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γ-Regular-Open Sets and γ-Extremally Disconnected Spaces

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Abstract
The aim of this paper is to introduce a new class of sets called γ-regular-open sets in topological spaces (X, τ) with an operation γ on τ together with its complement which is γ-regular-closed. Also to define a new space called γ-extremally disconnected, and to obtain several characterizations of γ-extremally disconnected spaces by utilizing γ-regular-open sets and γ-regular-closed sets. Further γ-locally indiscrete and γ-hyperconnected spaces have been defined.

Keywords: γ-regular-open set, γ-regular-closed set, γ-extremally disconnected space, γ-locally indiscrete space and γ-hyperconnected space

1. Introduction

In 1979, Kasahara defined the concept of an operation on topological spaces and introduced α-closed graphs of an operation. Ogata (1991) called the operation α as γ operation on τ and defined and investigated the concept of operation-open sets, that is, γ-open sets. He defined the complement of a γ-open subset of a space X as γ-closed. In addition, he also proved that the union of any collection of γ-open sets in a topological space (X, τ) is γ-open, but the intersection of any two γ-open sets in a space X need not be a γ-open set. Further study by Krishnan and Balachandran (2006*a*; 2006*b*) defined two types of sets called γ-preopen and γ-semiopen sets of a topological space (X, τ), respectively. Kalaivani and Krishnan (2009) defined the notion of α-γ-open sets. Basu, Afsan and Ghosh (2009) used the operation γ to introduce γ-β-open sets. Finally, Carpintero, Rajesh and Rosas (2012*a*) defined the notion of γ-*b*-open sets of a topological space (X, τ).

2. Preliminaries

Throughout this paper, (X, τ) (or simply X) will always denote topological spaces on which no separation axioms are assumed unless explicitly stated. A subset R of X is said to be regular open and regular closed if $R = Int(Cl(R))$ and $R = Cl(Int(R)))$, respectively (Steen & Seebach, 1978), where $Int(R)$ and $Cl(R)$ denotes the interior of R and the closure of R, respectively.

Now we recall some definitions and results which will be used in the sequel.

Definition 2.1 (Ogata, 1991)**:** Let (X, τ) be a topological space. An operation γ on the topology τ on X is a mapping $\gamma: \tau \to P(X)$ such that $U \subset \gamma(U)$ for each $U \in \tau$, where $P(X)$ is the power set of X and $\gamma(U)$ denotes the value of γ at U.

Definition 2.2 (Ogata, 1991)**:** A nonempty set R of X is said to be γ-open if for each $x \in R$, there exists an open set U such that $x \in U$ and $\gamma(U) \subset R$. The complement of a γ-open set is called a γ-closed.

Definition 2.3: Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then

1) the τ_γ-closure of R is defined as intersection of all γ-closed sets containing R. That is,
$\tau_\gamma\text{-}Cl(R) = \cap \{F : R \subset F, X \setminus F \in \tau_\gamma\}$ (Ogata, 1991).
2) the τ_γ-interior of R is defined as union of all γ-open sets contained in R. That is,
$\tau_\gamma\text{-}Int(R) = \cup \{U : U$ is a γ-open set and $U \subset R\}$ (Krishnan & Balachandran, 2006*b*).

Remark 2.4: For any subset R of a topological space (X, τ). Then:

1) R is γ-open if and only if $\tau_\gamma\text{-}Int(R) = R$ (Krishnan & Balachandran, 2006*b*).
2) R is γ-closed if and only if $\tau_\gamma\text{-}Cl(R) = R$ (Ogata, 1991).

Definition 2.5: Let (X, τ) be a topological space and γ be an operation on τ. A subset R of X is said to be:

1) α-γ-open if $R \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ (Kalaivani & Krishnan, 2009).
2) γ-preopen if $R \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ (Krishnan & Balachandran, 2006*a*).
3) γ-semiopen if $R \subset \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$ (Krishnan & Balachandran, 2006*b*).
4) γ-*b*-open if $R \subset \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ (Carpintero *et al*, 2012*a*).
5) γ-β-open if $R \subset \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))$ (Basu *et al*, 2009).
6) γ-clopen if it is both γ-open and γ-closed.

Definition 2.6: The complement of α-γ-open, γ-preopen, γ-semiopen, γ-*b*-open and γ-β-open set is said to be α-γ-closed (Kalaivani & Krishnan, 2009), γ-preclosed (Krishnan & Balachandran, 2006*a*), γ-semiclosed (Krishnan & Balachandran, 2006*b*), γ-*b*-closed (Carpintero *et al*, 2012*a*) and γ-β-closed (Basu *et al*, 2009), respectively.

The intersection of all γ-semiclosed sets of X containing a subset R of X is called the γ-semi-closure of R and it is denoted by $\tau_\gamma\text{-}sCl(R)$.

The family of all γ-open, α-γ-open, γ-preopen, γ-semiopen, γ-b-open, γ-β-open, γ-semiclosed, γ-b-closed, γ-β-closed and regular open subsets of a topological space (X, τ) is denoted by τ_γ, $\tau_{\alpha\text{-}\gamma}$, $\tau_\gamma PO(X)$, $\tau_\gamma SO(X)$, $\tau_\gamma BO(X)$, $\tau_\gamma \beta O(X)$, $\tau_\gamma SC(X)$, $\tau_\gamma BC(X)$, $\tau_\gamma \beta C(X)$ and $RO(X)$, respectively.

Definition 2.7 (Ogata, 1991)**:** Let (X, τ) be any topological space. An operation γ is said to be regular if for every open neighborhood U and V of each $x \in X$, there exists an open neighborhood W of x such that $\gamma(W) \subset \gamma(U) \cap \gamma(V)$.

Definition 2.8 (Ogata, 1991)**:** A topological space (X, τ) with an operation γ on τ is said to be γ-regular if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U) \subset V$.

Remark 2.9: If a topological space (X, τ) is γ-regular, then $\tau_\gamma = \tau$ (Ogata, 1991) and hence $\tau_\gamma Int(R) = Int(R)$ (Krishnan, 2003)

Theorem 2.10 (Krishnan & Balachandran, 2006*b*)**:** Let R be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then $\tau_\gamma sCl(R) = R \cup \tau_\gamma Int(\tau_\gamma Cl(R))$.

Lemma 2.11 (Krishnan & Balachandran, 2006*a*)**:** For any subset R of a topological space (X, τ) and γ be an operation on τ. The following statements are true.

1) $\tau_\gamma Int(X \backslash R) = X \backslash \tau_\gamma Cl(R)$ and $\tau_\gamma Cl(X \backslash R) = X \backslash \tau_\gamma Int(R)$.
2) $\tau_\gamma Cl(R) = X \backslash \tau_\gamma Int(X \backslash R)$ and $\tau_\gamma Int(R) = X \backslash \tau_\gamma Cl(X \backslash R)$.

Lemma 2.12: Let R and S be any subsets of a topological space (X, τ) and γ be an operation on τ. If $R \cap S = \varphi$, then $\tau_\gamma Int(R) \cap \tau_\gamma Cl(S) = \varphi$ and $\tau_\gamma Cl(R) \cap \tau_\gamma Int(S) = \varphi$.

Proof: Obvious.

Lemma 2.13 (Krishnan & Balachandran, 2006*a*)**:** Let (X, τ) be a topological space and γ be a regular operation on τ. Then for every γ-open set U and every subset R of X, we have $\tau_\gamma Cl(R) \cap U \subseteq \tau_\gamma Cl(R \cap U)$.

Definition 2.14 (Carpintero *et al*, 2012*b*)**:** A subset D of a topological space (X, τ) is said to be:

1) γ-dense if $\tau_\gamma Cl(D) = X$.
2) γ-semi-dense if $\tau_\gamma sCl(D) = X$.

3. γ-Regular-Open Sets

In this section, we introduce a new class of sets called γ-regular-open sets. This class of sets lies strictly between the classes of γ-clopen and γ-open sets.

Definition 3.1: A subset R of a topological space (X, τ) with an operation γ on τ is said to be γ-regular-open if $R = \tau_\gamma Int(\tau_\gamma Cl(R))$. The complement of a γ-regular-open set is γ-regular-closed. Or equivalently, a subset R of a space X is said to be γ-regular-closed if $R = \tau_\gamma Cl(\tau_\gamma Int(R))$. The class of all γ-regular-open and γ-regular-closed subsets of a topological space (X, τ) is denoted by $\tau_\gamma RO(X, \tau)$ or $\tau_\gamma RO(X)$ and $\tau_\gamma RC(X, \tau)$ or $\tau_\gamma RC(X)$, respectively.

Remark 3.2: It is clear from the definition that every γ-regular-open set is γ-open and every γ-clopen set is both γ-regular-open and γ-regular-closed.

Converses of the above remark are not true. It can be seen from the following example.

Example 3.3: Let $X = \{a, b, c\}$ and $\tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$. Define an operation $\gamma: \tau \to P(X)$ by: $\gamma(R) = R$ for every $R \in \tau$. Then $\tau_\gamma = \tau$ and $\tau_\gamma RO(X, \tau) = \{\varphi, X, \{a\}, \{b\}\}$. Then the set $\{a, b\}$ is γ-open, but it is not γ-regular-open. Also the sets $\{a\}$ and $\{a, c\}$ are γ-regular-open and γ-regular-closed, respectively, but they are not γ-clopen.

Remark 3.4: Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then:

1) R is γ-clopen if and only if it is γ-regular-open and γ-regular-closed.
2) R is γ-regular-open if and only if it is γ-preopen and γ-semiclosed.
3) R is γ-regular-closed if and only if it is γ-semiopen and γ-preclosed.

Proof: Follows from their definitions.

Theorem 3.5: The concept of γ-regular-open set and regular open set are independent. It is showing by the following example.

Example 3.6: Let $X = \{a, b, c\}$ and $\tau = \{\varphi, X, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Let $\gamma: \tau \to P(X)$ be an operation defined as follows: For every $R \in \tau$. Then

$$\gamma(R) = \begin{cases} R, & \text{if } R = \{b\} \\ R \cup \{a\}, & \text{if } R \neq \{b\} \end{cases}$$

Then $\tau_\gamma = \{\varphi, X, \{a\}, \{b\}, \{a, b\}\}$, τ_γ-$RO(X, \tau) = \{\varphi, X, \{a\}, \{b\}\}$ and $RO(X, \tau) = \{\varphi, X, \{a\}, \{b, c\}\}$. Then the set $\{b\}$ is γ-regular-open, but it is not regular open. Also the set $\{b, c\}$ is regular open, but it is not γ-regular-open.

Theorem 3.7: If (X, τ) be a γ-regular space, then the concept of γ-regular-open set and regular open set coincide.

Proof: Follows from Remark 2.9.

Remark 3.8: The union and intersection of any two γ-regular-open (respectively, γ-regular-closed) sets need not be γ-regular-open (respectively, γ-regular-closed) set. It is shown by the following example.

Example 3.9: In Example 3.6, the sets $\{a\}$ and $\{b\}$ (respectively, $\{a, c\}$ and $\{b, c\}$) are γ-regular-open (respectively, γ-regular-closed) sets, but $\{a\} \cup \{b\} = \{a, b\}$ (respectively, $\{a, c\} \cap \{b, c\} = \{c\}$) is not γ-regular-open (respectively, γ-regular-closed) set.

Theorem 3.10: Let (X, τ) be a topological space and γ be a regular operation on τ. Then:

1) The intersection of two γ-regular-open sets is γ-regular-open.
2) The union of two γ-regular-closed sets is γ-regular-closed.

Proof: Straightforward from Corollary 3.28 and Corollary 3.29 and the fact that every γ-regular-open set is γ-open and every γ-regular-closed set is γ-closed.

Theorem 3.11: For any subset R of a topological space (X, τ), then $R \in \tau_\gamma$-$\beta O(X)$ if and only if τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$.

Proof: Let $R \in \tau_\gamma$-$\beta O(X)$, then $R \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Then $R \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$ implies that τ_γ-$Cl(R) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) \subset \tau_\gamma$-$Cl(R)$. Therefore, τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$.

Conversely, suppose that τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. This implies that τ_γ-$Cl(R) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Then $R \subset \tau_\gamma$-$Cl(R) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Hence $R \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Therefore, $R \in \tau_\gamma$-$\beta O(X)$.

Theorem 3.12: For any subset R of a topological space (X, τ), then $R \in \tau_\gamma$-$SO(X)$ if and only if τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$.

Proof: The proof is similar to Theorem 3.11.

Proposition 3.13: For any subset R of a topological space (X, τ). If $R \in \tau_\gamma$-$SO(X)$, then τ_γ-$Cl(R) \in \tau_\gamma$-$RC(X)$.

Proof: The proof is immediate consequence of Theorem 3.12.

Corollary 3.14: For any subset R of a topological space (X, τ). Then

1) $R \in \tau_\gamma$-$SC(X)$ if and only if τ_γ-$Int(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.
2) if $R \in \tau_\gamma$-$SC(X)$, then τ_γ-$Int(R) \in \tau_\gamma$-$RO(X)$.

Theorem 3.15: Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then

1) $R \in \tau_\gamma$-$\beta O(X)$ if and only if τ_γ-$Cl(R) \in \tau_\gamma$-$RC(X)$.
2) $R \in \tau_\gamma$-$\beta O(X)$ if and only if τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X)$.
3) $R \in \tau_\gamma$-$\beta O(X)$ if and only if τ_γ-$Cl(R) \in \tau_\gamma$-$\beta O(X)$.
4) $R \in \tau_\gamma$-$\beta O(X)$ if and only if τ_γ-$Cl(R) \in \tau_\gamma$-$BO(X)$.

Proof:

1) The proof is immediate consequence of Theorem 3.11.

2) Let $R \in \tau_\gamma$-$\beta O(X)$, then by (1), τ_γ-$Cl(R) \in \tau_\gamma$-$RC(X) \subset \tau_\gamma$-$SO(X)$. So τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X)$. On the other hand, let τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X)$. Then by Theorem 3.12, τ_γ-$Cl(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Cl(R))))$ which implies that τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$ and hence by Theorem 3.11, $R \in \tau_\gamma$-$\beta O(X)$.

3) Let $R \in \tau_\gamma$-$\beta O(X)$, then by (2), τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X) \subset \tau_\gamma$-$\beta O(X)$. So τ_γ-$Cl(R) \in \tau_\gamma$-$\beta O(X)$. On the other hand, let τ_γ-$Cl(R) \in \tau_\gamma$-$\beta O(X)$, by (2), τ_γ-$Cl(\tau_\gamma$-$Cl(R)) \in \tau_\gamma$-$SO(X)$. Since τ_γ-$Cl(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Cl(R)$. Then τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X)$ and hence by (2), $R \in \tau_\gamma$-$\beta O(X)$.

4) Let $R \in \tau_\gamma$-$\beta O(X)$, then by (2), τ_γ-$Cl(R) \in \tau_\gamma$-$SO(X) \subset \tau_\gamma$-$BO(X)$. So τ_γ-$Cl(R) \in \tau_\gamma$-$BO(X)$. On the other hand, let τ_γ-$Cl(R) \in \tau_\gamma$-$BO(X) \subset \tau_\gamma$-$\beta O(X)$, then τ_γ-$Cl(R) \in \tau_\gamma$-$\beta O(X)$ and hence by (3), $R \in \tau_\gamma$-$\beta O(X)$.

From Theorem 3.11 and Theorem 3.15, we have the following corollary.

Corollary 3.16: For any subset R of a topological space (X, τ). Then

1) $R \in \tau_\gamma$-$\beta C(X)$ if and only if τ_γ-$Int(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))$.
2) $R \in \tau_\gamma$-$\beta C(X)$ if and only if τ_γ-$Int(R) \in \tau_\gamma$-$RO(X)$.
3) $R \in \tau_\gamma$-$\beta C(X)$ if and only if τ_γ-$Int(R) \in \tau_\gamma$-$SC(X)$.
4) $R \in \tau_\gamma$-$\beta C(X)$ if and only if τ_γ-$Int(R) \in \tau_\gamma$-$\beta C(X)$.
5) $R \in \tau_\gamma$-$\beta C(X)$ if and only if τ_γ-$Int(R) \in \tau_\gamma$-$BC(X)$.

Corollary 3.17: For any subset R of a topological space (X, τ). Then

1) If $R \in \tau_\gamma$-$RO(X)$, then τ_γ-$Cl(R) \in \tau_\gamma$-$RC(X)$ and τ_γ-$Int(R) \in \tau_\gamma$-$RO(X)$.
2) If $R \in \tau_\gamma$-$RC(X)$, then τ_γ-$Cl(R) \in \tau_\gamma$-$RC(X)$ and τ_γ-$Int(R) \in \tau_\gamma$-$RO(X)$.

Lemma 3.18: For any subset R of a topological space (X, τ), then τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$.

Proof: Since τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$. Then τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$. On the other hand, since τ_γ-$Int(R) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$ implies that τ_γ-$Int(R) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))$ and hence τ_γ-$Cl(\tau_\gamma$-$Int(R)) \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))))$. Therefore, τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))$.

Lemma 3.19: For any subset R of a topological space (X, τ), then τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.

Proof: The proof is similar to Lemma 3.18.

It is obvious from Lemma 3.18 and Lemma 3.19 that τ_γ-$Cl(\tau_\gamma$-$Int(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(R))$ are γ-regular-closed and γ-regular-open sets, respectively.

Lemma 3.20: Let R be any subset of a topological space (X, τ) and γ be a regular operation on τ. Then $R \in \tau_\gamma$-$PO(X)$ if and only if τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.

Proof: Let $R \in \tau_\gamma$-$PO(X)$, then $R \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)$ implies that τ_γ-$sCl(R) \subset \tau_\gamma$-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(R)) \in \tau_\gamma$-$RO(X)$. But τ_γ-$RO(X) \subset \tau_\gamma$-$SC(X)$ in general, then τ_γ-$Int(\tau_\gamma$-$Cl(R)) \in \tau_\gamma$-$SC(X)$ and hence τ_γ-$sCl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. So τ_γ-$sCl(R) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. On the other hand, by Theorem 2.10, we have τ_γ-$sCl(R) = R \cup \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. Then τ_γ-$Int(\tau_\gamma$-$Cl(R)) \subset \tau_\gamma$-$sCl(R)$. Therefore, τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$.
Conversely, let τ_γ-$sCl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$, then τ_γ-$sCl(R) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ and hence $R \subset \tau_\gamma$-$sCl(R) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ which implies that $R \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. Then $R \in \tau_\gamma$-$PO(X)$.

Proposition 3.21: Let P and R be subsets of a topological space (X, τ) and γ be an operation on τ. Then P is γ-preopen if and only if there exists a γ-regular open set R containing P such that τ_γ-$Cl(P) = \tau_\gamma$-$Cl(R)$.

Proof: Let P be a γ-preopen set, then $P \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$. Put $R = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$ is a γ-regular open set containing P. Since P is γ-preopen, then P is γ-β-open. By Theorem 3.11, τ_γ-$Cl(P) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) = \tau_\gamma$-$Cl(R)$.
Conversely, suppose R be a γ-regular-open set and P be any subset such that $P \subset R$ and τ_γ-$Cl(P) = \tau_\gamma$-$Cl(R)$. Then $P \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ and hence $P \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$. This means that P is γ-preopen set. This completes the proof.

Proposition 3.22: If S is both γ-semiopen and γ-semiclosed subset of a topological space (X, τ) with an operation γ on τ and τ_γ-$Cl(\tau_\gamma$-$Int(S)) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))$. Then S is both γ-regular-open and γ-regular-closed.

Proof: Clear.

Lemma 3.23: Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1) R is γ-regular-open.
2) R is γ-open and γ-semiclosed.
3) R is α-γ-open and γ-semiclosed.
4) R is γ-preopen and γ-semiclosed.
5) R is γ-open and γ-β-closed.
6) R is α-γ-open and γ-β-closed.

Proof:

(1) $\Rightarrow$ (2) Let R be γ-regular-open set. Since every γ-regular-open set is γ-open and every γ-regular open set is γ-semiclosed. Then R is γ-open and γ-semiclosed.

(2) $\Rightarrow$ (3) Let R be γ-open and γ-semiclosed set. Since every γ-open set is α-γ-open. Then R is α-γ-open and γ-semiclosed.

(3) $\Rightarrow$ (4) Let R be α-γ-open and γ-semiclosed set. Since every α-γ-open set is γ-preopen. Then A is γ-preopen and γ-semiclosed.

(4) $\Rightarrow$ (5) Let R be γ-preopen and γ-semiclosed set. Then $R \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ and τ_γ-$Int(\tau_\gamma$-$Cl(R)) \subset R$. Therefore, we have $R = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$. Then R is γ-regular-open set and hence it is γ-open. Since every γ-semiclosed set is γ-β-closed. Then R is γ-open and γ-β-closed.

(5) $\Rightarrow$ (6) It is obvious since every γ-open set is α-γ-open.

(6) $\Rightarrow$ (1) Let R be α-γ-open and γ-β-closed set. Then $R \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R)))$ and τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))) \subset R$. Then τ_γ-$Int(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))) = R$ and hence τ_γ-$Int(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(R))) = R$. Therefore, R is γ-regular-open set.

Corollary 3.24: Let S be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1) S is γ-regular-closed.
2) S is γ-closed and γ-semiopen.
3) S is α-γ-closed and γ-semiopen.
4) S is γ-preclosed and γ-semiopen.
5) S is γ-closed and γ-β-open.
6) S is α-γ-closed and γ-β-open.

Proof: Similar to Lemma 3.23 taking $R = X \backslash S$.

Lemma 3.25: Let R be any subset of a topological space (X, τ) and γ be an operation on τ. Then the following statements are equivalent:

1) R is γ-clopen.
2) R is γ-regular-open and γ-regular-closed.
3) R is γ-open and α-γ-closed.
4) R is γ-open and γ-preclosed.
5) R is α-γ-open and γ-preclosed.
6) R is α-γ-open and γ-closed.
7) R is γ-preopen and γ-closed.
8) R is γ-preopen and α-γ-closed.

Proof:

(1) $\Leftrightarrow$ (2) see Remark 3.4 (1).

The implications (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5), (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) are obvious (see Figure 1).

(5) $\Rightarrow$ (6) Let R be α-γ-open and γ-preclosed set. Then $R \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ and $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) \subset R$. This implies that $R = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R)))$ and hence $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))))$. By Lemma 3.18, we get $\tau_\gamma\text{-}Cl(R) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))$. Since $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(R))) \subset R$, then $\tau_\gamma\text{-}Cl(R) \subset R$. But in general $R \subset \tau_\gamma\text{-}Cl(R)$. Then $\tau_\gamma\text{-}Cl(R) = R$. It is obvious that R is γ-closed.

(8) $\Rightarrow$ (1) Let R be γ-preopen and α-γ-closed. Then $R \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) \subset R$. Therefore, we have $\tau_\gamma\text{-}Cl(R) \subset \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) \subset R$ and hence $\tau_\gamma\text{-}Cl(R) \subset R$. But in general $R \subset \tau_\gamma\text{-}Cl(R)$. Then $\tau_\gamma\text{-}Cl(R) = R$. It is obvious that R is γ-closed. Since $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) \subset R$ implies that $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)))) \subset \tau_\gamma\text{-}Int(R)$. Then by Lemma 3.19, $R \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \subset \tau_\gamma\text{-}Int(R)$ and hence $R \subset \tau_\gamma\text{-}Int(R)$. But in general $\tau_\gamma\text{-}Int(R) \subset R$. Then $\tau_\gamma\text{-}Int(R) = R$. It is obvious that R is γ-open. Therefore, R is γ-clopen.

Proposition 3.26: Let R and S be any two subsets of a topological space (X, τ) and γ be an operation on τ. Then:

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S)) \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$$

Proof: The proof is obvious and hence it is omitted.

The converse of the above proposition is true when the operation γ is regular operation on τ and if one of the set is γ-open in X, as shown by the following proposition.

Proposition 3.27: Let (X, τ) be a topological space and γ be a regular operation on τ. If R is γ-open subset of X and S is any subset of X, then

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$$

Proof: It is enough to prove $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$ since the converse is similar to Proposition 3.26. Since R is γ-open subset of a space X and γ is a regular operation on τ. Then by using Lemma 2.13, we have

$$\begin{aligned}\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) &\subset \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))] \\ &\subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl[R \cap \tau_\gamma\text{-}Cl(S)]) \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))\end{aligned}$$

So $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$. This completes the proof.

From Proposition 3.27, we have the following corollary.

Corollary 3.28: If R and S are γ-open subsets of a topological space (X, τ) and γ be a regular operation on τ, then

$$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S))$$

Corollary 3.29: If E and F are γ-closed subsets of a topological space (X, τ) and γ be a regular operation on τ, then

$$\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E)) \cup \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(F)) = \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(E \cup F))$$

Proof: the proof is similar to Corollary 3.28 taking $R = X \backslash E$ and $S = X \backslash F$.

4. γ-Extremally Disconnected Spaces

In this section, we introduce a new space called γ-extremally disconnected, and to obtain several characterizations of γ-extremally disconnected spaces by utilizing γ-regular-open sets and γ-regular-closed sets.

Definition 4.1: A topological space (X, τ) with an operation γ on τ is said to be γ-extremally disconnected if the τ_γ-closure of every γ-open set of X is γ-open in X. Or equivalently, a space X is γ-extremally disconnected if the τ_γ-interior of every γ-closed set of X is γ-closed in X.

In the following theorem, a space X is γ-extremally disconnected is equivalent to every two disjoint γ-open sets of X have disjoint τ_γ-closures.

Theorem 4.2: A space X is γ-extremally disconnected if and only if $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \varphi$ for every γ-open subsets R and S of X with $R \cap S = \varphi$.

Proof: Suppose R and S are two γ-open subsets of a γ-extremally disconnected space X such that $R \cap S = \varphi$. Then by Lemma 2.12, $\tau_\gamma\text{-}Cl(R) \cap S = \varphi$ which implies that $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Cl(S) = \varphi$ and hence $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \varphi$.

Conversely, let O be any γ-open subset of a space X, then $X \backslash O$ is γ-closed set and hence $\tau_\gamma\text{-}Int(X \backslash O)$ is γ-open set such that $O \cap \tau_\gamma\text{-}Int(X \backslash O) = \varphi$. Then by hypothesis, we have $\tau_\gamma\text{-}Cl(O) \cap \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(X \backslash O)) = \varphi$ which implies that $\tau_\gamma\text{-}Cl(O) \cap \tau_\gamma\text{-}Cl(X \backslash \tau_\gamma\text{-}Cl(O)) = \varphi$ and hence $\tau_\gamma\text{-}Cl(O) \cap X \backslash \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(O)) = \varphi$. This means that $\tau_\gamma\text{-}Cl(O) \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(O))$. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(O)) \subset \tau_\gamma\text{-}Cl(O)$ in general. Then $\tau_\gamma\text{-}Cl(O) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(O))$. So $\tau_\gamma\text{-}Cl(O)$ is γ-open set in X. Therefore, X is γ-extremally disconnected space.

Theorem 4.3: Let (X, τ) be a topological space and γ be a regular operation on τ. Then the following are equivalent:

1) X is γ-extremally disconnected.
2) $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \tau_\gamma\text{-}Cl(R \cap S)$ for every γ-open subsets R and S of X.
3) $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \tau_\gamma\text{-}Cl(R \cap S)$ for every γ-regular-open subsets R and S of X.
4) $\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F)$ for every γ-regular-closed subsets E and F of X.
5) $\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F)$ for every γ-closed subsets E and F of X.

Proof:

(1) $\Rightarrow$ (2) Let R and S be any two γ-open subsets of a γ-extremally disconnected space X. Then by Corollary 3.28, $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R \cap S)) = \tau_\gamma\text{-}Cl(R \cap S)$.

(2) $\Rightarrow$ (3) is clear since every γ-regular-open set is γ-open.

(3) $\Leftrightarrow$ (4) Let E and F be two γ-regular-closed subsets of X. Then $X \backslash E$ and $X \backslash F$ are γ-regular-open sets. By (3) and Lemma 2.11, we have

$$\begin{aligned} &\tau_\gamma\text{-}Cl(X \backslash E) \cap \tau_\gamma\text{-}Cl(X \backslash F) = \tau_\gamma\text{-}Cl(X \backslash E \cap X \backslash F) \\ \Leftrightarrow\ & X \backslash \tau_\gamma\text{-}Int(E) \cap X \backslash \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Cl(X \backslash (E \cup F)) \\ \Leftrightarrow\ & X \backslash (\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F)) = X \backslash \tau_\gamma\text{-}Int(E \cup F) \\ \Leftrightarrow\ & \tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F). \end{aligned}$$

(4) $\Rightarrow$ (5) Let E and F be two γ-closed subsets of X. Then $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{E}))$ and $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{F}))$ are γ-regular-closed sets. Then by (4) and Corollary 3.29, we get
$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{E}))) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{F}))) = \tau_\gamma\text{-}Int[\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{E})) \cup \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{F}))]$ and hence $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{E}))) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{F}))) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\mathrm{E} \cup \mathrm{F})))$.

Since E and F are γ-closed subsets of X. Then by Lemma 3.19, we obtain
$\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(E)) \cup \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(F)) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(E \cup F))$. This implies that $\tau_\gamma\text{-}Int(E) \cup \tau_\gamma\text{-}Int(F) = \tau_\gamma\text{-}Int(E \cup F)$.

(5) $\Leftrightarrow$ (2) the proof is similar to (3) $\Leftrightarrow$ (4).

(2) $\Rightarrow$ (1) let U be any γ-open subset of a space X, then $X \backslash U$ is γ-closed set and hence $\tau_\gamma\text{-}Int(X \backslash U)$ is γ-open set. Then by (2), we $\tau_\gamma\text{-}Cl(U) \cap \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(X \backslash U)) = \tau_\gamma\text{-}Cl(U \cap \tau_\gamma\text{-}Int(X \backslash U))$ which implies that $\tau_\gamma\text{-}Cl(U) \cap \tau_\gamma\text{-}Cl(X \backslash \tau_\gamma\text{-}Cl(U)) = \tau_\gamma\text{-}Cl(\varphi)$ since $U \cap \tau_\gamma\text{-}Int(X \backslash U) = \varphi$. Hence $\tau_\gamma\text{-}Cl(U) \cap X \backslash \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(U)) = \varphi$. This means that $\tau_\gamma\text{-}Cl(U) \subset \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(U))$. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(U)) \subset \tau_\gamma\text{-}Cl(U)$ in general. Then $\tau_\gamma\text{-}Cl(U) = \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(U))$. So $\tau_\gamma\text{-}Cl(U)$ is γ-open set in X. Therefore, a space X is γ-extremally disconnected.

Theorem 4.4: Let (X, τ) be a topological space and γ be a regular operation on τ. Then the following are equivalent:

1) X is γ-extremally disconnected.
2) $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \varphi$ for every γ-open subsets R and S of X with $R \cap S = \varphi$.
3) $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \varphi$ for every γ-regular-open subsets R and S of X with $R \cap S = \varphi$.

Proof:

(1) $\Leftrightarrow$ (2) see Theorem 4.3.

(2) $\Rightarrow$ (3) since every γ-regular-open set is γ-open. Then the proof is clear.

(3) $\Rightarrow$ (2) Let R and S be any two γ-open subsets of a space X such that $R \cap S = \varphi$. Then by Lemma 2.12, $R \cap \tau_\gamma\text{-}Cl(S) = \varphi$ implies that $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \varphi$ and hence $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R)) \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \varphi$. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$ are two γ-regular-open sets. Then by (3), we obtain $\tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))) \cap \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))) = \varphi$. Since $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$ are two γ-regular-open sets, then $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(R))$ and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$ are two γ-β-open sets. So by Theorem 3.11, we get $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(S) = \varphi$. This completes the proof.

Theorem 4.5: A space X is γ-extremally disconnected if and only if $\tau_\gamma\text{-}Cl(R) \cap \tau_\gamma\text{-}Cl(\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))) = \varphi$ for every γ-open subset R and every subset S of X with $R \cap S = \varphi$.

Proof: see Theorem 4.4, since R and $\tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S))$ are two γ-open subsets of X such that $R \cap \tau_\gamma\text{-}Int(\tau_\gamma\text{-}Cl(S)) = \varphi$.

Theorem 4.6: Let (X, τ) be a topological space and γ be a regular operation on τ. Then X is γ-extremally disconnected if and only if $\tau_\gamma Int(\tau_\gamma Cl(R)) \cup \tau_\gamma Int(\tau_\gamma Cl(S)) = \tau_\gamma Int(\tau_\gamma Cl(R \cup S))$ for every γ-open subsets R and S of X.

Proof: Let (X, τ) be a γ-extremally disconnected space and let R and S be any two γ-open subsets of X. Then $\tau_\gamma Cl(R)$ and $\tau_\gamma Cl(S)$ are γ-closed subsets of X. So by Theorem 4.3 (5), we have $\tau_\gamma Int(\tau_\gamma Cl(R)) \cup \tau_\gamma Int(\tau_\gamma Cl(S)) = \tau_\gamma Int(\tau_\gamma Cl(R) \cup \tau_\gamma Cl(S)) = \tau_\gamma Int(\tau_\gamma Cl(R \cup S))$.

Conversely, let E and F be two γ-closed subsets of X. Then $\tau_\gamma Int(E)$ and $\tau_\gamma Int(F)$ are γ-open subsets of X. So by hypothesis, $\tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Int(E))) \cup \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Int(F))) = \tau_\gamma Int(\tau_\gamma Cl[\tau_\gamma Int(E) \cup \tau_\gamma Int(F)]) = \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Int(E \cup F)))$. Since E and F are γ-closed subsets of X. Then by Lemma 3.19, $\tau_\gamma Int(\tau_\gamma Cl(E)) \cup \tau_\gamma Int(\tau_\gamma Cl(F)) = \tau_\gamma Int(\tau_\gamma Cl(E \cup F))$ and hence $\tau_\gamma Int(E) \cup \tau_\gamma Int(F) = \tau_\gamma Int(E \cup F)$. Therefore, by Theorem 4.3 (5), X is γ-extremally disconnected space.

Theorem 4.7: Let (X, τ) be a topological space and γ be a regular operation on τ. Then X is γ-extremally disconnected if and only if $\tau_\gamma Cl(\tau_\gamma Int(E)) \cap \tau_\gamma Cl(\tau_\gamma Int(F)) = \tau_\gamma Cl(\tau_\gamma Int(E \cap F))$ for every γ-closed subsets E and F of X.

Proof: Similar to Theorem 4.6 taking $R = X\backslash E$ and $S = X\backslash F$.

Theorem 4.8: A space X is γ-extremally disconnected if and only if $\tau_\gamma RO(X) = \tau_\gamma RC(X)$.

Proof: Obvious.

Theorem 4.9: Let (X, τ) be a topological space and γ be a regular operation on τ. Then the following are equivalent:

1) X is γ-extremally disconnected.
2) $R_1 \cap R_2$ is γ-regular-closed for all γ-regular-closed subsets R_1 and R_2 of X.
3) $R_1 \cup R_2$ is γ-regular-open for all γ-regular-open subsets R_1 and R_2 of X.

Proof: The proof is directly from Theorem 3.10 and Theorem 4.8.

Theorem 4.10: The following statements are equivalent for any topological space (X, τ).

1) X is γ-extremally disconnected.
2) Every γ-regular-closed subset of X is γ-open in X.
3) Every γ-regular-closed subset of X is α-γ-open in X.
4) Every γ-regular-closed subset of X is γ-preopen in X.
5) Every γ-semiopen subset of X is α-γ-open in X.
6) Every γ-semiclosed subset of X is α-γ-closed in X.
7) Every γ-semiclosed subset of X is γ-preclosed in X.
8) Every γ-semiopen subset of X is γ-preopen in X.
9) Every γ-β-open subset of X is γ-preopen in X.
10) Every γ-β-closed subset of X is γ-preclosed in X.
11) Every γ-b-closed subset of X is γ-preclosed in X.
12) Every γ-b-open subset of X is γ-preopen in X.
13) Every γ-regular-open subset of X is γ-preclosed in X.
14) Every γ-regular-open subset of X is γ-closed in X.
15) Every γ-regular-open subset of X is α-γ-closed in X.

Proof:

(1) $\Rightarrow$ (2) Let R be any γ-regular-closed subset of a γ-extremally disconnected space X. Then $R = \tau_\gamma Cl(\tau_\gamma Int(R))$. Since R is γ-regular-closed set, then it is γ-closed and hence $R = \tau_\gamma Cl(\tau_\gamma Int(R)) = \tau_\gamma Int(R)$. Therefore, R is γ-open set in X.

The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are clear since every γ-open set is α-γ-open and every α-γ-open set is γ-preopen.

(4) $\Rightarrow$ (5) Let S be a γ-semiopen set. Then $S \subset \tau_\gamma Cl(\tau_\gamma Int(S))$. Since $\tau_\gamma Cl(\tau_\gamma Int(S))$ is γ-regular-closed set. Then by (4), $\tau_\gamma Cl(\tau_\gamma Int(S))$ is γ-preopen and hence $\tau_\gamma Cl(\tau_\gamma Int(S)) \subset \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Cl(\tau_\gamma Int(S)))) = \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Int(S)))$. So $S \subset \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Int(S)))$. Therefore, S is α-γ-open set.

The implications (5) $\Leftrightarrow$ (6), (6) $\Rightarrow$ (7), (7) $\Leftrightarrow$ (8), (9) $\Leftrightarrow$ (10), (10) $\Rightarrow$ (11), (11) $\Leftrightarrow$ (12) and (14) $\Rightarrow$ (15) are obvious.

(8) $\Rightarrow$ (9) Let G be a γ-β-open set. Then by Theorem 3.15 (2), $\tau_\gamma Cl(G)$ is γ-semiopen set. So by (8), $\tau_\gamma Cl(G)$ is γ-preopen set. So $\tau_\gamma Cl(G) \subset \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Cl(G))) = \tau_\gamma Int(\tau_\gamma Cl(G))$ and hence $G \subset \tau_\gamma Int(\tau_\gamma Cl(G))$. Therefore, G is γ-preopen set in X.

(12) $\Rightarrow$ (13) Let H be a γ-regular-open set. Then H is γ-β-open set. By Theorem 3.15 (4), $\tau_\gamma Cl(H)$ is γ-b-open set. Then by (12), $\tau_\gamma Cl(H)$ is γ-preopen. So $\tau_\gamma Cl(H) \subset \tau_\gamma Int(\tau_\gamma Cl(\tau_\gamma Cl(H))) = \tau_\gamma Int(\tau_\gamma Cl(H))$. Since H is γ-

regular-open set. Then τ_γ-$Cl(H) \subset H$. Since $H \subset \tau_\gamma$-$Cl(H)$. Then τ_γ-$Cl(H) = H$. This means that H is γ-closed and hence it is γ-preclosed.

(13) ⇒ (14) Let U be a γ-regular-open set. Then by (13), U is γ-preclosed set. So τ_γ-$Cl(\tau_\gamma$-$Int(U)) \subset U$. Since U is γ-regular-open set, then U is γ-open. Hence τ_γ-$Cl(U) \subset U$. But in general $U \subset \tau_\gamma$-$Cl(U)$. Then τ_γ-$Cl(U) = U$. This means that U is γ-closed.

(15) ⇒ (1) Let V be any γ-open set of X. Then τ_γ-$Int(\tau_\gamma$-$Cl(V))$ is γ-regular-open set. By (15), τ_γ-$Int(\tau_\gamma$-$Cl(V))$ is α-γ-closed. So τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))))) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$. By Lemma 3.18, we get τ_γ-$Cl(V) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$. But τ_γ-$Int(\tau_\gamma$-$Cl(V)) \subset \tau_\gamma$-$Cl(V)$ in general. Then τ_γ-$Cl(V) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(V))$ and hence τ_γ-$Cl(V)$ is γ-open set of X. Therefore, X is γ-extremally disconnected space.

Theorem 4.11: The following conditions are equivalent for any topological space (X, τ).

1) X is γ-extremally disconnected.
2) The τ_γ-closure of every γ-β-open set of X is γ-regular-open in X.
3) The τ_γ-closure of every γ-b-open set of X is γ-regular-open in X.
4) The τ_γ-closure of every γ-semiopen set of X is γ-regular-open in X.
5) The τ_γ-closure of every α-γ-open set of X is γ-regular-open in X.
6) The τ_γ-closure of every γ-open set of X is γ-regular-open in X.
7) The τ_γ-closure of every γ-regular-open set of X is γ-regular-open in X.
8) The τ_γ-closure of every γ-preopen set of X is γ-regular-open in X.

Proof:

(1) ⇒ (2) Let R be a γ-β-open subset of a γ-extremally disconnected space X. Then by (1) and Lemma 3.11, we have τ_γ-$Cl(R) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R))$ implies that τ_γ-$Cl(R) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Cl(R)))$. Hence τ_γ-$Cl(R)$ in γ-regular-open set in X.

The implications (2) ⇒ (3), (3) ⇒ (4), (4) ⇒ (5), (5) ⇒ (6) and (6) ⇒ (7) are clear.

(7) ⇒ (8) Let P be any γ-preopen set of X, then τ_γ-$Int(\tau_\gamma$-$Cl(P))$ is γ-regular-open. By (7), τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)))$ is γ-regular-open set. So τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)))))$. Since every γ-preopen set is γ-β-open. Then by Theorem 3.11 and Lemma 3.19, we have τ_γ-$Cl(P) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Cl(P)))$. Hence τ_γ-$Cl(P)$ is γ-regular-open set in X.

(8) ⇒ (1) Let S be a γ-open set of X. Then S is γ-preopen and by (8), τ_γ-$Cl(S)$ is γ-regular-open set in X. Then τ_γ-$Cl(S)$ is γ-open. Therefore, X is γ-extremally disconnected space.

Remark 4.12: γ-regular-closed set can be replaced by γ-regular-open set in Theorem 4.11 (this is because of Theorem 4.8).

5. γ-Locally Indiscrete and γ-Hyperconnected Spaces

In this section, we introduce new types of spaces called γ-locally indiscrete and γ-hyperconnected. We give some properties and characterizations of these spaces.

Definition 5.1: A topoplogical space (X, τ) with an operation γ on τ is said to be:

1) γ-locally indiscrete if every γ-open subset of X is γ-closed, or every γ-closed subset of X is γ-open.
2) γ-hyperconnected if every nonempty γ-open subset of X is γ-dense.

Theorem 5.2: Let (X, τ) be a topological space and γ be an operation on τ. Then the following are holds:

1) If X is γ-locally indiscrete, then X is γ-extremally disconnected.
2) If X is γ-hyperconnected, then X is γ-extremally disconnected.

Proof: Follows from their definitions.

Theorem 5.3: If (X, τ) is γ-locally indiscrete space, then the following statements are true:

1) Every γ-semiopen subset of X is γ-open and hence it is γ-closed.
2) Every γ-semiclosed subset of X is γ-closed and hence it is γ-open.
3) Every γ-open subset of X is γ-regular-open and hence it is γ-regular-closed.
4) Every γ-closed subset of X is γ-regular-closed and hence it is γ-regular-open.
5) Every γ-semiopen subset of X is γ-regular-open and hence it is γ-regular-closed.
6) Every γ-semiclosed subset of X is γ-regular-closed and hence it is γ-regular-open.
7) Every γ-β-open subset of X is γ-preopen.
8) Every γ-β-closed subset of X is γ-preclosed.

Proof:

1) Let S be any γ-semiopen subset of a γ-locally indiscrete space (X, τ), then $S \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(S))$. Since τ_γ-$Int(S)$ is γ-open subset of X, then it is γ-closed. So τ_γ-$Cl(\tau_\gamma$-$Int(S)) = \tau_\gamma$-$Int(S)$ implies that $S \subset \tau_\gamma$-$Int(S)$. But τ_γ-$Int(S) \subset$ S. Then $S = \tau_\gamma$-$Int(S)$, this means that S is γ-open. Since a space X is γ-locally indiscrete, then S is γ-closed.

2) The proof is similar to part (1).

3) Let O be any γ-open subset of a γ-locally indiscrete space (X, τ). Since every γ-open set is γ-closed, then τ_γ-$Cl(\tau_\gamma$-$Int(O)) = O$. This implies that O is a γ-regular-closed set.
4) The proof is similar to part (3).
5) Follows directly from (1) and (3).
6) Follows directly from (2) and (4).
7) Let P be a γ-β-open subset of a γ-locally indiscrete space (X, τ). Then $P \subset \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(P))$ is γ-open set and hence it is γ-closed in γ-locally indiscrete space X. Then τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$. Then $P \subset \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))$. Hence P is γ-preopen set in X.
8) The proof is similar to part (7).

From Theorem 5.3, we have the following corollary.

Corollary 5.4: If (X, τ) is γ-locally indiscrete space, then

1) τ_γ-$RO(X) = \tau_\gamma = \tau_{\alpha\text{-}\gamma} = \tau_\gamma$-$SO(X)$.
2) τ_γ-$PO(X) = \tau_\gamma$-$BO(X) = \tau_\gamma$-$\beta O(X)$.

Theorem 5.5: A space (X, τ) is γ-hyperconnected if and only if τ_γ-$RO(X) = \{\varphi, X\}$.

Proof: In general φ and X are γ-regular-open subsets of a γ-hyperconnected space X. Let R be any nonempty proper subset of X which is γ-regular open. Then R is γ-open set. Since X is γ-hyperconnected space. So τ_γ-$Int(\tau_\gamma$-$Cl(R)) = \tau_\gamma$-$Int(X) = X$ and hence R is γ-regular-open set in X. Contradiction. Therefore, τ_γ-$RO(X) = \{\varphi, X\}$.
Conversely, suppose that τ_γ-$RO(X) = \{\varphi, X\}$ and let S be any nonempty γ-open subset of X. Then S is γ-β-open set. By Theorem 3.11, τ_γ-$Cl(S) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S)))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(S))$ is γ-regular-open set and S is nonempty γ-open set. Then τ_γ-$Int(\tau_\gamma$-$Cl(S))$ should be X. Therefore, τ_γ-$Cl(S) = \tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(S))) = \tau_\gamma$-$Cl(X) = X$. Then a space X is γ-hyperconnected.

Corollary 5.6: A space (X, τ) is γ-hyperconnected if and only if τ_γ-$RC(X) = \{\varphi, X\}$.

Proposition 5.7: If a space (X, τ) is γ-hyperconnected, then every nonempty γ-preopen subset of X is γ-semi-dense.

Proof: Let P be any nonempty γ-preopen subset of a γ-hyperconnected space X. By Lemma 3.19 and Lemma 3.20, we have τ_γ-$sCl(P) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)) = \tau_\gamma$-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))))$. Since τ_γ-$Int(\tau_\gamma$-$Cl(P))$ is a nonempty γ-open set and X is γ-hyperconnected space. Then τ_γ-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P))) = X$ and hence τ_γ-$Int(\tau_\gamma$-$Cl(\tau_\gamma$-$Int(\tau_\gamma$-$Cl(P)))) = \tau_\gamma$-$Int(X) = X$. Thus τ_γ-$sCl(P) = X$. This completes the proof.

6. Conclusion

In this paper, we introduce a new class of sets called γ-regular-open sets in a topological space (X, τ) together with its complement which is γ-regular-closed. Using these sets, we define γ-extremally disconnected space, and to obtain several characterizations of γ-extremally disconnected spaces. Finally, γ-locally indiscrete and γ-hyperconnected spaces have been introduced.

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By Remark 3.4, Definition 2.5 and Definition 2.6, we have the following figure.

Figure 1:

The relations between γ-regular-open set, γ-regular-closed set and various types of γ-sets.