Time Dependent Solution of Batch Arrival Queue with Second Optional Service, Optional Re-Service and Bernoulli Vacation

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Abstract

This paper deals with an $M^{[x]}/G/1$ queues with second optional service, optional re-service and Bernoulli vacations. Each customer undergoes first phase of service after completion of service, customer has the option to repeat or not to repeat the first phase of service and leave the system without taking the second phase or take the second phase service. Similarly after the second phase service he has yet another option to repeat or not to repeat the second phase service. After each service completion, the server may take a vacation with probability θ or may continue staying in the system with probability $1-\theta$. The service and vacation periods are assumed to be general. The time dependent probability generating functions have been obtained in terms of their Laplace transforms and the corresponding steady state results have been obtained explicitly. Also the average number of customers in the queue and the waiting time are also derived.

Keywords: Batch arrival, Second optional service, Optional re-service, Average queue size, Average waiting time.

1. Introduction

For the first time the concept of Bernoulli vacation were studied by Keilson and Servi [6]. In many applications such as hospital services, production systems, bank services, computer and communication networks; there is two phase of services such that the first phase is essential for all customers, but as soon as the essential services completed, it may leave the system or may immediately go for the second phase of service.

Recently, Madan [8] and Medhi [10] investigated an M/G/1 queueing system with a second optional service in which some of arrivals may require a second optional service immediately after completion of the first essential service.

Ke [5] studied an $M^{[x]}/G/1$ system with startup server and J additional options for service. Choudhury and Paul [3,4] studied a batch arrival queue with an additional service channel under N-policy.

At present, however most of the studies are devoted to batch arrival vacation models under different vacation policies because of its interdisciplinary character. Numerous researchers including Baba [1], Choudhury [2], Lee et al. [7], Madan and Choudhury [9] and many other have studied batch arrival queue under different vacation policies.

In this paper we consider batch arrival queue with second optional service in which the first phase of service is compulsory whereas the second phase is optional. Each customer undergoes first phase of service after completion of which customer has the option to repeat or not to repeat the first phase of service and leave the system without taking the second phase or take the second phase service. Similarly after the second phase service he has yet another option to repeat or not to repeat the second phase service. Further, we assume that this option of repeating the first phase or the second phase service can be availed only once.

The outline of the paper is as follows. The model description is given in section 2. Definitions and equations governing the system are given in section 3. The time dependent solution have been obtained in section 4 and corresponding steady state results have been derived explicitly in section 5. Average queue size and average waiting time are computed in section 6.

2. Model description

We assume the following to describe the queueing model of our study.

a) Customers arrive at the system in batches of variable size in a compound Poisson process and they are provided one by one service on a first come - first served basis. Let $\lambda c_i dt$ ($i \geq 1$) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t+dt]$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the arrival rate of batches.

b) There is a single server who provides the first phase service for all customers, as soon as the first service of a customer is completed, he may opt to repeat the first service with probability r_1 or may not repeat with probability $1-r_1$. After completing the first service, the customer may opt to take the second optional service with probability p or may leave the system without taking the second service with probability $1-p$. Similarly after taking the second phase service he may demand repetation of second phase service with probability r_2 or may leave the system without repeating the second phase service with probability 1- r_2. Further, we assume that this option of repeating the first phase or the second phase service can be availed only once.

c) The service time follows a general (arbitrary) distribution with distribution function $B_i(s)$ and density function $b_i(s)$. Let $\mu_i(x)dx$ be the conditional probability density of service completion during the interval $(x, x+dx]$, given that the elapsed time is x, so that

$$\mu_i(x) = \frac{b_i(x)}{1-B_i(x)}, \quad i = 1,2,$$

and therefore,

$$b_i(s) = \mu_i(s) e^{-\int_0^s \mu_i(x)dx}, \quad i = 1,2.$$

d) After each service completion, the server may take a vacation with probability θ or may continue staying in the system with probability $1-\theta$.

e) The server's vacation time follows a general (arbitrary) distribution with distribution function $V(t)$ and density function $v(t)$. Let $\gamma(x)dx$ be the conditional probability of a completion of a vacation during the interval $(x, x+dx]$ given that the elapsed vacation time is x, so that

$$\gamma(x) = \frac{v(x)}{1-V(x)},$$

and therefore,

$$v(t) = \gamma(t) e^{-\int_0^t \gamma(x)dx}$$

f) Various stochastic processes involved in the system are assumed to be independent of each other.

3. Definitions and Equations governing the system

We define $P_n^{(i)}(x,t)$ = Probability that at time t, the server is active providing i th phase service and there are n ($n \geq 0$) customers in the queue excluding the one being served and the elapsed service time for this customer is x. Consequently $P_n^{(i)}(t) = \int_0^{\infty} P_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer in the i th phase service irrespective of the value of x for i =1, 2. $R_n^{(i)}(x,t)$ = Probability that at time t, the server is active providing i th phase repeating service and there are n ($n \geq 0$) customers in the queue excluding one customer who is repeating i th phase service and the elapsed service time for this customer is x Consequently $R_n^{(i)}(t) = \int_0^\infty R_n^{(i)}(x,t)dx$ denotes the probability that at time t there are n customers in the queue excluding one customer who is repeating i th phase service irrespective of the value of x for i =1, 2.

$V_n(x,t)$ = Probability that at time t, the server is under vacation with elapsed vacation time x and there are n ($n \geq 0$) customers in the queue. Accordingly $V_n(t) = \int_0^\infty V_n(x,t)dx$ denotes the probability that at time t there are n customers in the queue and the server is under vacation irrespective of the value of x.

Q(t) = Probability that at time t, there are no customers in the queue and the server is idle but available in the system.

The system is then governed by the following set of differential-difference equations:

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t) + \frac{\partial}{\partial t}P_0^{(1)}(x,t) + [\lambda + \mu_1(x)]P_0^{(1)}(x,t) = 0 \quad (1)$$

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t) + \frac{\partial}{\partial t}P_n^{(1)}(x,t) + [\lambda + \mu_1(x)]P_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(1)}(x,t), \quad n \geq 1 \quad (2)$$

$$\frac{\partial}{\partial x}P_0^{(2)}(x,t) + \frac{\partial}{\partial t}P_0^{(2)}(x,t) + [\lambda + \mu_2(x)]P_0^{(2)}(x,t) = 0 \quad (3)$$

$$\frac{\partial}{\partial x}P_n^{(2)}(x,t) + \frac{\partial}{\partial t}P_n^{(2)}(x,t) + [\lambda + \mu_2(x)]P_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k P_{n-k}^{(2)}(x,t), \quad n \geq 1 \quad (4)$$

$$\frac{\partial}{\partial x}R_0^{(1)}(x,t) + \frac{\partial}{\partial t}R_0^{(1)}(x,t) + [\lambda + \mu_1(x)]R_0^{(1)}(x,t) = 0 \quad (5)$$

$$\frac{\partial}{\partial x}R_n^{(1)}(x,t) + \frac{\partial}{\partial t}R_n^{(1)}(x,t) + [\lambda + \mu_1(x)]R_n^{(1)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(1)}(x,t), \quad n \geq 1 \quad (6)$$

$$\frac{\partial}{\partial x}R_0^{(2)}(x,t) + \frac{\partial}{\partial t}R_0^{(2)}(x,t) + [\lambda + \mu_2(x)]R_0^{(2)}(x,t) = 0 \quad (7)$$

$$\frac{\partial}{\partial x}R_n^{(2)}(x,t) + \frac{\partial}{\partial t}V_n^{(2)}(x,t) + [\lambda + \mu_2(x)]R_n^{(2)}(x,t) = \lambda\sum_{k=1}^{n} c_k R_{n-k}^{(2)}(x,t), \quad n \geq 1 \quad (8)$$

$$\frac{\partial}{\partial x}V_0(x,t) + \frac{\partial}{\partial t}V_0(x,t) = -[\lambda + \gamma(x)]V_0(x,t) \quad (9)$$

$$\frac{\partial}{\partial x}V_n(x,t) + \frac{\partial}{\partial t}V_n(x,t) = -[\lambda + \gamma(x)]V_n(x,t) + \lambda\sum_{k=1}^{n} c_k V_{n-k}(x,t), n \geq 1 \quad (10)$$

$$\begin{aligned}\frac{d}{dt}Q(t) + \lambda Q(t) &= (1-\theta)(1-r_2)\int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx \\ &+ (1-\theta)\int_0^\infty R_0^{(2)}(x,t)\mu_2(x)dx + (1-\theta)(1-p)\int_0^\infty R_0^{(1)}(x,t)\mu_1(x)dx \\ &+ (1-\theta)(1-p)(1-r_1)\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty V_0(x,t)\gamma(x)dx \quad (11)\end{aligned}$$

The above equations are to be solved subject to the following boundary conditions

$$P_n^{(1)}(0,t) = \lambda c_{n+1}Q(t) + (1-\theta)(1-p)(1-r_1)\int_0^\infty P_{n+1}^{(1)}(x,t)\mu_1(x)dx$$

$$+(1-\theta)(1-r_2)\int_0^\infty P_{n+1}^{(2)}(x,t)\mu_2(x)dx+\int_0^\infty V_{n+1}(x,t)\gamma(x)dx$$
$$+(1-\theta)(1-p)\int_0^\infty R_{n+1}^{(1)}(x,t)\mu_1(x)dx+(1-\theta)\int_0^\infty R_{n+1}^{(2)}(x,t)\mu_2(x)dx$$
$$n\geq 0 \tag{12}$$

$$P_n^{(2)}(0,t)=p(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx+p\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx,$$
$$n\geq 0 \tag{13}$$

$$R_n^{(1)}(0,t)=r_1\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx,\ n\geq 0 \tag{14}$$

$$R_n^{(2)}(0,t)=r_2\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx,\ n\geq 0 \tag{15}$$

$$V_n(0,t)=(1-p)\theta(1-r_1)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx$$
$$+\theta(1-r_2)\int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx+(1-p)\theta\int_0^\infty R_n^{(1)}(x,t)\mu_1(x)dx$$
$$+\theta\int_0^\infty R_n^{(2)}(x,t)\mu_2(x)dx,\ n\geq 0 \tag{16}$$

We assume that initially there are no customers in the system and the server is idle. So the initial conditions are

$$P_n^{(i)}(0)=R_n^{(i)}(0)=0 \text{ for } n\geq 0 \text{ and } Q(0)=1. \tag{17}$$

4. Probability generating functions of the queue length: the time - dependent solution

We define the probability generating functions ,

$$P^{(i)}(x,z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(x,t);P^{(i)}(z,t)=\sum_{n=0}^{\infty}z^nP_n^{(i)}(t),C(z)=\sum_{n=1}^{\infty}c_nz^n,$$

$$R^{(i)}(x,z,t)=\sum_{n=0}^{\infty}z^nR_n^{(i)}(x,t);R^{(i)}(z,t)=\sum_{n=0}^{\infty}z^nR_n^{(i)}(t) \text{ for } i=1,2. \tag{18}$$

$$V(x,z,t)=\sum_{n=0}^{\infty}z^nV_n(x,t);\quad V(z,t)=\sum_{n=0}^{\infty}z^nV_n(t),\ x>0 \tag{19}$$

which are convergent inside the circle given by $z \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\bar{f}(s)=\int_0^\infty e^{-st}f(t)dt,\ \Re(s)>0. \tag{20}$$

We take the Laplace transform of equations (1) - (16) and use (17) to obtain

$$\frac{\partial}{\partial x}\bar{P}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x))\bar{P}_0^{(1)}(x,s)=0 \tag{21}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x))\bar{P}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(1)}(x,s),n\geq 1 \tag{22}$$

$$\frac{\partial}{\partial x}\bar{P}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x))\bar{P}_0^{(2)}(x,s)=0 \tag{23}$$

$$\frac{\partial}{\partial x}\bar{P}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x))\bar{P}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\bar{P}_{n-k}^{(2)}(x,s),n\geq 1 \tag{24}$$

$$\frac{\partial}{\partial x}\overline{R}_0^{(1)}(x,s)+(s+\lambda+\mu_1(x))\overline{R}_0^{(1)}(x,s)=0 \tag{25}$$

$$\frac{\partial}{\partial x}\overline{R}_n^{(1)}(x,s)+(s+\lambda+\mu_1(x))\overline{R}_n^{(1)}(x,s)=\lambda\sum_{k=1}^{n}c_k\overline{R}_{n-k}^{(1)}(x,s),n\geq 1 \tag{26}$$

$$\frac{\partial}{\partial x}\overline{R}_0^{(2)}(x,s)+(s+\lambda+\mu_2(x))\overline{R}_0^{(2)}(x,s)=0 \tag{27}$$

$$\frac{\partial}{\partial x}\overline{R}_n^{(2)}(x,s)+(s+\lambda+\mu_2(x))\overline{R}_n^{(2)}(x,s)=\lambda\sum_{k=1}^{n}c_k\overline{R}_{n-k}^{(2)}(x,s),n\geq 1 \tag{28}$$

$$\frac{\partial}{\partial x}\overline{V}_0(x,s)+[s+\lambda+\gamma(x)]\overline{V}_0(x,s)=0 \tag{29}$$

$$\frac{\partial}{\partial x}\overline{V}_n(x,s)+[s+\lambda+\gamma(x)]\overline{V}_n(x,s)=\lambda\sum_{k=1}^{n}c_k\overline{V}_{n-k}(x,s),\ n\geq 1 \tag{30}$$

$$\begin{aligned}(s+\lambda)\overline{Q}(s)=1&+(1-p)(1-r_1)(1-\theta)\int_0^\infty \overline{P}_0^{(1)}(x,s)\mu_1(x)dx\\&+(1-r_2)(1-\theta)\int_0^\infty \overline{P}_0^{2}(x,s)\mu_2(x)dx+(1-\theta)\int_0^\infty \overline{R}_0^{(2)}(x,s)\mu_2(x)dx\\&+(1-\theta)(1-p)\int_0^\infty \overline{R}_0^{(1)}(x,s)\mu_1(x)dx+\int_0^\infty V_0(x,s)\gamma(x)dx\end{aligned} \tag{31}$$

$$\begin{aligned}\overline{P}_n^{(1)}(0,s)=\lambda c_{n+1}\overline{Q}(s)&+(1-\theta)(1-p)(1-r_1)\int_0^\infty P_{n+1}^{(1)}(x,s)\mu_1(x)dx\\&+(1-\theta)(1-r_2)\int_0^\infty P_{n+1}^{(2)}(x,s)\mu_2(x)dx+(1-\theta)(1-p)\int_0^\infty R_{n+1}^{(1)}(x,s)\mu_1(x)dx\\&+(1-\theta)\int_0^\infty R_{n+1}^{(2)}(x,s)\mu_2(x)dx+\int_0^\infty V_{n+1}(x,s)\gamma(x)dx,\ n\geq 0\end{aligned} \tag{32}$$

$$\overline{P}_n^{(2)}(0,s)=p(1-r_1)\int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx+p\int_0^\infty R_n^{(1)}(x,s)\mu_1(x)dx,\quad n\geq 0 \tag{33}$$

$$\overline{R}_n^{(1)}(0,s)=r_1\int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx,\ n\geq 0 \tag{34}$$

$$\overline{R}_n^{(2)}(0,s)=r_2\int_0^\infty P_n^{(2)}(x,s)\mu_2(x)dx,\ n\geq 0 \tag{35}$$

$$\begin{aligned}V_n(0,s)=\theta(1-r_1)(1-p)&\int_0^\infty P_n^{(1)}(x,s)\mu_1(x)dx\\&+\theta(1-p)\int_0^\infty R_n^{(1)}(x,s)\mu_1(x)dx+\theta(1-r_2)\int_0^\infty P_n^{(2)}(x,s)\mu_2(x)dx\\&+\theta\int_0^\infty R_n^{(2)}(x,s)\mu_2(x)dx,\ n\geq 0\end{aligned} \tag{36}$$

Now multiplying equations (22), (24), (26) (28) and (30) by z^n and summing over n from 0 to ∞, adding to equations (21), (23), (25) and (27) using the generating functions defined in (18) and (19) we get

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\overline{P}^{(1)}(x,z,s)=0 \tag{37}$$

$$\frac{\partial}{\partial x}\overline{P}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z))+\mu_2(x)]\overline{P}^{(2)}(x,z,s)=0 \tag{38}$$

$$\frac{\partial}{\partial x}\overline{R}^{(1)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_1(x)]\overline{R}^{(1)}(x,z,s)=0 \tag{39}$$

$$\frac{\partial}{\partial x}\overline{R}^{(2)}(x,z,s)+[s+\lambda-\lambda C(z)+\mu_2(x)]\overline{R}^{(2)}(x,z,s)=0 \tag{40}$$

$$\frac{\partial}{\partial x}\overline{V}(x,z,s)+[s+\lambda-\lambda C(z)+\gamma(x)]\overline{V}(x,z,s)=0 \tag{41}$$

For the boundary conditions, we multiply both sides of equation (32) by z^n sum over n from 0 to ∞, and use the equation (31), we get

$$\begin{aligned} z\overline{P}^{(1)}(0,z,s) &= [1-(s+\lambda)\overline{Q}(s)]+\lambda C(z)\overline{Q}(s) \\ &+(1-\theta)(1-r_1)(1-p)\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)dx \\ &+(1-\theta)(1-p)\int_0^\infty \overline{R}^{(1)}(x,z,s)\mu_1(x)dx+(1-\theta)\int_0^\infty \overline{R}^{(2)}(x,z,s)\mu_2(x)dx \\ &+(1-\theta)(1-r_2)\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx+\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx \end{aligned} \tag{42}$$

Performing similar operation on equations (33) to (36) we get,

$$\overline{P}^{(2)}(0,z,s)=(1-r_1)p\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)dx+p\int_0^\infty \overline{R}^{(1)}(x,z,s)\mu_1(x)dx \tag{43}$$

$$\overline{R}^{(1)}(0,z,s)=r_1\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)dx \tag{44}$$

$$\overline{R}^{(2)}(0,z,s)=r_2\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx \tag{45}$$

$$\begin{aligned} V(0,z,s) &= \theta(1-r_1)(1-p)\int_0^\infty P^{(1)}(x,z,s)\mu_1(x)dx \\ &+\theta(1-r_2)\int_0^\infty P^{(2)}(x,z,s)\mu_2(x)dx+\theta(1-p)\int_0^\infty R^{(1)}(x,z,s)\mu_1(x)dx \\ &+\theta\int_0^\infty R^{(2)}(x,z,s)\mu_2(x)dx,\ n\ge 0 \end{aligned} \tag{46}$$

Integrating equation (37) between 0 to x, we get

$$\overline{P}^{(1)}(x,z,s)=\overline{P}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_1(t)dt} \tag{47}$$

where $P^{(1)}(0,z,s)$ is given by equation (42).

Again integrating equation (47) by parts with respect to x yields,

$$\overline{P}^{(1)}(z,s)=\overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{48}$$

where

$$\overline{B}_1(s+\lambda-\lambda C(z))=\int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_1(x)$$

is the Laplace-Stieltjes transform of the first phase service time $B_1(x)$. Now multiplying both sides of equation (47) by $\mu_1(x)$ and integrating over x we obtain

$$\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_1(x)dx=\overline{P}^{(1)}(0,z,s)\overline{B}_1[s+\lambda(1-c(z))] \tag{49}$$

Similarly, on integrating equations (38) to (41) from 0 to x, we get

$$\overline{P}^{(2)}(x,z,s)=\overline{P}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_2(t)dt} \tag{50}$$

$$\overline{R}^{(1)}(x,z,s) = \overline{R}^{(1)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_1(t)dt} \tag{51}$$

$$\overline{R}^{(2)}(x,z,s) = \overline{R}^{(2)}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \mu_2(t)dt} \tag{52}$$

$$\overline{V}(x,z,s) = \overline{V}(0,z,s)e^{-[s+\lambda-\lambda C(z)]x-\int_0^x \gamma(t)dt} \tag{53}$$

where $\overline{P}^{(2)}(0,z,s), \overline{R}^{(1)}(0,z,s)$, $\overline{R}^{(2)}(0,z,s)$ and $\overline{V}(0,z,s)$ are given by equations (43) to (46). Again integrating equations (50) to (53) by parts with respect to x yields,

$$\overline{P}^{(2)}(z,s) = \overline{P}^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{54}$$

$$\overline{R}^{(1)}(z,s) = \overline{R}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{55}$$

$$\overline{R}^{(2)}(z,s) = \overline{R}^{(2)}(0,z,s)\left[\frac{1-\overline{B}_2(s+\lambda(1-C(z)))}{s+\lambda-\lambda C(z)}\right] \tag{56}$$

$$\overline{V}(z,s) = \overline{V}(0,z,s)\left[\frac{1-\overline{V}(s+\lambda-\lambda C(z))}{s+\lambda-\lambda C(z)}\right] \tag{57}$$

where

$$\overline{B}_2(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dB_2(x)$$

$$\overline{V}(s+\lambda-\lambda C(z)) = \int_0^\infty e^{-[s+\lambda-\lambda C(z)]x}dV(x)$$

are the Laplace-Stieltjes transform of the second phase service time and vacation time. Now multiplying both sides of equations (50) to (53) by $\mu_1(x)$, $\mu_2(x)$ and $\gamma(x)$ integrating over x , we obtain

$$\int_0^\infty \overline{P}^{(2)}(x,z,s)\mu_2(x)dx = \overline{P}^{(2)}(0,z,s)\overline{B}_2[s+\lambda-\lambda C(z)] \tag{58}$$

$$\int_0^\infty \overline{R}^{(1)}(x,z,s)\mu_1(x)dx = \overline{R}^{(1)}(0,z,s)\overline{B}_1[s+\lambda-\lambda C(z)] \tag{59}$$

$$\int_0^\infty \overline{R}^{(2)}(x,z,s)\mu_2(x)dx = \overline{R}^{(2)}(0,z,s)\overline{B}_2[s+\lambda-\lambda C(z)] \tag{60}$$

$$\int_0^\infty \overline{V}(x,z,s)\gamma(x)dx = \overline{V}(0,z,s)\overline{V}[s+\lambda-\lambda C(z)] \tag{61}$$

Using equation (58) in (45), we get

$$\overline{R}^{(2)}(0,z,s) = r_2\overline{B}_2(a)\overline{P}^{(2)}(0,z,s) \tag{62}$$

By using equation (49) in (44), we get

$$\overline{R}^{(1)}(0,z,s) = r_1\overline{B}_1(a)\overline{P}^{(1)}(0,z,s) \tag{63}$$

Using equations (49), (59) and using (63) in (43), we get

$$\overline{P}^{(2)}(0,z,s) = p\overline{B}_1(a)[1-r_1+r_1\overline{B}_1(a)]\overline{P}^{(1)}(0,z,s) \tag{64}$$

Using equations (49), (58), (59) and (60) in (46), we get

$$\overline{V}(0,z,s) = \theta p\overline{B}_1(a)\overline{B}_2(a)[1-r_1+r_1\overline{B}_1(a)][1-r_2+r_2\overline{B}_2(a)]\overline{P}^{(1)}(0,z,s)$$

$$+\theta(1-p)\overline{B}_2(a)\overline{P}^{(1)}(0,z,s) \qquad (65)$$

Using equations (49), (58) to (61) in (42), we get

$$z\overline{P}^{(1)}(0,z,s)=[1-s\overline{Q}(s)]+\lambda(C(z)-1)\overline{Q}(s)$$
$$+\overline{B}_1(a)(1-r_1+r_1\overline{B}_1(a))(1-\theta+\theta\overline{V}(a))$$
$$[1-p+p\overline{B}_2(a)(1-r_2+r_2\overline{B}_2(a))]\overline{P}^{(1)}(0,z,s) \qquad (66)$$

Similarly using equations (62) to (65), in (66), we get

$$\overline{P}^{(1)}(0,z,s)=\frac{\lambda(C(z)-1)\overline{Q}(s)+(1-s\overline{Q}(s))}{Dr} \qquad (67)$$

where

$$Dr=z-\overline{B}_1(a)(1-r_1+r_1\overline{B}_1(a))(1-\theta+\theta\overline{V}(a))[1-p+p\overline{B}_2(a)(1-r_2+r_2\overline{B}_2(a))],$$

and $a=s+\lambda-\lambda C(z)$.

Substituting (67) into equations (62) to (65), we get

$$\overline{P}^{(2)}(0,z,s)=p\overline{B}_1(a)(1-r_1+r_1\overline{B}_1(a))\frac{[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)]}{Dr} \qquad (68)$$

$$\overline{R}^{(1)}(0,z,s)=r_1\overline{B}_1(a)\frac{[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)]}{Dr} \qquad (69)$$

$$\overline{R}^{(2)}(0,z,s)=r_2p\overline{B}_1(a)\overline{B}_2(a)(1-r+r\overline{B}_1(a))\frac{[(1-s\overline{Q}(s))+\lambda(C(z)-1)\overline{Q}(s)]}{Dr} \qquad (70)$$

$$\overline{V}(0,z,s)=\frac{\theta}{Dr}\overline{B}_1(a)(1-r_1+r_1\overline{B}_1(a))(1-p+p\overline{B}_2(a)(1-r_2+r_2\overline{B}_2(a)))$$
$$[\lambda(C(z)-1)\overline{Q}(s)+(1-s\overline{Q}(s))] \qquad (71)$$

Using equations (67) to (71) in (48) and (54) to (57), we get $\overline{P}^{(1)}(z,s),\overline{P}^{(2)}(z,s),\overline{R}^{(1)}(z,s)$, $\overline{R}^{(2)}(z,s)$ and $\overline{V}(z,s)$. Thus which completes the proof of the theorem.

5. Conclusion

In this paper we have studied a batch arrival queue with two phases of service and optional re-service with Bernoulli vacation. This paper clearly analyzes the transient solution, steady state results , Mean number of customer in the queue and the system of our queueing system.

References

[1] Baba,Y. (1986). On the $M^{[x]}/G/1$ queue with vacation time, *Operation Research Letter*, 5, 93-98.
[2] Choudhury, G.. (2002). A batch arrival queue with a vacation time under single vacation policy, *Computer Operation Research*, 29, 1941-1955.
[3] Choudhury, G.., & Paul, M. (2004). A batch arrival queue with an additional service channel under N-policy, *Appllied Mathematical Computation*, 156, 115-130.
[4] Choudhury, G.., & Paul, M. (2006). A batch arrival queue with a second optional service channel under N-policy, *Stochastic Analysis and Applications*, 24, 1-22.
[5] Ke, J. (2008). An $M^{[x]}/G/1$ system with startup server and J additional options for service, *Appllied Mathematical Modelling*, 32, 443-458.
[6] Keilson, J., & Servi, L. D. (1987). Dynamic of the M/G/1 vacation model, *Operation Research*, 35(4).
[7] Lee, S.S., et al. (1995). Batch arrival queue with N-policy and single vacation, *Computer Operation Research*, 22, 173-189.
[8] Madan, K.C. (2000). An M/G/1 queue with second optional service, *Queueing System*, 34, 79-98.
[9] Madan, K.C. & Choudhury, G.. (2005). A single server queue with two phases of heterogenous service under Bernoulli schedule and a general vcation time, *Information and Managment Science*, 16(2), 1-16.
[10] Medhi, J. (2002). A single server Poisson input queue with a second optional channel, *Queueing System*, 42, 239-242.