a_3+2a_4+2a_5)p(u,v)\leq p(u,v).\end{aligned}$$

We deduce

$$\Psi(p(u,v))\leq\Psi(p(u,v)-\ \varphi\ (\theta\ (u,v))\ \text{i.e}\ \theta(u,v)=0$$

So $p(u,v)=0$ meaning that $u=v$, that is the uniqueness of the fixed point of T.

2. Suppose that u and v are not comparable. Choose and element $w\in X$ comparable with both of them. Then also $u=T^nu$ is comparable is T^nw for each n (Since T is nondecreasing) Appling (2.1) one obtain that

$$\begin{aligned}\Psi(p(u,T^nw))&=\Psi(p(TT^{n-1}u,\ TT^{n-1}w))\\&\leq\Psi(\theta(T^{n-1}u,\ T^{n-1}w))-\ \varphi\ (\theta(T^{n-1}u,\ T^{n-1}w))\\&=\Psi(\theta(u,\ T^{n-1}w))-\ \varphi\ (\theta(u,\ T^{n-1}w))\end{aligned}$$

Where

$$\begin{aligned}\theta(u,T^{n-1}w)&=a_1p\ (u,\ T^{n-1}w)+\ a_2p(u,T\ T^{n-1}u)+a_3p(T^{n-1}w,T\ T^{n-1}w)\\&+a_4[p(u,T\ T^{n-1}w)+p(T^{n-1}w,\ Tu)]+a_5[p((T^{n-1}w,T\ T^{n-1}w)+p(u,\ TT^{n-1}u)]\\&=a_1p\ (u,\ T^{n-1}w)+\ a_2p(u,u)+a_3p(T^{n-1}w,\ T^nw)\\&+a_4[p(u,\ T^nw)+p(T^{n-1}w,\ u)]+a_5[p((T^{n-1}w,\ T^nw)+p(u,u)]\\&=(a_1+a_4)\ p\ (u,\ T^{n-1}w)+a_3p(T^{n-1}w,\ T^nw)+a_4p(u,\ T^nw)+a_5p(T^{n-1}w,\ T^nw)\\&=(a_1+a_3+a_4+a_5)p(u,T^{n-1}w)+\ (a_3+a_4+a_5)p(u,T^nw)\end{aligned}$$

Similarly as in the proof of theorem (2.1). It can be shown that under the condition $(a_1+2a_3+2a_4+2a_5)\leq 1$

$P(u,T^n w)\leq p(u,T^{n-1}w)$

Note that when we consider

$\Psi(p(T^n w,u))\leq \Psi(\theta(T^{n-1}w,u))- \varphi\ (\theta(T^{n-1}w,u))$

Where

$\theta(T^{n-1}w,u))= (a_1 +a_2)p(u, T^{n-1}w)+a_2 p\ (T^{n-1}w, T^n w)+ a_4 p(u, T^n w)+ a_5\, p(T^{n-1}w, T^n w)$

$\leq(a_1 +a_2+a_4+a_5)p(u,T^{n-1}w)+(a_2+a_4+a_5)\ p(u, T^n w))$

Hence one finds under $(a_1 +2a_2+2a_4+2a_5) \leq$ *that* $p(T^n w,u)\leq p(T^{n-1}w,u)$

In each case, it follows that the sequence $\{p(u,f^n w)\}$ is non increasing and it has a limit $l\geq 0$ adjusting again in the proof of theorem (2.1). one can finds that $l=0$ in the same way it can be deduced that $p(v,T^n w)\rightarrow 0$ as $n\rightarrow+\infty$Now passing to the limit in $p(u,v)\leq p(u,T^n w)+ p(T^n w,v)$ it follow that $P(u,v)=0$ so u=v, and the uniqueness of the fixed point is proved.

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