Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.3, No.13, 2013 www.iiste.org

Numerical method for pricing American options under regime-switching jump-diffusion models

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Abstract

Our concern in this paper is to solve the pricing problem for American options in a Markov-modulated jump-diffusion model, based on a cubic spline collocation method. In this respect, we solve a set of coupled partial integro-differential equations PIDEs with the free boundary feature by using the horizontal method of lines to discretize the temporal variable and the spatial variable by means of Crank-Nicolson scheme and a cubic spline collocation method, respectively. This method exhibits a second order of convergence in space, in time and also has an acceptable speed in comparison with some existing methods. We will compare our results with some recently proposed approaches.

Keywords: American Option, Regime-Switching, Crank-Nicolson scheme, Spline collocation, Free Boundary Value Problem.

1. Introduction

Options form a very important and useful class of financial securities in the modern financial world. They play a very significant role in the investment, financing and risk management activities of the finance and insurance markets around the globe. In many major international financial centers, such as New York, London, Tokyo, Hong Kong, and others, options are traded very actively and it is not surprising to see that the trading volume of options often exceeds that of their underlying assets. A very important issue about options is how to determine their values. This is an important problem from both theoretical and practical perspectives

Recently, there has been a considerable interest in applications of regime switching models driven by a Markov chain to various financial problems. For an overview of Markov Chains

The Markovian regime-switching paradigm has become one of the prevailing models in mathematical finance. It is now widely known that under the regime-switching model, the market is incomplete and so the option valuation problem in this framework will be a challenging task of considerable importance for market practitioners and academia. In an incomplete market, the payoffs of options might not be replicated perfectly by portfolios of primitive assets. This makes the option valuation problem more difficult and challenging. Among the many researchers that have addressed the option pricing problem under the regime-switching model, we must mention the following: [4] develop a new numerical schemes for pricing American option with regime-switching. [20] provides a general framework for pricing of perpetual American and real options in regime-switching Levy models. [20] investigate the pricing of both European and American-style options when the price dynamics of the underlying risky assets are governed by a Markov-modulated constant elasticity of variance process. [17] develop a new tree method for pricing financial derivatives in a regimes-witching mean-reverting model. [22] develop a flexible model to value longevity bonds under stochastic interest rate and mortality with regime-switching.

The paper is organized as follows. In Section 2, we describe briefly the problem for American options in a Markov-modulated jump-diffusion model. Then, we discuss time semi-discretization in Section 3. Section 4 is devoted to the spline collocation method for pricing American options under regime-switching jump-diffusion models using a cubic spline collocation method. Next, the error bound of the spline solution is analyzed. In order to validate the theoretical results presented in this paper, we present numerical tests on three known examples in Section 5. The obtained numerical results are compared to the ones given in [2]. Finally, a conclusion is given in Section 6.

2. Regime-switching Lévy processes

Markov chains are frequently used for capturing random shifts between different states. In this section, we review the most important definitions from continuous-time Markov chains, Lévy processes with regime-switching (or Markov-modulated) parameters and also option pricing in this framework (see Chourdakis [11] for a comprehensive treatment).

Let α_t be a continuous-time Markov chain taking values among H different states, where H is the total number of states considered in the economy. Each state represents a particular regime and is labeled by an integer i between 1 and H. Hence the state space of α_t is given by $M = 1,...,H$. Let matrix $Q = (q_{ij})_{H\times H}$ denote the generator of α_t. From Markov chain theory (see for example, Yin and Zhang [9]), the entries q_{ij} in Q satisfy: (I) $q_{ij} \geq 0$ if $i \neq j$; (II) $q_{ii} \leq 0$ and $q_{ii} = -\sum_{j\neq i} q_{ij}$ for each $i = 1,...,H.$

Let W_t be a standard Brownian motion defined on a risk-neutral probability space $(\Omega, \hat{\mathrm{o}}, \mathbf{P})$ and assume that this process is independent of the Markov chain α_t. We consider the following regime-switching exponential Lévy model for describing the underlying asset price dynamics:

$$S_t = S_0 e^{X_t}.$$

The log-price process X_t will be constructed in the following manner: Consider a collection of independent Lévy processes $\{Y_t^i\}_{i=1}^H$ indexed by i. The increments of the log-price process will switch between these H different Lévy processes, depending on the state at α_t:

$$dX_t = dY_t^{\alpha_t}.$$

Each Lévy process Y_t^i assumed to have a Lévy-Itô decomposition of the form

$$dY_t^i = \mu_t^i dt + \sigma_t^i dW_t + \int_{\mathbb{R}} z N^i(dz, dt), \quad i = 1, 2, ..., H,$$

in which μ^i is the drift and σ^i is the diffusion coefficient of the i-th Lévy process. In this equation, $N^i(.,t)$ is a Poisson random measure defined on Borel subsets of $\mathbb{R}$ with $\nu^i(.)$ as its associated Lévy measure, describing the discontinuities.

We now consider the pricing of an American put option written on the underlying asset $\{S_t\}_{t\geq 0}$ with strike price K and maturity date T. To obtain an equation with constant coefficients for the price of this option in each regime, we switch to log-prices and let $x = \log(S_t/S_0)$. Then the transformed option price $V_i(x,t)$ at time $0 \leq t \leq T$ and regime at $\alpha_t = i$ will satisfy the following equation, due to the risk-neutral pricing principle (see for example Karatzas and Shreve [10]):

$$V_i(x,t) = \sup_{t\leq\tau\leq T} \mathbb{E}\left[e^{-r(\tau-t)}(K - e^{X_\tau})^+ \mid X_t = x, \alpha_t = i\right]$$

In the above equation, τ is a stopping time satisfying $t \leq \tau \leq T$ and $\mathbb{E}$ is the expectation operator with respect to equivalent martingale measure $\mathbf{P}$. This is the optimal stopping formulation of the American option pricing problem. We must note here that in writing these equations and all subsequent ones, the parameters μ^i and σ^i in (2.1) are taken regime-independent and constant and also the fixed interest rate r is absorbed into the constant μ term to simplify the presentation of the material. One can now show that $V_i(x,\tau)$ for $i = 1,2,...,H$ satisfy the following system of free boundary value problems [4]:

$$\begin{cases} \dfrac{\partial V_i}{\partial \tau} - \mathrm{L}^i V_i + \sum_{j=1}^{H} q_{ij} V_j & = & 0, & x > \bar{x}_i(\tau), i = 1,2,...,H, \\ V_i(x,\tau) & = & K - e^x, & x \le \bar{x}_i(t), i = 1,2,...,H, \\ V_i(x,T) & = & (K - e^x)^+, & i = 1,2,...,H, \\ \lim_{x \to \bar{x}_i(\tau)} V_i(x,\tau) & = & K - e^{\bar{x}_i(\tau)}, & i = 1,2,...,H, \\ \lim_{x \to \bar{x}_i(\tau)} \dfrac{\partial V_i}{\partial x} & = & -1, & i = 1,2,...,H, \\ \bar{x}_i(T) & = & K, & i = 1,2,...,H, \end{cases} \tag{1}$$

in which $\bar{x}_i(\tau)$ for $i = 1,...,H$ denote the optimal exercise boundaries and L^i is the infinitesimal generator of the i-th Lévy process of the form

$$\mathrm{L}^i V_i = -\frac{1}{2}\sigma^2 \partial_{xx} V_i - (r - \frac{1}{2}\sigma^2 - \lambda^i \xi)\partial_x V_i + (r + \lambda^i) V_i - \int_{\mathrm{R}} V_i(x+z,\tau)\nu^i(dz),$$

with

$$\lambda^i \text{ stands for the jump intensity at state i},$$
$$\xi = \int_{\mathrm{R}} (e^z - 1) dF(z),$$

for the function F which is the distribution of jumps sizes. This is a set of coupled partial integro-differential equations with H free boundaries due to the regime-switching feature introduced in the underlying asset model. The analytical solution of the above system of PIDEs is not available at hand and so the need for efficient numerical approaches seems a necessity. In the sequel, we introduce our approach to solve this set of equations.
Remark: One should notice that if we set $\lambda = 0$ and $H = 1$; (1) will become original Black-Scholes PDE.

3. Time and Spatial discretization

Our aim in this section is to use a cubic spline collocation method to find an approximate solution for the set of Eqs. (1). By using the change of variables $t = T - \tau$ and applying the Crank-Nicolson scheme in time, we can use the collocation method in each time step to find a continuous approximation in the whole interval. It is obvious that $V_i(x,t)$ for $i = 1,...,H$ satisfy the following set of coupled PIDEs in operator form:

$$\frac{\partial V_i}{\partial t} + \mathrm{L} V_i - \sum_{j=1}^{H} q_{ij} V_j = 0, \qquad i = 1,...,H,$$

which is valid in the space-time domain $[-\infty, +\infty] \times [0,T]$. In order to numerically approximate the solution, let us truncate the x-domain into the sub domain $\Omega_x = [x_{min}, x_{max}]$.

Taking $V = [V_1, V_2, ..., V_H]^T$ and $\Omega = \Omega_x \times [0,T]$,

$$\begin{cases} \dfrac{\partial V}{\partial t} - P\dfrac{\partial^2 V}{\partial x^2} - R\dfrac{\partial V}{\partial x} + (G - Q)V & = & I(V(x,t)), & (x,t) \in \Omega, \\ V(x,0) & = & K.I_H, & x \in \Omega_x, \\ V(x_{min},t) & = & \max(K - e^x, 0).I_H, & t \in [0,T], \\ V(x_{max},t) & = & 0.I_H, & t \in [0,T], \end{cases} \tag{2}$$

where

$$
\begin{aligned}
I_H &= [1,1,...,1]^T \in \mathbb{R}^H, \\
P &= \left(diag(\frac{\sigma^2}{2} I_{j,j}) \right)_{j=1,...H}, \\
R &= \left(diag(r - \frac{\sigma^2}{2} - \lambda^j \xi) \right)_{j=1,...H}, \\
G &= \left(diag(r + \lambda^j) \right)_{j=1,...H}, \\
I(V) &= \left(diag(\lambda^j) \right)_{j=1,...H} \int_{\mathbb{R}} V(x+z,t) f(z) dz,
\end{aligned}
$$

with $Q - G$ is a continuous, bounded, symmetric matrix function and each function of the matrix $G - Q$ is $\geq \tilde{\gamma} > 0$ on $\overline{\Omega}$ and $\max(K - e^x, 0)$ is sufficiently smooth function.

Here we assume that the problem satisfies sufficient regularity and compatibility conditions which guarantee that the problem has a unique solution $V \in C(\overline{\Omega}) \cap C^{2,1}(\Omega)$ satisfying (see, [13, 1, and 14]):

$$
\left| \frac{\partial^{i+j} V(x,t)}{\partial x^i \partial t^j} \right| \leq k \ on\ \overline{\Omega}; \quad 0 \leq j \leq 3 \ and\ 0 \leq i + j \leq 4, \tag{3}
$$

where k is a constant in $\mathbb{R}^H$.

3.1. Time discretization and description of the Crank-Nicolson scheme

Discretize the time variable by setting $t^m = m\Delta t$ for $m = 0,1,...,M$, in which $\Delta t = \frac{T}{M}$ and then define

$$
V^m(x) = V(x, t^m), \ m = 0,1,...,M.
$$

Now by applying the Crank-Nicolson scheme on (2), we arrive at the following equation

$$
\frac{V^{m+1}(x) - V^m(x)}{\Delta t} - \frac{1}{2} \mathrm{L}(V^{m+1} + V^m) = \frac{1}{2} \left(I(V^{m+1}) + I(V^m) \right)
$$

One way is to replace V^{m+1} with V^m in the linear terms. This leads to the following modified system:

$$
V^{m+1}(x) - \frac{\Delta t}{2} \mathrm{L} V^{m+1} = \frac{\Delta t}{2} \mathrm{L} V^m + V^m + \Delta t I(V^m). \tag{4}
$$

For $m = 0,1,...,M$. The final price of the American option at time level m will be of the form:

$$
\begin{cases}
P \dfrac{\partial^2 V^{m+1}}{\partial x^2} + R \dfrac{\partial V^{m+1}}{\partial x} + L V^{m+1} = J\ (V^m), & \forall x \in \Omega_x, \\
V^0(x) = \phi_0(x).I_H, & \forall x \in \Omega_x, \\
V^{m+1}(x_{min}) = \psi.I_H, & 0 \leq m < M, \\
V^{m+1}(x_{max}) = 0.I_H, & 0 \leq m < M.
\end{cases} \tag{5}
$$

Where, for any $m \geq 0$ and for any $x \in \Omega_x$, we have

$$\begin{aligned}
L &= \left(Q-G-\frac{2}{\Delta t}I\right),\\
J\,(V^m) &= \mathsf{L}V^m-\frac{2}{\Delta t}V^m-2I(V^m),\\
\mathsf{L} &= P\frac{\partial^2}{\partial x^2}+R\frac{\partial}{\partial x}-(G-Q)I,\\
\phi_0(x) &= (K-e^x)^+,\\
\psi &= K-e^{x_{min}},
\end{aligned}$$

V^{m+1} is solution of (5), at the $(m+1)$ th-time level.

The following theorem proves the order of convergence of the solution V^m to $V(x,t)$.

Theorem 3.1 Problem (5) is second order convergent .i.e.

$$\left\|V(x,t_m)-V^m\right\|_H \le C(\Delta t)^2.$$

Proof: We introduce the notation $e_m=V(x,t_m)-V^m$ the error at step m and

$$\left\|e_m\right\|_H=\sup_{x\in\Omega_x}\max_{1\le i\le H}|e_m^i(x)|$$

By Taylor series expansion of $V,$ we have

$$V(x,t_{m+1})=V(x,t_{m+\frac{1}{2}})+\frac{\Delta t}{2}\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}})+\frac{(\Delta t)^2}{8}\frac{\partial^2 V}{\partial t^2}(x,t_{m+\frac{1}{2}})+O((\Delta t)^3).I_H,$$

$$V(x,t_m)=V(x,t_{m+\frac{1}{2}})-\frac{\Delta t}{2}\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}})+\frac{(\Delta t)^2}{8}\frac{\partial^2 V}{\partial t^2}(x,t_{m+\frac{1}{2}})+O((\Delta t)^3).I_H.$$

By using these expansions, we get

$$\frac{V(x,t_{m+1})-V(x,t_m)}{\Delta t}=\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}})+O((\Delta t)^2).I_H, \tag{6}$$

and by Taylor series expansion of $\dfrac{\partial V}{\partial t}$, we have

$$\frac{\partial V}{\partial t}(x,t_{m+1})=\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}})+\frac{\Delta t}{2}\frac{\partial^2 V}{\partial t^2}(x,t_{m+\frac{1}{2}})+\frac{(\Delta t)^2}{8}\frac{\partial^3 V}{\partial t^3}(x,t_{m+\frac{1}{2}})+O((\Delta t)^3).I_H,$$

$$\frac{\partial V}{\partial t}(x,t_m)=\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}})-\frac{\Delta t}{2}\frac{\partial^2 V}{\partial t^2}(x,t_{m+\frac{1}{2}})+\frac{(\Delta t)^2}{8}\frac{\partial^3 V}{\partial t^3}(x,t_{m+\frac{1}{2}})+O((\Delta t)^3).I_H.$$

By using these expansions, and $\left|\dfrac{\partial^3 V}{\partial t^3}\right|\le c.I_H$ on $\overline{\Omega}$ (see relation (3)), we have

$$\frac{1}{2}\frac{\partial}{\partial t}[V(x,t_{m+1})+V(x,t_m)]=\frac{\partial}{\partial t}V(x,t_{m+\frac{1}{2}})+O((\Delta t)^2).I_H.$$

This implies

$$\begin{aligned}
\frac{\partial V}{\partial t}(x,t_{m+\frac{1}{2}}) &= \frac{1}{2}\frac{\partial}{\partial t}[V(x,t_{m+1})+V(x,t_m)]+O((\Delta t)^2).I_H\\
&=\frac{1}{2}[\mathsf{L}V(x,t_{m+1})+I(V(x,t_{m+1}))+\mathsf{L}V(x,t_m)+I(V(x,t_m))]+O((\Delta t)^2).I_H.
\end{aligned}$$

By using this relation in (6) we get

$$(1-\frac{\Delta t}{2}\mathsf{L})V(x,t_{m+1})=(1+\frac{\Delta t}{2}\mathsf{L})V(x,t_m)+\frac{\Delta t}{2}[I(V(x,t_{m+1}))+I(V(x,t_m))]+O((\Delta t)^3).I_H,$$

by (4). Then, we obtain

$$(1-\frac{\Delta t}{2}\mathsf{L})e_{m+1}=(1+\frac{\Delta t}{2}\mathsf{L})e_m+\frac{\Delta t}{2}\left[I(e_{m+1})+I(e_m)\right]+O((\Delta t)^3).I_H.$$

We may bound the term $O((\Delta t)^3)$ by $c(\Delta t)^3$ for some $c>0$, and this upper bound is valid uniformly throughout $[0,T]$. Therefore, it follows from the triangle inequality that

$$\left\|(I-\frac{\Delta t}{2}\mathsf{L})e_{m+1}\right\|_H\le\left\|(I+\frac{\Delta t}{2}\mathsf{L})e_m\right\|_H+\frac{\Delta t}{2}\left(\|I(e_m)\|_H+\|I(e_{m+1})\|_H\right)+c(\Delta t)^3.$$

We use the cross-correlation function (see [3]) defined by

$$R_{fe_m^i}(x)=f(x)**e_m^i(x)=\int_{\mathsf{R}}f(z)e_m^i(x+z)dz,\ for\ i=1,...,H,$$

we have

$$\left\|R_{fe_m}\right\|_H\le\|f\|_H\|e_m\|_H.$$

$$\begin{aligned}\left\|(I-\frac{\Delta t}{2}\mathsf{L})e_{m+1}\right\|_H&\le\left\|(I+\frac{\Delta t}{2}\mathsf{L})e_m\right\|_H+\frac{\Delta t}{2}\left(\|I(e_m)\|_H+\|I(e_{m+1})\|_H\right)+c(\Delta t)^3,\\&\le\left\|(I+\frac{\Delta t}{2}\mathsf{L})\right\|_H\|e_m\|_H+\frac{\Delta t}{2}\|\lambda\|_H\|f\|_H\left(\|e_m\|_H+\|e_{m+1}\|_H\right)+c(\Delta t)^3.\end{aligned}$$

Clearly, the operator $\left(I_H\pm\frac{\Delta t}{2}\mathsf{L}\right)$ satisfies a maximum principle (see, [7, 5]) and consequently

$$\left\|\left(I_H\pm\frac{\Delta t}{2}\mathsf{L}\right)^{-1}\right\|_H\le\left(\frac{1}{1+\frac{\Delta t}{2}\tilde{\gamma}}\right).$$

Since we are ultimately interested in letting $\Delta t\to 0$, there is no harm in assuming that $\Delta t.\eta<2$, with $\eta=\left(\|\mathsf{L}\|_H+\|\lambda\|_H\|f\|_H\right)$. We can thus deduce that

$$\|e_{m+1}\|_H\le\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)\|e_m\|_H+\left(\frac{c}{1-\frac{1}{2}\Delta t.\eta}\right)(\Delta t)^3. \tag{7}$$

We now claim that

$$\|e_m\|_H\le\frac{c}{\eta}\left[\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)^m-1\right](\Delta t)^2. \tag{8}$$

The proof is by induction on m. When $m=0$ we need to prove that $\|e_0\|_H\le 0$ and hence that $e_0=0$. This is certainly true, since at $t_0=0$ the numerical solution matches the initial condition and the error is zero.

For general $m\ge 0$, we assume that (8) is true up to m and use (7) to argue that

$$\|e_{m+1}\|_H \leq \frac{c}{\eta}\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)\left[\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)^m - 1\right](\Delta t)^2 + \left(\frac{c}{1-\frac{1}{2}\Delta t.\eta}\right)(\Delta t)^3$$

$$\leq \frac{c}{\eta}\left[\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)^{m+1} - 1\right](\Delta t)^2.$$

This advances the inductive argument from m to $m+1$ and proves that (8) is true. Since $0<\Delta t.\eta<2,$ it is true that

$$\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right) = 1+\left(\frac{\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right) \leq \sum_{l=0}^{\infty}\frac{1}{l!}\left(\frac{\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)^l = exp\left(\frac{\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right).$$

Consequently, relation (8) yields

$$\|e_m\|_H \leq \frac{c(\Delta t)^2}{\eta}\left(\frac{1+\frac{1}{2}\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right)^m \leq \frac{c(\Delta t)^2}{\eta}exp\left(\frac{m\Delta t.\eta}{1-\frac{1}{2}\Delta t.\eta}\right).$$

This bound is true for every nonnegative integer m such that $m\Delta t < T$. Therefore

$$\|e_m\|_H \leq \frac{c(\Delta t)^2}{\eta}exp\left(\frac{T.\eta}{1-\frac{1}{2}\Delta t.\eta}\right).$$

We deduce that

$$\left\|V(x,t_m)-V^m\right\|_H \leq C(\Delta t)^2.$$

In other words, problem (5) is second order convergent.
For any $m \geq 0$, problem (5) has a unique solution and can be written on the following form:

$$\begin{cases} PV''(x)+RV'(x)+LV(x) &= \tilde{f}(x)\in \mathbb{R}^H, \quad \forall x\in\Omega_x, \\ V(x_{min}) &= \psi.I_H, \\ V(x_{max}) &= 0, \end{cases} \tag{9}$$

In the sequel of this paper, we will focus on the solution of problem (9).

3.2. Spatial discretization and cubic spline collocation method

Let $\otimes$ denotes the notation of Kronecker product, $\|.\|$ the Euclidean norm on $\mathbb{R}^{n+1+H}$ and $S^{(k)}$ the k^{th} derivative of a function S .
In this section we construct a cubic spline which approximates the solution V of problem (9), in the interval $\Omega_x \subset \mathbb{R}$.

Let $\Theta=\{x_{min}=x_{-3}=x_{-2}=x_{-1}=x_0<x_1<\cdots<x_{n-1}<x_n=x_{n+1}=x_{n+2}=x_{n+3}=x_{max}\}$ be a subdivision of the interval Ω_x . Without loss of generality, we put $x_i=a+ih$, where $0\leq i\leq n$

and $h=\frac{x_{max}-x_{min}}{n}$. Denote by $\mathsf{S}_4(\Omega_x,\Theta)=\mathsf{P}_3^2(\Omega_x,\Theta)$ the space of piecewise polynomials of degree less than or equal to 3 over the subdivision Θ and of class C^2 everywhere on Ω_x. Let B_i, $i=-3,\cdots,n-1$, be the B-splines of degree 3 associated with Θ. These B-splines are positives and form a basis of the space $S_4(\Omega_x,\Theta)$.

Consider the local linear operator Q_3 which maps the function V onto a cubic spline space $\mathsf{S}_4(\Omega_x,\Theta)$ and which has an optimal approximation order. This operator is the discrete C^2 cubic quasi-interpolant (see [15]) defined by

$$Q_3V=\sum_{i=-3}^{n-1}\mu_i(V)B_i,$$

where the coefficients $\mu_j(V)$ are determined by solving a linear system of equations given by the exactness of Q_3 on the space of cubic polynomial functions $\mathsf{P}_3(\Omega_x)$. Precisely, these coefficients are defined as follows:

$$\begin{cases}\mu_{-3}(V)=V(x_0)=V(x_{min}),\\ \mu_{-2}(V)=\frac{1}{18}(7V(x_0)+18V(x_1)-9V(x_2)+2V(x_3)),\\ \mu_j(V)=\frac{1}{6}(-V(x_{j+1})+8V(x_{j+2})-V(x_{j+3})),\text{for } j=-1,\dots,n-3,\\ \mu_{n-2}(V)=\frac{1}{18}(2V(x_{n-3})-9V(x_{n-2})+18V(x_{n-1})+7V(x_n)),\\ \mu_{n-1}(V)=V(x_n)=V(x_{max}).\end{cases}$$

It is well known (see e.g. [16], chapter 5) that there exist constants C_k, $k=0,1,2,3,$ such that, for any function $V\in C^4(\Omega_x)$,

$$\left\|V^{(k)}-Q_3V^{(k)}\right\|_H\le C_kh^{4-k}\left\|V^{(4-k)}\right\|_H,\quad k=0,1,2,3, \tag{10}$$

By using the boundary conditions of problem (9), we obtain $\mu_{-3}(V)=Q_3V(x_{min})=V(x_{min})=\psi.I_H$ and $\mu_{n-1}(V)=Q_3V(x_{max})=V(x_{max})=0.I_H$. Hence

$$Q_3V=z_1+S,$$

where

$$z_1=\psi B_{-3}I_H\quad and\quad S=\left[\sum_{j=-2}^{n-2}\mu_j(V_1)B_j,\cdots,\sum_{j=-2}^{n-2}\mu_j(V_H)B_j\right]^T.$$

From equation: (10), we can easily see that the spline S satisfies the following equation

$$PS^{(2)}(x_j)+RS^{(1)}(x_j)+LS^{(0)}(x_j)=g(x_j)+O(h^2).I_H,\ j=0,\dots,n \tag{11}$$

with

$$g(x_j)=\tilde{f}(x_j)-(Pz_1^{(2)}(x_j)+Rz_1^{(1)}(x_j)+Lz_1^{(0)}(x_j))\in\mathsf{R}^H,\ j=0,\dots,n.$$

The goal of this section is to compute a cubic spline collocation $\tilde{Sp}_i=\sum_{j=-3}^{n-1}\tilde{c}_{j,i}B_j$, $i=1,\dots,H$ which satisfies the equation (9) at the points τ_j, $j=0,\dots,n+2$ with $\tau_0=x_0$, $\tau_j=\frac{x_{j-1}+x_j}{2}$, $j=1,\cdots,n$, $\tau_{n+1}=x_{n-1}$ and $\tau_{n+2}=x_n$.

Then, it is easy to see that

$$\tilde{c}_{-3,i} = \psi \qquad and \qquad \tilde{c}_{n-1,i} = 0,\ for\ i = 1,\dots,H.$$

Hence

$$\tilde{S}p_i = z_1 + \tilde{S}_i, \quad where\ \ \tilde{S}_i = \sum_{j=-2}^{n-2} \tilde{c}_{j,i} B_j,\ for\ i = 1,\dots,H$$

and the coefficients $\tilde{c}_{j,i}$, $j = -2,\dots,n-2$ and $i = 1,\dots,H$ satisfy the following collocation conditions:

$$P\tilde{S}^{(2)}(\tau_j) + R\tilde{S}^{(1)}(\tau_j) + L\tilde{S}^{(0)}(\tau_j) = g(\tau_j),\ j = 1,\dots,n+1, \tag{12}$$

where

$$\tilde{S} = [\tilde{S}_1,\dots,\tilde{S}_H]^T,$$
$$g(\tau_j) = \tilde{f}(\tau_j) - (Pz_1^{(2)}(\tau_j) + Rz_1^{(1)}(\tau_j) + Lz_1^{(0)}(\tau_j)) \in \mathrm{R}^H,\ j = 1,\dots,n+1.$$

Taking

$$C = \left[\mu_{-2}(V_1),\dots,\mu_{n-2}(V_1),\dots,\mu_{-2}(V_H),\dots,\mu_{n-2}(V_H)\right]^T \in \mathrm{R}^{n+1+H},$$
$$\tilde{C} = \left[\tilde{c}_{-2,1},\dots,\tilde{c}_{n-2,1},\dots,\tilde{c}_{-2,H},\dots,\tilde{c}_{n-2,H}\right]^T \in \mathrm{R}^{n+1+H},$$

and using equations (11) and (12), we get:

$$\left((P\otimes A_h^{(2)}) + (R\otimes A_h^{(1)}) + (L\otimes A_h^{(0)})\right) C = F + E \tag{13}$$

and

$$\left((P\otimes A_h^{(2)}) + (R\otimes A_h^{(1)}) + (L\otimes A_h^{(0)})\right)\tilde{C} = F, \tag{14}$$

with

$$F = [g_1,\dots,g_{n+1}]^T \text{ and } g_j = \frac{1}{\Delta t} g(\tau_j) \in \mathrm{R}^H,$$
$$E = [O(\frac{h^2}{\Delta t}),\dots,O(\frac{h^2}{\Delta t})]^T \in \mathrm{R}^{n+1+H},$$
$$A_h^{(k)} = (B_{-3+p}^{(k)}(\tau_j))_{1\le j,p\le n+1},\quad k = 0,1,2,$$

It is well known that $A_h^{(k)} = \frac{1}{h^k} A_k$ for $k = 0,1,2$ where matrices A_0, A_1 and A_2 are independent of h, with the matrix A_2 is invertible [8].

Then, relations (13) and (14) can be written in the following form

$$(P\otimes A_2)\left(I + U + V\right)C = h^2 F + h^2 E, \tag{15}$$

$$(P\otimes A_2)\left(I + U + V\right)\tilde{C} = h^2 F_{\tilde{C}}, \tag{16}$$

with

$$U = h(P\otimes A_2)^{-1}(R\otimes A_1), \tag{17}$$
$$V = h^2(P\otimes A_2)^{-1}(L\otimes A_0). \tag{18}$$

In order to determine the bounded of $\| C - \tilde{C} \|_\infty$, we need the following Lemma.

Lemma 3.1 If $h^2\rho < \dfrac{\Delta t}{4}$, then $I+U+V$ is invertible, where $\rho = \| (P\otimes A_2)^{-1} \|_\infty$.

Proof: From the relation (17), we have

$$\| U \|_\infty \le h \| (P\otimes A_2)^{-1} \|_\infty \| (R\otimes A_1) \|_\infty$$
$$\le h\rho \| (R\otimes A_1) \|_\infty .$$

For h sufficiently small, we conclude

$$\| U \|_\infty < \frac{1}{4}. \tag{19}$$

From the relation (18) and $\| A_0 \|_\infty \le 1$, we have

$$\begin{aligned}
\| V \|_\infty &\le h^2 \| (P\otimes A_2)^{-1} \|_\infty \| L\otimes A_0 \|_\infty \\
&\le h^2 \| (P\otimes A_2)^{-1} \|_\infty \| L \|_\infty \\
&\le h^2\rho\left(\| Q-G \|_\infty + \frac{2}{\Delta t}\right) \\
&\le h^2\rho \| Q-G \|_\infty + \frac{2h^2\rho}{\Delta t}.
\end{aligned}$$

For h sufficiently small, we conclude that $h^2\rho \| Q-G \|_\infty < \dfrac{1}{4}$. Then

$$\| Q-G \|_\infty < \frac{1}{4} + \frac{2h^2\rho}{\Delta t}. \tag{20}$$

As $\dfrac{2h^2\rho}{\Delta t} < \dfrac{1}{2}$. So, $\| U+V \| \le \| U \| + \| V \| < 1,$ and therefore $I+U+V$ is invertible.

Proposition 3.1 If $h^2 \le \dfrac{\Delta t}{4\rho}$, then there exists a constant cte which depends only on the functions p, q, l and g such that

$$\| C - \tilde{C} \| \le cte.h^2. \tag{21}$$

Proof: Assume that $h^2 \le \dfrac{\Delta t}{4\rho}$. From (15) and (16), we have

$$C - \tilde{C} = h^2 (I+U+V)^{-1} (P\otimes A_2)^{-1} E.$$

Since $E = O(\dfrac{h^2}{\Delta t}),$ then there exists a constant K_1 such that $\| E \| \le K_1 \dfrac{h^2}{\Delta t}$. This implies that

$$\begin{aligned}
\| C - \tilde{C} \| &\le h^2 \| (I+U+V)^{-1} \|_\infty \| (P\otimes A_2)^{-1} \|_\infty \| E \| \\
&\le \frac{h^2\rho}{\Delta t} \| (I+U+V)^{-1} \|_\infty \ K_1 h^2 \\
&\le \frac{1}{4} \| (I+U+V)^{-1} \|_\infty \ K_1 h^2.
\end{aligned}$$

On the other hand, from (19) and (20), we get $\| U+V \|_\infty < 1$. Thus

$$\| (I+U+V)^{-1} \|_{\infty} \leq \frac{1}{1-\| U+V \|_{\infty}} = cte.$$

Finally, we deduce that

$$\| C-\tilde{C} \| \leq \; cte.h^2.$$

Now, we are in position to prove the main theorem of our work.

Proposition 3.2 The spline approximation $\tilde{Sp}$ converges quadratically to the exact solution V of problem (2), i.e., $\left\| V-\tilde{Sp} \right\|_H = O(h^2)$.

Proof: From the relation (10), we have
$\left\| V-Q_3(V) \right\|_H = O(h^4)$, so $\left\| V-Q_3(V) \right\|_H \leq Kh^4$, where K is a positive constant. On the other hand we have

$$Q_3(V_i(x)) - \tilde{Sp}_i(x) = \sum_{j=-2}^{n-2} (\mu_j(V_i) - \tilde{c}_{j,i}) B_j(x), \;\; for \;\; i=1,...,H.$$

Therefore, by using (21) and $\sum_{j=-2}^{n-2} B_j(x) \leq 1$, we get

$$| Q_3(V_i(x)) - \tilde{Sp}_i(x) | \leq \| C-\tilde{C} \| \;\; \sum_{j=-2}^{n-2} B_j(x) \leq \| C-\tilde{C} \| \leq K_1 h^2, \;\; for \;\; i=1,...,H.$$

Since $\| Q_3(V) - \tilde{Sp} \|_H \leq \| V - Q_3(V) \|_H + \| Q_3(V) - \tilde{Sp} \|_H$, we deduce the stated result.

4. Numerical examples

In this section we verify experimentally theoretical results obtained in the previous section. If the exact solution is known, then at time $t \leq T$ the maximum error E^{max} can be calculated as:

$$E^{max} = \max_{x\in[x_{min},x_{max}], t\in[0,T], 1\leq i\leq H} | S_i^{M,N}(x,t) - V_i(x,t) |.$$

Otherwise it can be estimated by the following double mesh principle:

$$E_{M,N}^{max} = \max_{x\in[x_{min},x_{max}], t\in[0,T], 1\leq i\leq H} | S_i^{M,N}(x,t) - S_i^{2M,2N}(x,t) |,$$

where $S_i^{M,N}(x,t)$ is the numerical solution on the $M+1$ grids in space and $N+1$ grids in time, and $S_i^{2M,2N}(x,t)$ is the numerical solution on the $2M+1$ grids in space and $2N+1$ grids in time, for $1 \leq i \leq H$.

We need to estimate the integral $\int_{\mathbb{R}} V_i^m(x+z)\nu^i dz$ and for this purpose we use a Gaussian quadrature formula in a bounded interval of the form $[z_{min}, z_{max}]$ to arrive at

$$\int_{\mathbb{R}} V_i^m(x+z)\nu^i dz \approx \lambda^i \int_{z_{min}}^{z_{max}} V_i^m(x+z) f(z) dz \approx \lambda^i \sum_{k=1}^{p} w_k V_i^m(x+z_k) f(z_k), \qquad (22)$$

for $i=1,..,H$ in which the w_k 's are the Gaussian quadrature coefficients; cf. [21, 6] for details.

We present two examples to better illustrate the use of the switching Lévy approach and the proposed pricing methodology in concrete situations. These examples are concerned with American put options in three and five world states respectively. In the first example, we assume that the stock price follows a Merton jump-diffusion process with an intensity parameter governed by a three-state hidden Markov chain. In the second one, we consider the Kou jump diffusion model with jump intensities having a discrete five-state Markov dynamics.

4.1. Example 1

In this example, we assume a three-regime economy in which the dynamics of the underlying stock price in the i-th regime obeys a Merton jump-diffusion process with the Lévy measure

$$\nu^i(z) = \frac{\lambda^i}{\sqrt{2\pi}\sigma_j} exp\{-\frac{(z-\mu_j)^2}{2\sigma_j^2}\},$$

where the intensity vector is given by: $\lambda = [0.3, 0.5, 0.7]^T$,
the generator matrix is defined by

$$Q = \begin{bmatrix} -0.8 & 0.6 & 0.2 \\ 0.2 & -1 & 0.8 \\ 0.1 & 0.3 & -0.4 \end{bmatrix}.$$

and the a priori state probabilities are given to be $p = [0.2, 0.3, 0.5]^T$.
Other useful data are provided in the following table:
Table 1. Data used to value American options under regime-switching jump-diffusion models.

Parameter	*values*
S	100
K	100
T	1
σ	0.15
r	0.05
σ_j	0.45
μ_j	-0.5

For this problem, we use a uniform distribution of points in the interval $[x_{\min}, x_{\max}] = [-6, 6]$ for the collocation process and truncate the integration domain in (22) according to

$$z_{\max} = (-2\sigma_j^2 \log(\varepsilon\sigma_j\sqrt{2\pi}/2)) + \mu_j,$$
$$z_{\min} = -z_{\max},$$

with $\varepsilon = 10e^{-12}$. We must note here that using these two bounds forces the total truncation error to be uniformly bounded by ε and the derivation of them is described in full detail in [12] and [19].

The comparison of the maximum error values between the method developed in this paper with the one developed in [2] will be taken at five different values of the number of space steps $N = 256, 512, 1024, 2048,$ and time steps $M = 128, 256, 512, 1024,$.

We conduct experiments on different values of N, M and σ. Table 2 show values of the maximum error (max_error) obtained in our numerical experiments and the one obtained in [2]. We see that the values of maximum error obtained by our method improve the ones in [2].

Table 2. Numerical results for three world states

N	M	Our max_error	max_error in [2]
256	128	0.83×10^{-3}	2.86×10^{-3}
512	256	0.20×10^{-3}	1.78×10^{-3}
1024	512	0.52×10^{-4}	0.88×10^{-3}
2048	1024	0.13×10^{-4}	0.36×10^{-3}

4.2. Example 2

In this example, we assume that the stock price process follows the Kou jump-diffusion model where the jumps arrive at Poisson times and are distributed according to the law

$$\nu^i(z) = \lambda^i(p\eta_1 e^{-\eta_1 z} 1_{z\geq 0} + (1-p)\eta_2 e^{\eta_2 z} 1_{z<0}).$$

We assume that our five-state Markov chain has a generator of the form:

$$Q = \begin{bmatrix} -1 & 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & -1 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & -1 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & -1 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 & -1 \end{bmatrix},$$

and that the economy switches between different jump intensities described by the vector $\lambda = [0.1, 0.3, 0.5, 0.7, 0.9]^T$.

In this case, we suppose that the market could be in any of the five regimes with equal probability. Other corresponding information is given in the following table:

Table 3. Data used to value American options under regime-switching jump-diffusion models.

Parameter	*values*
S	100
K	100
T	0.25
σ	0.5
r	0.05
p	0.5
η_1	3
η_2	2

We use a uniform distribution of points in the interval $[x_{\min}, x_{\max}] = [-6, 6]$ as collocation points and use the following bounds for the truncation process in (22):

$$z_{\max} = \log(\varepsilon/p)/(1-\eta_1),$$
$$z_{\min} = -\log(\varepsilon/(1-p))/(1-\eta_2),$$

where we use the value of $\varepsilon = 10e^{-12}$. We refer the reader to [19] to see a full derivation of these bounds in order to obtain uniform truncation error bounds. Table 3 contains the option prices corresponding to each intensity regime reported for different values of N and M.

The comparison of the maximum error values between the method developed in this paper with the one developed in [2] will be taken at five different values of the number of space steps $N = 512, 1024, 8198$ and time steps $M = 256, 512, 512$ for $\lambda_1 = 0.1,\ \lambda_2 = 0.3,\ \lambda_3 = 0.5,\ \lambda_4 = 0.7$ and $\lambda_5 = 0.9$.

We conduct experiments on different values of N, M and λ. Table 4 show values of the maximum error (max_error) obtained in our numerical experiments and the one obtained in [2]. We see that the values of maximum error obtained by our method improve the ones in [2].

5. Conclusion

In this paper, a cubic spline collocation approach is introduced to price American options in a regime-switching Lévy context. After a brief review of the process of deriving the set of coupled PIDEs describing the prices in different regimes, we present the details of our methodology which consists of first discretizing in time (by Crank-Nicolson scheme) and then collocating in space (by a cubic spline ollocation method). Then, we have shown provided an error estimate of order $O(h^2)$ with respect to the maximum norm $\| \ \|_H$. In our paper we consider a cubic spline space defined by multiple knots in the boundary and we propose a simple and efficient new collocation method by considering as collocation points the mid-points of the knots of the cubic spline space. It is observed that the collocation method developed in this paper, when applied to some examples, can improve the results obtained by the collocation methods given in the literature. The two test problems which are studied in this paper demonstrate that this approach has an efficient alternative to the one proposed in [2].

Table 4. Numerical results for different intensity regimes and discretization parameters.

N	M	Our max_error	max_error in [2]
For $\lambda_1 = 0.1$			
512	256	0.00150	0.0356
1024	512	0.00037	0.0132
8198	512	0.00030	0.0060
For $\lambda_2 = 0.3$			
512	256	0.00146	0.0348
1024	512	0.00036	0.0136
8198	512	0.00029	0.0066
For $\lambda_3 = 0.5$			
512	256	0.00143	0.0341
1024	512	0.00035	0.0138
8198	512	0.00028	0.0070
For $\lambda_4 = 0.7$			
512	256	0.00139	0.0339
1024	512	0.00034	0.0140
8198	512	0.00027	0.0071
For $\lambda_5 = 0.9$			
512	256	0.00133	0.0338
1024	512	0.00033	0.0142
8198	512	0.00026	0.0072

Acknowledgment

We are grateful to the reviewers for their constructive comments that helped to improve the paper.

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